

Direct approach to the problem of strong local minima in calculus of variations

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The proof of Theorem 5 does not prove the statement in Theorem 5. The problem is that the cut-off functions $\theta_k^r(\mathbf{x})$ that vanish on $\partial\Omega$ cannot correctly recover the function $\mathcal{I}(\mathbf{x}_0, \mathbf{x}_0)$ at the points $\mathbf{x}_0 \in \partial\Omega_1$, where the measure $\tilde{\pi}$ has a non-zero mass. This problem is easily fixed by using cut-off functions $\theta_k^r \in C_0^\infty(B(\mathbf{x}_0, r))$ that do not vanish on $\partial\Omega$, if $\mathbf{x}_0 \in \partial\Omega$. We require that $\theta_k^r(\mathbf{x}) \rightarrow \chi_{B(\mathbf{x}_0, r)}(\mathbf{x})$, as $k \rightarrow \infty$ for all $\mathbf{x} \in \mathbb{R}^d$. This change corrects the Proof of Theorem 5 but makes the arguments in Step 4 in the Proof of Theorem 2 (p. 82) invalid, since $\theta_k^r(\mathbf{x})\mathbf{v}_n(\mathbf{x})$ no longer has to vanish on $\partial B_\Omega(\mathbf{x}_0, r) \cap \partial\Omega$. This problem is fixed by augmenting the statement of the Decomposition Lemma (Lemma 1). We claim that it is possible to modify sequences \mathbf{z}_n and \mathbf{v}_n in such a way that they vanish on $\partial\Omega_1$, while satisfying all the other properties required by Lemma 1. Using the modified functions in the consequent analysis leads now to the functions $\theta_k^r(\mathbf{x})\mathbf{v}_n(\mathbf{x})$ that do vanish on $\partial B_\Omega(\mathbf{x}_0, r)$.

Theorem 1 *Suppose the sequence of functions $\psi_n \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ that vanish on $\partial\Omega_1$ (relatively open subset of a C^1 boundary $\partial\Omega$ of Ω) is bounded in $W^{1,2}(\Omega; \mathbb{R}^m)$. Suppose $\alpha_n \rightarrow 0$, as $n \rightarrow \infty$ is a sequence of positive numbers such that $\alpha_n \psi_n \rightarrow \mathbf{0}$ in $W^{1,\infty}$ weak-*. Suppose \mathbf{z}_n and \mathbf{v}_n are as in Lemma 1, i.e.*

- (a) $\psi_n(\mathbf{x}) = \mathbf{z}_n(\mathbf{x}) + \mathbf{v}_n(\mathbf{x})$;
- (b) For all $\mathbf{x} \in \Omega \setminus R_n$ we have $\mathbf{z}_n(\mathbf{x}) = \psi_n(\mathbf{x})$ and $\nabla \mathbf{z}_n(\mathbf{x}) = \nabla \psi_n(\mathbf{x})$;
- (c) The sequence $\{|\nabla \mathbf{z}_n|^2\}$ is equiintegrable;

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- (d) $\mathbf{v}_n \rightharpoonup 0$ weakly in $W^{1,2}(\Omega; \mathbb{R}^m)$;
- (e) $|R_n| \rightarrow 0$ as $n \rightarrow \infty$;
- (f) $\alpha_n \nabla \mathbf{z}_n$ is bounded in L^∞ ;

Then there exists modified versions $\tilde{\mathbf{z}}_n$ and $\tilde{\mathbf{v}}_n$ of \mathbf{z}_n and \mathbf{v}_n , respectively, such that they satisfy all the properties (a)–(f) and additionally the following two properties

- (g) $\alpha_n \tilde{\mathbf{z}}_n \rightarrow \mathbf{0}$, as $n \rightarrow \infty$ uniformly in $\mathbf{x} \in \Omega$;
- (h) $\tilde{\mathbf{z}}_n(\mathbf{x}) = \tilde{\mathbf{v}}_n(\mathbf{x}) = \boldsymbol{\psi}_n(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \partial\Omega_1$.

Proof Step 1. Let $\mathbf{w}_n(\mathbf{x}) = (w_n^{(1)}(\mathbf{x}), \dots, w_n^{(m)}(\mathbf{x}))$ be defined by

$$w_n^{(i)}(\mathbf{x}) = \min\{z_n^{(i)}(\mathbf{x}), \|\boldsymbol{\psi}_n\|_{L^\infty(\Omega \setminus R_n)}\}, \quad i = 1, \dots, m.$$

Then $\mathbf{w}_n(\mathbf{x})$ is Lipschitz continuous with $|\nabla \mathbf{w}_n(\mathbf{x})| \leq |\nabla \mathbf{z}_n(\mathbf{x})|$ for a.e. $\mathbf{x} \in \Omega$, and $\mathbf{w}_n(\mathbf{x}) = \boldsymbol{\psi}_n(\mathbf{x})$ for almost all $\mathbf{x} \in \Omega \setminus R_n$. Similarly, let $\mathbf{u}_n(\mathbf{x}) = (u_n^{(1)}(\mathbf{x}), \dots, u_n^{(m)}(\mathbf{x}))$ be defined by

$$u_n^{(i)}(\mathbf{x}) = \max\{w_n^{(i)}(\mathbf{x}), -\|\boldsymbol{\psi}_n\|_{L^\infty(\Omega \setminus R_n)}\}, \quad i = 1, \dots, m.$$

