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FUZZY SYSTEMS AND THEIR MATHEMATICS

Quantale-valued quasi-neighborhood systems: fundamentals and application to L-quasi-topologies, L-quasi-uniformities, and rough approximation operators

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Abstract

In accordance with Rodabaugh's approach to L-quasi-topology, the aim of this paper is to use the concept of semi-quantales as a theoretical basis to construct and study the notion of quantale-valued quasi-neighborhood systems as a generalized form of the resent Höhle–Šostak's L-neighborhood systems. Some properties of such notion and relationships with L-quasi-topologies, L-quasi-fuzzy topologies, L-quasi-uniform structures and L-fuzzy rough approximation operators are established.

Keywords Quantale · Semi-quantale · L-quasi-topologies · L-quasi-fuzzy topologies · L-quasi-neighborhood system · L-quasi-interior operator · L-fuzzy rough approximation operators

1 Introduction

In 1986, a non-commutative extension of the concept of frames (locale) is proposed by Mulvey (1986) under the name quantale which is an algebraic structure with a strong connection to Mathematical logic, and so with the purpose of studying the foundations of quantum mechanics and the spectrum of non-commutative C^*-algebras. In 2007, Rodabaugh (2007) introduced the concept semi-quantale as a generalization of Mulvey's quantale and used it as a lattice-theoretic and algebraic basis for studying the lattice-valued topological spaces and powerset theories from the view-point of algebraic theories. The notion of semi-quantale provides a useful tool to gather various lattice-theoretic notions, which have been extensively studied in non-commutative structures; it has a wide application, especially in studying the non-commutative lattice-valued quasi-topology (Rodabaugh 2007; Höhle 2015; El-Saady 2016a, b; Zhang 2018).

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The problem of characterizing lattice-valued topologies and fuzzy topologies by means of suitable local structures has been investigated by many authors since the end of the seventies. It is easily observed that Pu-Liu's quasi-coincident neighborhood system (Liu and Luo 1997) and its generalization by Fang (2004, 2006), Wang's remote-neighborhood system (Wang 1988) and its generalization (Yang and Li 2012; Yao 2012), Shi's neighborhood system (Pang and Shi 2014; Shi 2009) and Höhle–Šostak's L-neighborhood system and L-fuzzy neighborhood system (Höhle and Šostak 1999) are important tools to study lattice-valued topologies and fuzzy topologies. Recently, the notion of quantale-valued generalized neighborhood systems is proposed and used to define a theory of rough set, called rough approximation operators based on L-generalized neighborhood systems (Zhao et al. 2019, 2018). Such quantale-valued generalized neighborhood system-based on approximation operators include the generalized neighborhood system-based approximation operators (Syau and Lin 2014; Zhang et al. 2015; Zhao and Li 2018), the L-fuzzy relation-based approximation operators (Hao and Huang 2017; She and Wang 2009) and some of L-fuzzy covering-based approximation operators (Li et al. 2017, 2008) as their special case.

In this paper, we aim to introduce the concept quantale-valued quasi-neighborhood systems as a generalization of the well-known Höhle–Šostak's L-neighborhood systems and providing a common framework for the equivalent notions: L-interior operators and L-neighborhood systems.

The induced notion can be considered as another way for obtaining L-quasi-topologies (Rodabaugh 2007). Some of their properties will be studied as well as their relationships with L-quasi-topologies, L-quasi-fuzzy topologies, L-quasi-uniform structures, and L-fuzzy rough approximation operators will be studied.

The remaining part of this paper is organized as follows. In Sect. 2, we recall some basic needed concepts. In Sect. 3, the concepts of L-quasi-interior operators and L-quasi-neighborhood systems are introduced as well as their relationships with L-quasi-topologies and L-quasi-fuzzy topologies are established. In Sect. 4, in accordance with Rodabaugh's L-quasi-topology, the concept quantic L-quasi-uniformity is introduced as a generalized form of the well known L-quasi-uniformity (Gutiérrez García et al. 2003). A relationship between such notion and L-quasi-neighborhood systems is established. In Sect. 5, the concept of rough approximation operators based on L-quasi-neighborhood systems is introduced.

2 Preliminaries

Definition 2.1 (Rodabaugh 2007) A semi-quantale $(L, \leq, \bigvee, \otimes)$ defined to be a complete lattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \longrightarrow L$, with no additional assumptions. As convention, we denote the join, meet, top and bottom elements in the complete lattice $(L, \leq)$ by $\bigvee, \bigwedge, \top_L$ and $\bot_L$, respectively.

Definition 2.2 A semi-quantale $L = (L, \leq, \otimes)$ is called:

(1) a unital semi-quantale (Rodabaugh 2007, 2008) if the binary operation $\otimes$ has an identity element $e \in L$ called the unit. If the unit e of the groupoid $(L, \otimes)$ coincides with the top element $\top$ of L, then a unital semi-quantale is called a strictly two-sided semi-quantale.
(2) a commutative semi-quantale (Rodabaugh 2007) if $\otimes$ is commutative, i.e., $a \otimes b = b \otimes a$ for every $a, b \in L$.
(3) a quantale (Rosenthal 1990) if the binary operation $\otimes$ is associative and satisfies

$$a \otimes \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \otimes b_i) \text{ and } \left(\bigvee_{i\in I} b_i\right) \otimes a = \bigvee_{i\in I}(b_i \otimes a), \text{ for all } a \in L, \{b_i\}_{i\in I} \subseteq L.$$

A preserving tensor product mapping $h : M \longrightarrow L$, between semi-quantales $(L, \leq, \otimes)$ and $(M, \leq, \odot)$, is said to be a semi-quantale morphism (Rodabaugh 2007) if $h(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} h(a_i)$; for $\{a_i\}_{i\in I} \subseteq L$.

If a semi-quantale morphism $h : L \longrightarrow M$ additionally preserves the top (resp., unit) element, i.e., $h(\top_L) = \top_M$ (resp., $h(e_L) = e_M$), then it is said to be strong (resp., unital).

The category **SQuant** comprises all semi-quantales together with semi-quantale morphisms. The non-full subcategory **UnSQuant** of **SQuant** comprises all unital semi-quantales and all unital semi-quantale morphisms. **Quant** is the full subcategory of **SQuant**, which has as objects all quantales.

StQuant is the full subcategory of **Quant**, which has as objects all strictly two-sided (or integral) quantales, i.e., unital quantales with $e = \top$.

Every quantale L is left- and right-residuated—i.e., there exist binary operations $\searrow$ and $\swarrow$ on L satisfying the following axioms:

$$a\otimes b \leq c \Leftrightarrow b \leq a \searrow c, \text{ and } b\otimes a \leq c \Leftrightarrow b \leq c \swarrow a$$

In particular, $\searrow$ and $\swarrow$ are determined by $a \searrow b = \bigvee\{c : a \otimes c \leq b\}$ and $b \swarrow a = \bigvee\{c : c \otimes a \leq b\}$, respectively, providing a single residuum $\rightarrow$ in case of a commutative multiplication (resulting complete residuated lattices of Denniston et al. 2013).

Lemma 2.3 (Bělohlávek and Vychodil 2005; Blount and Tsinakis 2003; Fang 2010; Georgescu and Popescu 2003; Rosenthal 1990; Solovyov 2013, 2016) *Let $(L, \leq, \otimes) \in |$**Quant**$|$. For each $a, b, c, d, a_i, b_i \in L$, the following properties hold:*

(1) $a \otimes (a \searrow b) \leq b$,
(2) $(a \searrow b) \otimes (c \searrow d) \leq (a \otimes c) \searrow (b \otimes d)$ *and* $(d \swarrow c) \otimes (b \swarrow a) \leq (d \otimes b) \swarrow (c \otimes a)$.
(3) $b \leq c$ *implies* $a \searrow b \leq a \searrow c$; *and* $c \searrow a \leq b \searrow a$;
(4) $a \searrow c \leq (b \otimes a) \searrow (b \otimes c)$ *and* $c \swarrow b \leq (c \otimes a) \swarrow (b \otimes a)$;
(5) $\bigvee_{i\in I}(a_i \searrow b) \leq \bigwedge_{i\in I} a_i \searrow b$ *and* $\bigvee_{i\in I}(a \searrow b_i) \leq a \searrow (\bigvee_{i\in I} b_i)$;
(6) $(\bigvee_{i\in I} a_i) \searrow b = \bigwedge_{i\in I}(a_i \searrow b)$;
(7) $a \otimes \bigwedge_{i\in I} b_i \leq \bigwedge_{i\in I}(a \otimes b_i)$.

A commutative $(L, \leq, \otimes) \in |$**Quant**$|$ is said to satisfy the double negation law if for any $a \in L$, $(a \longrightarrow \bot_L) \longrightarrow \bot_L = a$. In the following, we use $\neg a$ to denote $a \longrightarrow \bot_L$. Furthermore, for any $a, b \in L$, we define $a \oplus b = \neg(\neg a \otimes \neg b)$.

Let X be a non-empty set and $L \in |$**SQuant**$|$. An L-fuzzy subset (or L-subset) of X is a mapping $A : X \longrightarrow L$. The family of all L-fuzzy subsets on X will be denoted by L^X. The smallest element and the largest element in L^X are denoted by $\underline{\bot}$ and $\underline{\top}$, respectively. By $\underline{\alpha}$, we mean the constant function $\underline{\alpha} : X \rightarrow L$ such that $\underline{\alpha}(x) = \alpha$. If $L \in |$**UnSQuant**$|$with a unit $e \in L$, by $\underline{e}$, we mean the constant function $\underline{e} : X \rightarrow L$

with value e. The algebraic and lattice-theoretic structures can be extended from the semi-quantale $(L, \leq, \otimes)$ to L^X pointwisely:

- $A \leq B \Leftrightarrow A(x) \leq B(x)$,
- $(A \otimes B)(x) = A(x) \otimes B(x)$,

for all $x \in X$.

Obviously, $(L^X, \leq, \otimes)$ is again a semi-quantale with respect to the multiplication $\otimes$ and the joins of a subset $\{A_i\}_{i \in I}$ of L^X is given by

$$\left(\bigvee_{i \in I} A_i\right)(x) = \bigvee_{i \in I} A_i(x) \ \ \forall x \in X.$$

In the case L is unital with unit e, then L^X becomes a unital semi-quantale with the unit $\underline{e}$.

For a commutative quantale $(L, \leq, \otimes)$ and any $A, B \in L^X$ the subsethood degree (Bělohlávek 2002; Georgescu and Popescu 2003) $S : L^X \times L^X \longrightarrow L$, of A in B (and the intersection degree (Chen and Li 2007) $T : L^X \times L^X \longrightarrow L$, of A and B) given, for any $A, B \in L^X$, by $S(A, B) = \bigwedge_{x \in X}(A(x) \longrightarrow B(x))$ and $T(A, B) = \bigvee_{x \in X}(A(x) \otimes B(x))$.

The following lemma collects some properties of subsethood degree. They can be founded in many literatures such as Bělohlávek (2002), Bělohlávek and Vychodil (2005), Chen and Li (2007), Fang (2010).

Lemma 2.4 *Let $(L, \leq, \otimes)$ be a commutative unital quantale. For all $A, B, C \in L^X$ the following properties hold:*

(S1) $A \leq B \Leftrightarrow S(A, B) \geq e$;
(S2) $S(A, B) \leq S(C, A) \longrightarrow S(C, B)$;
(S3) *If* $A \leq B$, *then* $S(C, A) \leq S(C, B)$ *and* $S(B, C) \leq S(A, C)$;
(S4) $S(A, B) \otimes S(C, D) \leq S(A \otimes C, B \otimes D)$;
(S5) $S(A, B) \otimes S(B, C) \leq S(A, C)$;
(S6) $S(A, \alpha \otimes B) \geq \alpha \otimes S(A, B)$.

For an ordinary mapping $f : X \longrightarrow Y$, one can define the mappings $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\}$ for every $A \in L^X$ and every $y \in Y$, $f_L^{\leftarrow}(B) = B \circ f$ for every $B \in L^Y$, respectively. For more details, we refer to Rodabaugh (1983, 2007).

Lemma 2.5 (Fang 2010) *Let $(L, \leq, \otimes)$ be a commutative quantale, and let $h : X \longrightarrow Y$ be an ordinary mapping. Then, for $A, B \in L^X$, $C, D \in L^Y$,*

$$S(A, B) \ \leq S(h^{\rightarrow}(A), h^{\rightarrow}(B)) \ \textit{and}$$
$$S(C, D) \ \leq S(h^{\leftarrow}(C), h^{\leftarrow}(D)).$$

Definition 2.6 (Zhao et al. 2019, 2018) Let $(L, \leq, \otimes)$ be a commutative quantale. A function $N : X \longrightarrow L^{L^X}$ is called an L-generalized neighborhood system operator on X, where $\forall x \in X, N(x) = N_x$ is non-empty, i.e., $\bigvee_{K \in L^X} N_x(K) = \top_L$. Usually, N_x is called an L-generalized neighborhood system of x, and $N_x(K)$ is interpreted as the degree of K being a neighborhood of x.