Then $\mathbf{u}_n(\mathbf{x})$ is Lipschitz continuous with $|\nabla \mathbf{u}_n(\mathbf{x})| \leq |\nabla \mathbf{z}_n(\mathbf{x})|$ for a.e. $\mathbf{x} \in \Omega$, and $\mathbf{u}_n(\mathbf{x}) = \boldsymbol{\psi}_n(\mathbf{x})$ for almost all $\mathbf{x} \in \Omega \setminus R_n$. Therefore $|\nabla \mathbf{u}_n|^2$ is also equiintegrable and

$$\alpha_n \|\mathbf{u}_n\|_\infty \leq \alpha_n \|\boldsymbol{\psi}_n\|_\infty \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Step 2. Let $\boldsymbol{\psi}_0$ be a $W^{1,2}$ -weak limit of (a subsequence of) $\boldsymbol{\psi}_n$. Then there exists a sequence $\widehat{\boldsymbol{\psi}}_n \in C^1(\bar{\Omega}; \mathbb{R}^m)$ such that $\widehat{\boldsymbol{\psi}}_n$ vanishes on $\partial\Omega_1$, converges to $\boldsymbol{\psi}_0$ in the $W^{1,2}$ norm and additionally satisfies

$$\lim_{n \rightarrow \infty} \alpha_n \|\widehat{\boldsymbol{\psi}}_n\|_{1,\infty} = 0.$$

It follows from $|R_n| \rightarrow 0$, as $n \rightarrow \infty$ that $\boldsymbol{\psi}_n - \mathbf{u}_n \rightharpoonup \mathbf{0}$ weakly in $W^{1,2}$. Thus, by Rellich’s lemma,

$$\lim_{n \rightarrow \infty} \|\mathbf{u}_n - \widehat{\boldsymbol{\psi}}_n\|_2 = 0.$$

Let $\eta_n(\mathbf{x})$ be a Lipschitz cut-off function such $0 \leq \eta_n(\mathbf{x}) \leq 1$ and

$$\eta_n(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \partial\Omega, \\ 0, & \text{dist}(\mathbf{x}, \partial\Omega) \geq \delta_n. \end{cases} \tag{1}$$

It is possible to do so, while ensuring that

$$\|\nabla \eta_n\|_\infty \leq \frac{C}{\delta_n}, \tag{2}$$

where C depends only on Ω and $\delta_n \rightarrow 0$, as $n \rightarrow \infty$ so slowly that

$$\lim_{n \rightarrow \infty} \frac{\alpha_n \|\mathbf{u}_n\|_\infty}{\delta_n} = \lim_{n \rightarrow \infty} \frac{\alpha_n \|\widehat{\boldsymbol{\psi}}_n\|_{1,\infty}}{\delta_n} = \lim_{n \rightarrow \infty} \frac{\|\mathbf{u}_n - \widehat{\boldsymbol{\psi}}_n\|_2}{\delta_n} = 0.$$

Step 3. Let

$$\tilde{\mathbf{z}}_n = (1 - \eta_n)\mathbf{u}_n + \eta_n \widehat{\boldsymbol{\psi}}_n.$$

It is obvious that $\tilde{\mathbf{z}}_n(\mathbf{x})$ vanishes on $\partial\Omega_1$ and that $\alpha_n \tilde{\mathbf{z}}_n(\mathbf{x}) \rightarrow 0$, as $n \rightarrow \infty$ uniformly in $\mathbf{x} \in \Omega$. Defining $\tilde{\mathbf{v}}_n = \boldsymbol{\psi}_n - \tilde{\mathbf{z}}_n$, proves (a), (g) and (h). By definition $\tilde{\mathbf{z}}_n(\mathbf{x}) = \mathbf{u}_n(\mathbf{x})$ for all

\mathbf{x} , such that $\text{dist}(\mathbf{x}, \partial\Omega) \geq \delta_n$. Hence, (b) and (e) are also established. The property (d) is a consequence of (b), (c) and (e). Let us now establish (f).

$$\nabla \tilde{z}_n = (1 - \eta_n)\nabla \mathbf{u}_n + \eta_n \nabla \widehat{\psi}_n + (\widehat{\psi}_n - \mathbf{u}_n) \otimes \nabla \eta_n. \tag{3}$$

Hence,

$$\alpha_n \|\nabla \tilde{z}_n\|_\infty \leq \alpha_n \|\nabla z_n\|_\infty + \alpha_n \|\widehat{\psi}_n\|_{1,\infty} + \frac{\alpha_n \|\widehat{\psi}_n\|_{1,\infty}}{\delta_n} + \frac{\alpha_n \|\mathbf{u}_n\|_\infty}{\delta_n}.$$

The property (f) is proved. To prove property (c), we observe that

$$\|\nabla \tilde{z}_n - \nabla \mathbf{u}_n\|_2 \leq \|\eta_n(\nabla \mathbf{u}_n - \nabla \widehat{\psi}_n)\|_2 + \frac{\|\mathbf{u}_n - \widehat{\psi}_n\|_2}{\delta_n} \rightarrow 0, \text{ as } n \rightarrow \infty$$

because $|\nabla \mathbf{u}_n - \nabla \widehat{\psi}_n|^2$ is equiintegrable and η_n is bounded and supported on a set of small measure. Hence, (c) follows from the equiintegrability of $|\nabla \mathbf{u}_n|^2$. □

Another correction has to be made to the statements of Theorems 5 and 6. Theorem 5 has to apply to points in $\Omega \cup \partial\Omega_1$, while Theorem 6 to points in $\overline{\partial\Omega_2}$. Finally, no additional smoothness assumptions are required of $\partial\Omega_1$. We only need $\partial\Omega_1$ to be relatively open and coincide with the interior of its closure. If this condition is not satisfied we should just redefine $\partial\Omega_1$ as the interior of the closure of the original $\partial\Omega_1$. The set $\partial\Omega_2$ is then defined as $\partial\Omega \setminus \overline{\partial\Omega_1}$.