An L-generalized neighborhood system operator $N : X \longrightarrow L^{L^X}$ is said to be:

(1) serial, if for any $x \in X$ and $A \in L^X, N_x(A) \leq \bigvee_{y \in X} A(y)$;
(2) reflexive, if for any $x \in X$ and $A \in L^X$, $N_x(A) \leq A(x)$;
(3) unary, if for any $x \in X$ and $A, B \in L^X$,

$$N_x(A) \otimes N_x(B) \leq \bigvee_{G \in L^X} \{N_x(G) \otimes S(G, A \otimes B)\};$$

(3) transitive, if for any $x \in X$ and $A \in L^X$,

$$N_x(A) \leq \bigvee_{B \in L^X} \left\{N_x(B) \otimes \bigwedge_{y \in X}\left((B(y)) \longrightarrow \bigvee_{B_y \in L^X} (N_y(B_y) \otimes S(B_y, A))\right)\right\}.$$

Definition 2.7 (Zhao et al. 2019, 2018) Let $(L, \leq, \otimes)$ be a commutative quantale. Let $N : X \longrightarrow L^{L^X}$ be an L-generalized neighborhood system operator on X. Then, for each $A \subseteq L^X$, the upper and lower approximation operators $\overline{N}(A)$ and $\underline{N}(A)$ are defined as follows: for any $x \in X$,

$$\overline{N}(A)(x) = \bigwedge_{K \in L^X} \{N_x(K) \longrightarrow T(K, A)\};$$
$$\underline{N}(A)(x) = \bigvee_{K \in L^X} \{N_x(K) \otimes S(K, A)\},$$

respectively.

Theorem 2.8 (Zhao et al. 2019) *Let $N : X \longrightarrow L^{L^X}$ be an L-generalized neighborhood system operator on X and the quantale $(L, \leq, \otimes)$ satisfies the double negative law, then*

$$\underline{N}(A) = \neg\overline{N}(\neg A) \ \textit{and} \ \overline{N}(A) = \neg\underline{N}(\neg A).$$

Definition 2.9 (Demirci 2010; Rodabaugh 2007) For $L \in |\mathbf{SQuant}|$, an L-quasi-topology on a non-empty set X, is a subfamily $\tau \subset L^X$ satisfying the following axioms:

(QT_1) $A \otimes B \in \tau$, for all $A, B \in \tau$.
(QT_2) $\bigvee_{i \in I} A_i \in \tau$, for all $\{A_i\}_{i \in I} \subseteq \tau$.

An L-quasi-topology τ on X is called strong if and only if it satisfies the following axiom:

(QT_3) $\underline{\top} \in \tau$.

If L is a unital semi-quantale with unit e, a subunital semi-quantale τ of L^X is called an L-topology on X (Rodabaugh 2008); i.e., τ satisfies (QT_1),(QT_2) and the following:

(QT_4) $\underline{e} \in \tau$.

If $\tau \subset L^X$ is an L-quasi-topology (resp., L-topology), then the pair (X, τ) is said to be an L-quasi-topological (resp., L-topological) space. A mapping $f : (X, \tau_1) \to (Y, \tau_2)$, between L-quasi-topological spaces, is said to be L-continuous if $\{\mu \circ f : \mu \in \tau_2\} \subseteq \tau_1$.

Definition 2.10 (Rodabaugh 2007) Let $(L, \leq, \otimes) \in |\mathbf{SQuant}|$, and X be a non-empty set.

(i) A map $\tau : L^X \longrightarrow L$ is called an L-quasi-fuzzy topology on X iff the next conditions are satisfied for all $A, B \in L^X$ and $\{A_i\}_{i \in I} \subseteq L^X$:

$(QT1)$ $\tau(A) \otimes \tau(B) \leq \tau(A \otimes B)$,
$(QT2)$ $\bigwedge_{i \in I} \tau(A_i) \leq \tau(\bigvee_{i \in I} A_i)$.

(ii) An L-quasi-fuzzy topology is said to be strong iff $\tau(\underline{\top}) = \top$.
(iii) If L is a unital semi-quantale with unit e. An L-quasi-fuzzy topology is then called an L-fuzzy topology iff $\tau(\underline{e}) = e$.
(iv) The pair (X, τ) is called an L-quasi-fuzzy (resp., strong L-quasi-fuzzy, L-fuzzy) topological space if τ is an L-quasi-fuzzy (resp., strong L-quasi-fuzzy, L-fuzzy) topology on X.
(iv) An L-quasi-fuzzy (resp., strong L-quasi-fuzzy, L-fuzzy) topology τ on X (Höhle and Šostak 1999) is called enriched iff τ satisfies the subsequent axiom:

$\tau(A) \leq \tau(\underline{\alpha} \otimes A), \forall \alpha \in L, A \in L^X$.

3 Quantale-valued quasi-neighborhood systems

In accordance with Rodabaugh's approach to L-quasi-topology, we aim to present the concept of L-quasi-neighborhood systems as a generalization of the Höhle–Šostak's L-neighborhood systems (Höhle and Šostak 1999) and consider it as another way for obtaining L-quasi-topologies (Rodabaugh 2007).

Before going ahead to consider this concept, we aim to present the notion of L-quasi-interior operator as generalization of the well-known L-interior operator (Höhle and Šostak 1999) and as an equivalence concept of the L-quasi-neighborhood systems.

For the rest of this section and furthers, if not otherwise specified, $L = (L, \leq, \otimes)$ is always assumed to be a quantale.

Definition 3.1 For a non-empty set X, the mapping $\mathcal{I} : L^X \to L^X$ is called:

(i) An L-quasi-interior operator on X iff $\mathcal{I}$ satisfies the following conditions:
for all $A, B \in L^X$.
(I_1) $\mathcal{I}(A) \leq \mathcal{I}(B)$ whenever $A \leq B$;
(I_2) $\mathcal{I}(A) \leq A$;
(I_3) $\mathcal{I}(A) \otimes \mathcal{I}(B) \leq \mathcal{I}(A \otimes B)$;
(I_4) $\mathcal{I}(A) \leq \mathcal{I}(\mathcal{I}(A))$.
(ii) A strong L-quasi-interior operator if it satisfy the following condition:
(I_5) $\mathcal{I}(\underline{\top}) = \underline{\top}$.
(iii) An L-interior operator if $L \in |\mathbf{UnQuant}|$ with unit e and the following condition is satisfied:
(I_6) $\mathcal{I}(\underline{e}) = \underline{e}$.
In case where L is a strictly two-sided semi-quantale (i.e., $e = \top$), the above strong L-quasi-interior operator coincided with the Höhle's L-interior operator (Höhle and Šostak 1999).

In the following, we shall characterize the relationship between L-quasi-topologies and L-quasi-interior operators as in classical topology.

Proposition 3.2 *Every L-quasi-interior operator $\mathcal{I} : L^X \longrightarrow L^X$ induces an L-quasi-topology $\tau_{\mathcal{I}}$ on X given by*

$$\tau_{\mathcal{I}} = \{A \in L^X : \mathcal{I}(A) = A\}.$$

Conversely, every L-quasi-topology t on X induces an L-quasi-interior operator $\mathcal{I}_\tau : L^X \longrightarrow L^X$ defined by

$$\mathcal{I}_\tau(A) = \bigvee\{B \in \tau : B \leq A\}.$$

Proof Straightforward and, therefore, omitted. □

Remark 3.3 (1) A strong L-quasi-interior operator $\mathcal{I} : L^X \longrightarrow L^X$ induces a strong L-quasi-topology $\tau_{\mathcal{I}}$ on X since $\mathcal{I}(\underline{\top}) = \underline{\top}$ implies $\underline{\top} \in \tau_{\mathcal{I}}$. In addition, a strong L-quasi-topology $\tau_{\mathcal{I}}$ on X induces a strong L-quasi-interior operator $\mathcal{I} : L^X \longrightarrow L^X$.

(2) For $L \in |\mathbf{UnSQuant}|$. An L-interior operator $\mathcal{I} : L^X \longrightarrow L^X$ induces an L-topology $\tau_{\mathcal{I}}$ on X since $\mathcal{I}(\underline{e}) = \underline{e}$ implies $\underline{e} \in \tau_{\mathcal{I}}$.

As a consequence of the above, for $L \in |\mathbf{UnSQuant}|$ with $e = \top$, we have that the relations $\mathcal{I}_{\tau_{\mathcal{I}}} = \mathcal{I}$ and $\tau_{\mathcal{I}_\tau} = \tau$ hold - i.e. L-quasi-interior (resp., strong L-quasi-interior, L-interior) operators and L-quasi-topologies (resp., strong L-quasi-topologies, L-topologies) are equivalent concepts.

To give another example of the relationship between L-quasi-interior operators and L-quasi-topologies, let us, at first consider an arbitrary class $\tau \subset L^X$ and define

$$\mathcal{I}_\tau(A) = \bigvee_{B \in \tau} \{B \otimes S(B, A)\}, \ \forall A \in L^X \tag{1}$$

(In particular, $\mathcal{I}_\tau(A) = \perp$ if no $B \in \tau$ satisfies $S(B, A) = \top$).

Lemma 3.4 *For $(L, \leq, \otimes)$ be a commutative unital quantale, the operator $\mathcal{I}_\tau : L^X \longrightarrow L^X$ satisfies, for all $A, B \in L^X$:*

(1) $S(A, B) \leq S(\mathcal{I}_\tau(A), \mathcal{I}_\tau(B))$;
(2) $\mathcal{I}_\tau(A) \leq A$;
(3) $\mathcal{I}_\tau(A) \leq \mathcal{I}_\tau(\mathcal{I}_\tau(A))$.

Proof (1)

$$\begin{aligned}
&S(\mathcal{I}_\tau(A), \mathcal{I}_\tau(B)) \\
&= \bigwedge_{x \in X} (\mathcal{I}_\tau(A)(x) \longrightarrow \mathcal{I}_\tau(B)(x)) \\
&= \bigwedge_{x \in X} \Bigg(\bigvee_{A_1 \in \tau} (A_1(x) \otimes S(A_1, A)) \\
&\quad \longrightarrow \bigvee_{B_1 \in \tau} (B_1(x) \otimes S(B_1, B)) \Bigg) \\
&= \bigwedge_{x \in X} \bigwedge_{A_1 \in \tau} (A_1(x) \otimes S(A_1, A) \\
&\quad \longrightarrow \bigvee_{B_1 \in \tau} (B_1(x) \otimes S(B_1, B))) \ (\textit{Lemma } 2.3(6)) \\
&\geq \bigwedge_{x \in X} \bigwedge_{A_1 \in \tau} (S(A_1, A) \otimes A_1(x) \\
&\quad \longrightarrow S(A_1, B) \otimes A_1(x)) \\
&\geq \bigwedge_{x \in X} \bigwedge_{A_1 \in \tau} (S(A_1, A) \\
&\quad \longrightarrow S(A_1, B)) \ (\text{by } \textit{Lemma } 2.3(4)) \\
&\geq S(A, B) \ (\text{by } \textit{Lemma } 2.4(S2))
\end{aligned}$$

(2) Note that $B \otimes S(B, A) \leq A$ hold for all $A, B \in L^X$. Therefore, $\mathcal{I}_\tau(A) = \bigvee_{B \in \tau} (B \otimes S(B, A)) \leq A$ is true for all $A \in L^X$.

(3) $\mathcal{I}_\tau(\mathcal{I}_\tau(A)) = \bigvee_{B \in \tau} (B \otimes S(B, \mathcal{I}_\tau(A))) \geq \mathcal{I}_\tau(A) \otimes S(\mathcal{I}_\tau(A), \mathcal{I}_\tau(A)) \overset{S1}{\geq} \mathcal{I}_\tau(A)$.

□

Proposition 3.5 *For $(L, \leq, \otimes)$ be a commutative quantale. An L-quasi-topology $\tau \subset L^X$ induces an L-quasi-interior operator $\mathcal{I}_\tau : L^X \longrightarrow L^X$ defined by*

$$\mathcal{I}_\tau(A) = \bigvee_{B \in \tau} \{B \otimes S(B, A)\}, \ \forall A \in L^X. \tag{2}$$

Proof According to the above lemma, we have to verify only that the operator $\mathcal{I}_\tau : L^X \longrightarrow L^X$ satisfies the item (I_3) as follows:

$$\begin{aligned}
&(I_3)\ \mathcal{I}_\tau(A) \otimes \mathcal{I}_\tau(B) \\
&= \bigvee_{A_1 \in \tau} \{A_1 \otimes S(A_1, A)\} \otimes \bigvee_{B_1 \in \tau} \{B_1 \otimes S(B_1, B)\}, \ \forall A, B \in L^X \\
&= \bigvee_{A_1, B_1 \in \tau} \{A_1 \otimes B_1 \otimes S(A_1, A) \otimes S(B_1, B)\} \\
&\leq \bigvee_{A_1 \otimes B_1 \in \tau} \{A_1 \otimes B_1 \otimes S(A_1 \otimes B_1, A \otimes B)\} \\
&\quad (\text{by } \textit{Lemma } 2.4(S4)) \\
&= \mathcal{I}_\tau(A \otimes B).
\end{aligned}$$

Remark 3.6 For $L \in |\mathbf{UnQuant}|$ and any $A \in \tau$, we observe that

$$\begin{aligned}
\mathcal{I}_\tau(A) &= \bigvee_{B \in \tau} \{B \otimes S(B, A)\} \geq A \otimes S(A, A) \\
&\overset{S1}{\geq} A \text{ or equivalently, } \mathcal{I}_\tau(A) = A.
\end{aligned}$$

So that

(1) An L-topology $\tau_{\mathcal{I}}$ on X induces an L-interior operator since $\underline{e} \in \tau$ implies $\mathcal{I}(\underline{e}) = \underline{e}$.
(2) A strong L-quasi-topology $\tau \subset L^X$ induces a strong L-quasi-interior operator.

Now, we are in a position to present the concept of L-quasi-neighborhood systems as a generalization of the Höhle–Šostak's L-neighborhood systems (Höhle and Šostak 1999).

Definition 3.7 Let X be a non-empty set.

(i) A map $N : X \longrightarrow L^{L^X}$ is called an L-quasi-neighborhood system on X with $N(x) = N_x$ for each $x \in X$, if N_x satisfies the following conditions: for all $A, B \in L^X$

(N_1) $N_x(A) \leq N_x(B)$ whenever $A \leq B$;

(N_2) $N_x(A) \leq A(x)$ for all $A \in L^X$;
(N_3) $(N_x(A)) \otimes (N_x(B)) \leq N_x(A \otimes B)$;
(N_4) $N_x(A) \leq \bigvee_{B \in L^X} \{N_x(B) \otimes S(B, N_-(A))\}$ for all $A \in L^X$,

where $N_-(A) \in L^X$ is defined by $[N_-(A)](x) = N_x(A)$ for each $x \in X$.

(iii) An L-quasi-neighborhood system $N : X \longrightarrow L^{L^X}$ is said to be a strong if

$$N_x(\underline{\top}) = \top.$$

(iv) If $(L, \leq, \otimes) \in |\mathbf{UnQuant}|$ with unit e. An L-quasi-neighborhood system on X is then called an L-neighborhood system iff $N_x(\underline{e}) = e$.
(v) In case L is an strictly two-sided quantale (i.e., $e = \top$), the above strong L-quasi-neighborhood system coincided with the Höhle's L-neighborhood system (Höhle and Šostak 1999).

The pair (X, N) is called an L-quasi (resp., strong L-quasi, L)-neighborhood space if N is an L-quasi (resp., strong L-quasi, L)-neighborhood system on X.

An L-quasi (resp., strong L-quasi, L)-neighborhood system on X is called stratified if

(vi) $N_x(\underline{\alpha} \otimes A) \geq \alpha \otimes N_x(A)$ for all $A \in L^X$ and $\alpha \in L$.

Now, we give the following easily proven proposition concerning the equivalence between L-quasi-interior (resp., a strong L-quasi-interior, L-interior) operators and L-quasi-neighborhood (resp., a strong L-quasi-neighborhood, L-neighborhood) systems.

Proposition 3.8 (see Höhle and Šostak 1999) *An L-quasi-interior (resp., strong L-quasi-interior, L-interior) operator $\mathcal{I} : L^X \to L^X$, on a non-empty set X, induces an L-quasi-neighborhood (resp., strong L-quasi-neighborhood, L-neighborhood) system $N^{\mathcal{I}} : X \to L^{L^X}$ defined by*

$$N^{\mathcal{I}}_x(A) = [\mathcal{I}(A)](x);\ \forall A \in L^X, \forall x \in X$$

Conversely, an L-quasi-neighborhood (resp., strong L-quasi-neighborhood, L-neighborhood) system on X induces an L-quasi-interior (resp., strong L-quasi-interior, L-interior) operators $\mathcal{I}_N : L^X \to L^X$ defined by

$$[\mathcal{I}_N(A)](x) = N_x(A);\ \forall A \in L^X, \forall x \in X.$$

Remark 3.9 Axiom (N_4) can obviously be reformulated in the following form:
$(N_4^{'})$ $N_x(A) \leq N_x(N_-(A))$ for all $A \in L^X$.

Since

$$\begin{aligned}
N_x(A) &\leq \bigvee_{B \in L^X} \{N_x(B) \otimes S(B, N_-(A))\} \text{ for all } A \in L^X \\
&\leq \bigvee_{B \in L^X} \{N_x(B) \otimes S(N_-(B), N_-(N_-(A)))\} \\
&= \bigvee_{B \in L^X} \left\{N_x(B) \otimes \bigwedge_{x \in X} (N_x(B) \longrightarrow N_x(N_-(A)))\right\} \\
&\leq \bigvee_{B \in L^X} \{N_x(B) \otimes (N_x(B) \longrightarrow N_x(N_-(A)))\} \\
&= \{N_x(B) \otimes (N_x(B) \longrightarrow N_x(N_-(A)))\} \\
&\leq N_x(N_-(A)) \text{ (by } \textit{Lemma } 2.3(1)).
\end{aligned}$$

Remark 3.10 In the case $(L, \leq, \otimes) \in |\mathbf{UnQuant}|$ with $e = \top$, it is known that, for all $a, b \in L$, $a \leq b \Leftrightarrow S(a, b) = \top$ and $a \otimes \top = a$, so axiom (N_4) can obviously be reformulated in the following form:

$$(N_4^{''})\ N_x(A) \leq \bigvee_{B \in L^X} \{N_x(B) : B(y) \leq N_y(A), \forall y \in X\}$$

for all $A \in L^X$.

Definition 3.11 Let (X, N) and (Y, N^*) be two L-quasi-neighborhood spaces. A function $h : (X, N) \to (Y, N^*)$ is called N-continuous at $x \in X$ iff $N^*_{h(x)}(A) \leq N_x(h^{\leftarrow}(A))$ for all $A \in L^Y$. A function h is N-continuous if it is N-continuous at every $x \in X$.

Example 3.12 Every L-quasi-neighborhood system $N : X \longrightarrow L^{L^X}$ with $\bigvee_{K \in L^X} N_x(K) = \top$, is a reflexive, unary and transitive L-generalized neighborhood system (Zhao et al. 2019, 2018).

Proof From items (N_2) and (N_3), we have that $N : X \longrightarrow L^{L^X}$ is a reflexive and unary L-generalized neighborhood system.

The transitivity given as follows: For $A \in L^X$ and from item (N_4), we have that

$$\begin{aligned}
&N_x(A) \\
&\leq \bigvee_{B \in L^X} \{N_x(B) \otimes S(B, N_-(A)\} \\
&= \bigvee_{B \in L^X} \left\{N_x(B) \otimes \bigwedge_{y \in X} (B(y) \longrightarrow N_y(A))\right\} \\
&\overset{N_4}{\leq} \bigvee_{B \in L^X} \left\{N_x(B) \otimes \bigwedge_{y \in X} (B(y) \right. \\
&\qquad \left. \longrightarrow \bigvee_{B_y \in L^X} (N_y(B_y) \otimes S(B_y, N_-(A)))) \right\}
\end{aligned}$$

$$\overset{N_2}{\leq} \bigvee_{B \in L^X} \left\{ N_x(B) \otimes \bigwedge_{y \in X} (B(y) \longrightarrow \bigvee_{B_y \in L^X} (N_y(B_y) \otimes S(B_y, A))) \right\}$$

□

In the sequel, we will introduce the relationship between L-quasi-neighborhood systems and L-quasi-topologies.

Proposition 3.13 *Let $N : X \longrightarrow L^{L^X}$ be an L-quasi (resp., stratified L-quasi) neighborhood system. A subfamily $\tau_N \subset L^X$ defined by*

$$\tau_N = \{A \in L^X : A(x) = N_x(A), \forall x \in X\}.$$

is an L-quasi (resp., stratified L-quasi) topology on X

Proof (1) Let $N : X \longrightarrow L^{L^X}$ be an L-quasi-neighborhood system. Then,

(QT_1) Let $A_1,\ A_2 \in \tau_N$, then

$$\begin{aligned}(A_1 \otimes A_2)(x) &= A_1(x) \otimes A_2(x)\\ &= N_x(A_1) \otimes N_x(A_2)\\ &\leq N_x(A_1 \otimes A_2)\end{aligned}$$

In addition, from (N_3), we have

$$N_x(A_1 \otimes A_2) \leq (A_1 \otimes A_2)(x).$$

Then, $N_x(A_1 \otimes A_2) = (A_1 \otimes A_2)(x)$, which means that $A_1 \otimes A_2 \in \tau_N$.

(QT_2) Let $A_i \in \tau_N$ for all $i \in I$, then $A_i(x) = N_x(A_i)$. Therefore,

$$\bigvee_{i \in I} A_i(x) = \bigvee_{i \in I} N_x(A_i) \leq N_x\left(\bigvee_{i \in I} A_i\right).$$

In addition, from (N_2), we have

$$N_x\left(\bigvee_{i \in I} A_i\right) \leq \left(\bigvee_{i \in I} A_i\right)(x),$$

and therefore,

$$N_x\left(\bigvee_{i \in I} A_i\right) = \left(\bigvee_{i \in I} A_i\right)(x).$$

Then, $\bigvee_{i \in I} A_i \in \tau_N$.

Therefore, τ_N is an L-quasi-topology on X and thus (X, τ_N) is an L-quasi-topological space.

(2) For the case $N : X \longrightarrow L^{L^X}$ is a stratified L-quasi-neighborhood system, let $A \in \tau_N$. Since $N_x(\underline{\alpha} \otimes A) \geq \alpha \otimes N_x(A) = \alpha \otimes A(x) = (\underline{\alpha} \otimes A)(x)$ and by the condition (N_3), we have $\underline{\alpha} \otimes A \in \tau_N$. □

Remark 3.14 (1) If $N : X \longrightarrow L^{L^X}$ is a strong L-quasi-neighborhood system on X, then for each $x \in X$, τ_N is a strong L-quasi-topology, since $N_x(\underline{\top}) = \top$, implies that $\underline{\top} \in \tau_N$.

(2) If $L \in |\mathbf{UnQuant}|$ with unit e and if $N : X \longrightarrow L^{L^X}$ is an L-neighborhood system, then $N_x(\underline{e}) = e$, for each $x \in X$, and this implies that $\underline{e} \in \tau_N$ which means that τ_N is an L-topology.

As a consequence of the above Propositions 3.5 and 3.8, we have the following result:

Proposition 3.15 *Let (X, τ) be an L-quasi-(resp., enriched L-quasi-) topological space. A mapping $N^\tau : X \to L^{L^X}$ defined by*

$$N^\tau_x(A) = \bigvee_{B \in \tau} \{B(x) \otimes S(B, A)\}.$$

is an L-quasi (resp., stratified L-quasi) neighborhood system on X.

In the following proposition, we will study the relationship between L-quasi-neighborhood systems and L-quasi-fuzzy topologies.

Proposition 3.16 *Let $N : X \longrightarrow L^{L^X}$ be an L-quasi-neighborhood system. A mapping $\tau_N : L^X \to L$, defined by*

$$\tau_N(A) = S(A, N_-(A)).$$

is an L-quasi-fuzzy topology on X. Moreover, if N is stratified, then τ_N is enriched.

Proof For the case of L-quasi-neighborhood system:

(i)

$$\begin{aligned}&\tau_N(A \otimes B)\\ &= S((A \otimes B), N_-(A \otimes B)).\\ &= \bigwedge_{x \in X} ((A \otimes B)(x) \longrightarrow N_x(A \otimes B)).\\ &\geq \bigwedge_{x \in X} ((A(x) \otimes B(x)) \longrightarrow (N_x(A) \otimes N_x(B))).\\ &\geq \bigwedge_{x \in X} ((A(x) \longrightarrow N_x(A)) \otimes (B(x) \longrightarrow N_x(B)))\end{aligned}$$

(*Lemma* 2.3(2)).

$$\geq \bigwedge_{x\in X}(A(x)\longrightarrow N_x(A))\otimes\bigwedge_{x\in X}(B(x)\longrightarrow N_x(B)).$$
$$= S(A, N_-(A))\otimes S(B, N_-(B)).$$
$$\geq \tau_N(A)\otimes\tau_N(B).$$

(ii) For $\{A_i : i \in I\} \subseteq L^X$, since $A_i \leq \bigvee_{i\in I} A_i$, we have

$$\tau_N\left(\bigvee_i A_i\right)$$
$$= S\left(\left(\bigvee_i A_i\right), N_-\left(\bigvee_i A_i\right)\right).$$
$$= \bigwedge_{x\in X}\left(\left(\bigvee_i A_i\right)(x)\longrightarrow N_x\left(\bigvee_i A_i\right)\right).$$
$$\geq \bigwedge_{x\in X}\left(\bigvee_i A_i(x)\longrightarrow\bigvee_i N_x(A_i)\right).$$
$$\geq \bigwedge_{x\in X}\bigwedge_i(A_i(x)\longrightarrow N_x(A_i))$$
$$= \bigwedge_i\bigwedge_{x\in X}(A_i(x)\longrightarrow N_x(A_i)) = \bigwedge_i \tau_N(A_i).$$

For the case of stratified L-quasi-neighborhood system:

$$\tau_N(\underline{\alpha}\otimes A)$$
$$= S(\underline{\alpha}\otimes A), N_-(\underline{\alpha}\otimes A))$$
$$= \bigwedge_{x\in X}((\underline{\alpha}\otimes A)(x)\longrightarrow N_x(\underline{\alpha}\otimes A)).$$
$$\geq \bigwedge_{x\in X}(\alpha\otimes A(x))\longrightarrow(\alpha\otimes N_x(A))$$
$$\geq \bigwedge_{x\in X}(A(x)\longrightarrow N_x(A))\ (\text{by } \textit{Lemma } 2.3(4)).$$
$$= S(A, N_-(A)),$$
$$= \tau_N(A).$$

□

As consequences of the above proposition, we have the following special cases:

(1) In the case where $L \in |\mathbf{UnSQuant}|$ with a unit e and if N is an L-neighborhood system on X, then it is clear that

$$\tau_N(\underline{e}) = S(\underline{e}, N_-(\underline{e})) = S(\underline{e}, \underline{e}) \geq e.$$

which means that τ_N is an L-fuzzy topology on X.

(2) In the case $(L, \leq, \otimes) \in |\mathbf{UnQuant}|$ with $e = \top$ and if $N : X \longrightarrow L^{L^X}$ be a strong L-quasi-neighborhood system, then

$$\tau_N(\underline{\top}) = S(\underline{\top}, N_-(\underline{\top})) = S(\underline{\top}, \underline{\top}) = \top.$$

which means that the corresponding L-quasi-fuzzy topology τ_N on X is strong.

Proposition 3.17 *Let (X, N) and (Y, N^*) be two L-quasi-neighborhood spaces. If a mapping $h : (X, N) \to (Y, N^*)$ is N-continuous, then $h : (X, \tau_N) \to (Y, \tau_{N^*})$ is L-continuous*

Proof Since $N^*_{h(x)}(A) \leq N_x(h^{\leftarrow}(A))$ for all $A \in L^Y$, by Proposition 3.16, we have

$$\tau_{N^*}(A) = S(A, N^*_-(A))$$
$$= \bigwedge_{y\in Y}(A(y)\longrightarrow N^*_y(A))$$
$$\leq \bigwedge_{x\in X}(h^{\leftarrow}(A)(x)\longrightarrow N^*_{h(x)}(A))$$
$$\leq \bigwedge_{x\in X}(h^{\leftarrow}(A)(x)\longrightarrow N_x(h^{\leftarrow}(A)))$$
$$= S(h^{\leftarrow}(A), N_-(h^{\leftarrow}(A)))$$
$$= \tau_N(h^{\leftarrow}(A)).$$

□

4 *L*-quasi-uniformities and *L*-quasi-neighborhood systems

In this section, we will present a concept of a fuzzy (quasi-)uniformity which is in accordance with Rodabaugh's L-quasi-topology and study the relationship between it and L-quasi-neighborhood systems.

Definition 4.1 Let X be a non-empty set.

(i) A mapping $\mathcal{U} : L^{X\times X} \longrightarrow L$ is called a quantic L-quasi-uniformity on $X \times X$ if it satisfies the following conditions, for $d, d_1, d_2 \in L^{X\times X}$:
(U_1) If $d_1 \leq d_2$, then $\mathcal{U}(d_1) \leq \mathcal{U}(d_2)$;
(U_2) $\mathcal{U}(d_1 \otimes d_2) \geq \mathcal{U}(d_1) \otimes \mathcal{U}(d_2)$;
(U_3) $\mathcal{U}(d) \leq \bigwedge_{x\in X}(d(x, x))$;
(U_4) $\mathcal{U}(d) \leq \bigvee\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes S((d_2 \circ d_1), d)\}$,
(where $(d_2 \circ d_1)(x, y) = \bigvee_{z\in X}\{d_1(x, z) \otimes d_2(z, y)\}$).

(ii) A quantic L-quasi-uniformity on $X \times X$ is said to be a strong if it satisfies the additional axiom:
(U_5) $\mathcal{U}(\top_{X\times X}) = \top$.

(iii) If $L \in |\mathbf{UnSQuant}|$ with unit e, a quantic L-quasi-uniformity on $X \times X$ is called L-quasi-uniformity if it satisfies the following condition:
(U_6) $\mathcal{U}(e_{X\times X}) \geq e$.
In case $(L, \leq, \otimes)$ is an strictly two-sided semi-quantale, strong quantic L-quasi-uniformity coincides with L-quasi-uniformity in the sense of Gutiérrez García et al. (2003).

(iii) A quantic L-quasi-uniformity (resp., a strong quantic L-quasi-uniformity, an L-quasi-uniformity) $\mathcal{U}$ on X is called a quantic L-uniformity (resp., a strong quantic L-uniformity, L-uniformity) if

$$\mathcal{U}(d) \leq \mathcal{U}(d^{-1}) \ for\ each\ d \in L^{X\times X},$$
where $d^{-1}(x, y) = d(y, x)$.

In this case, the pair $(X, \mathcal{U})$ is called a quantic L-uniform (resp., a strong quantic L-uniform, L-uniform) space.

Remark 4.2 (see Gutiérrez García (2000)) Not that for any quantic L-quasi-uniformity $\mathcal{U} : L^{X\times X} \longrightarrow L$, item (U_4) can be rewritten in the following form for any $d \in L^{X\times X}$:

$(U_4^{'})$ $\mathcal{U}(d) \leq \bigvee\{\mathcal{U}(d_1)\otimes\mathcal{U}(d_2)\otimes\alpha : d_1, d_2 \in L^{X\times X},\ \alpha \in L \text{ and } (d_2 \circ d_1) \otimes \alpha \leq d\}$,

Proof For any $\alpha \in L$ satisfying $(d_2 \circ d_1) \otimes \alpha \leq d$, we have

$$\alpha \leq \bigwedge_{x,y}(d_2 \circ d_1)(x, y) \longrightarrow d(x, y) = S((d_2 \circ d_1), d)$$

and so

$$\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes \alpha \leq \mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes S((d_2 \circ d_1), d).$$

which means that $(U_4^{'})$ implies (U_4).

On the other hand, since $(d_2 \circ d_1) \otimes S((d_2 \circ d_1), d) \leq d$, then (U_4) implies $(U_4^{'})$. □

In the following theorem, we shall see in which way a quantic L-quasi-uniformity $\mathcal{U} : L^{X\times X} \longrightarrow L$ induces an L-quasi-topology. In order to do it, we consider the collection $\{\mathcal{N}_x^{\mathcal{U}}\}_{x\in X}$ defined for each an L-set $A \in L^X$ as follows:

$$\mathcal{N}_x^{\mathcal{U}}(A) = \bigvee \{\mathcal{U}(d) \otimes S(d(x, -), A)\}\,,\ d \in L^{X\times X},$$

(where $d(x, -)(y) = d(x, y)$ for each $y \in X$).

Theorem 4.3 *Let $\mathcal{U} : L^{X\times X} \longrightarrow L$ be a quantic L-quasi-uniformity on X, the mapping $\mathcal{N}^{\mathcal{U}} : X \to L^{L^X}$ defined by*

$$\mathcal{N}_x^{\mathcal{U}}(A) = \bigvee \{\mathcal{U}(d) \otimes S(d(x, -), A)\}\,,\ d \in L^{X\times X}$$

for each $x \in X$ and $A \in L^X$ is an L-quasi-neighborhood system on X.

Proof

(N_1)

$$\begin{aligned}
&S(\mathcal{N}_-^{\mathcal{U}}(A), \mathcal{N}_-^{\mathcal{U}}(B))\\
&= \bigwedge_{x\in X} \left(\mathcal{N}_x^{\mathcal{U}}(A) \longrightarrow \mathcal{N}_x^{\mathcal{U}}(B)\right)\\
&= \bigwedge_{x\in X} \Big\{\bigvee(\mathcal{U}(d) \otimes S(d(x, -), A))\\
&\quad \longrightarrow \bigvee(\mathcal{U}(d) \otimes S(d(x, -), B))\Big\}\\
&= \bigwedge_{x\in X} \bigwedge \{(\mathcal{U}(d) \otimes S(d(x, -), A))\\
&\quad \longrightarrow \bigvee(\mathcal{U}(d) \otimes S(d(x, -), B))\Big\}\\
&\text{(by } \textit{Lemma } 2.3(6))\\
&\geq \bigwedge_{x\in X} \bigwedge \{(\mathcal{U}(d) \otimes S(d(x, -), A))\\
&\quad \longrightarrow (\mathcal{U}(d) \otimes S(d(x, -), B))\}\\
&\text{(by } \textit{Lemma } 2.3(5))\\
&\geq \bigwedge_{x\in X} \bigwedge\{S(d(x, -), A) \longrightarrow S(d(x, -), B)\}\\
&\text{(by } \textit{Lemma } 2.3(4))\\
&\geq S(A, B)\} \text{ (by } \textit{Lemma } 2.4(S2)).
\end{aligned}$$

(N_2)

$$\begin{aligned}
&\mathcal{N}_x^{\mathcal{U}}(A) \otimes \mathcal{N}_x^{\mathcal{U}}(B)\\
&= \bigvee \{\mathcal{U}(d_1) \otimes S(d_1(x, -), A)\}\\
&\quad \otimes \bigvee \{\mathcal{U}(d_2) \otimes S(d_2(x, -), B)\}\\
&= \bigvee \{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes S(d_1(x, -), A)\\
&\quad \otimes S(d_2(x, -), B)\}\\
&\leq \bigvee\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes S(d_1(x, -)\\
&\quad \otimes d_2(x, -), A \otimes B)\}\text{(by } \textit{Lemma } 2.4(S4))\\
&\leq \bigvee \{\mathcal{U}(d_1 \otimes d_2) \otimes S(d_1 \otimes d_2(x, -), A \otimes B)\}\\
&= \mathcal{N}_x^{\mathcal{U}}(A \otimes B)
\end{aligned}$$

(N_3)

$$\begin{aligned}
&\mathcal{N}_x^{\mathcal{U}}(A)\\
&= \bigvee \{\mathcal{U}(d) \otimes S(d(x, -), A)\}\\
&\leq \bigvee \{d(x, x) \otimes S(d(x, -), A)\}\\
&= \bigvee \left\{d(x, x) \otimes \bigwedge_{x\in X} d(x, x) \longrightarrow A(x))\right\}\\
&\leq \bigvee \bigwedge_{x\in X} \{(d(x, x) \otimes (d(x, x) \longrightarrow A(x))\}\\
&\leq A(x) \text{ (by } \textit{Lemma } 2.3(1)).
\end{aligned}$$

(N_4) According to Remark 3.9, we will prove the that $\mathcal{N}_x^{\mathcal{U}}(A) \leq \mathcal{N}_x^{\mathcal{U}}(\mathcal{N}_-^{\mathcal{U}}(A))$ for all $A \in L^X$:

$$\mathcal{N}_x^{\mathcal{U}}(\mathcal{N}_-^{\mathcal{U}}(A))$$

$$= \bigvee_{d_1 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes S(d_1(x,-), \mathcal{N}_{-}^{\mathcal{U}}(A))\right\}$$

$$= \bigvee_{d_1 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \bigwedge_{z\in X} \left(d_1(x,z) \longrightarrow \mathcal{N}_{z}^{\mathcal{U}}(A)\right)\right\}$$

$$= \bigvee_{d_1 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \bigwedge_{z\in X} \left(d_1(x,z) \longrightarrow \left[\bigvee_{d_2\in L^{X\times X}} \{\mathcal{U}(d_2) \otimes S(d_2(z,-), A)\}\right]\right)\right\}$$

$$\overset{(S6)}{\geq} \bigvee_{d_1,d_2 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes \bigwedge_{z\in X} (d_1(x,z) \longrightarrow S(d_2(z,-), A))\right\}$$

$$= \bigvee_{d_1,d_2 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes \bigwedge_{z\in X} \left(d_1(x,z) \longrightarrow \bigwedge_{y\in X} (d_2(z,y) \longrightarrow A(y))\right)\right\}$$

$$= \bigvee_{d_1,d_2 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes \bigwedge_{y\in X} \left(\bigwedge_{z\in X} [d_1(x,z) \otimes d_2(z,y)] \longrightarrow A(y)\right)\right\}$$

$$\geq \bigvee_{d_1,d_2 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes \bigwedge_{y\in X} \left(\bigvee_{z\in X} [d_1(x,z) \otimes d_2(z,y)] \longrightarrow A(y)\right)\right\}$$

(by *Lemma* 2.3(3))

$$= \bigvee_{d_1,d_2 \in L^{X\times X}} \left\{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes \bigwedge_{y\in X} ([d_2 \circ d_1](x,y) \longrightarrow A(y))\right\}$$

$$= \bigvee_{d_1,d_2 \in L^{X\times X}} \{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes S((d_2 \circ d_1)(x,-), A))\}$$

$$\overset{(S5)}{\geq} \bigvee_{d_1,d_2,d \in L^{X\times X}} \{\mathcal{U}(d_1) \otimes \mathcal{U}(d_2) \otimes S((d_2 \circ d_1)(x,-), d(x,-)) \otimes S(d(x,-), A)\}$$

$$\geq \bigvee_{d \in L^{X\times X}} \{\mathcal{U}(d) \otimes S(d(x,-), A)\} = \mathcal{N}_x^{\mathcal{U}}(A).$$

□

As a consequence of the above theorem and by Proposition 3.16, for a quantic L-quasi-uniformity $\mathcal{U}$ on a non-empty set X a mapping $\tau_{\mathcal{U}} : L^X \to L$, defined by

$$\tau_{\mathcal{U}}(A) = S(A, N_{-}^{\mathcal{U}}(A))$$

is an L-quasi-fuzzy topology on X.

Definition 4.4 Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quantic L-quasi-uniform spaces. A map $h : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is called quasi-uniformly continuous if for $\nu \in L^{Y\times Y}$, $\mathcal{V}(\nu) \leq \mathcal{U}((h \times h)^{\leftarrow}(\nu))$.

Proposition 4.5 *Let $(X, \mathcal{U})$ and $(Y, \mathcal{U})$ be two quantic L-quasi-uniform spaces. If a map $h : (X, \mathcal{U}) \to (Y, \mathcal{U})$ is quasi-uniformly continuous, then the mapping $h : (X, N_x^{\mathcal{U}}) \to (Y, N_{h(x)}^{\mathcal{V}})$ is N-continuous.*

Proof

$$\begin{aligned} N_{h(x)}^{\mathcal{V}}(A) &= \bigvee \{\mathcal{V}(d) \otimes S(d(h(x), -), A)\} \\ &\leq \bigvee \{\mathcal{U}((h \times h)^{\leftarrow}(d)) \\ &\qquad \otimes S((h \times h)^{\leftarrow}(d)(x,-), h^{\leftarrow}(A))\} \\ &\leq N_x^{\mathcal{U}}(h^{\leftarrow}(A)). \end{aligned}$$

□

5 Rough approximation operators based on *L*-quasi-neighborhood systems

In this section, we conclude this paper by introducing an application of L-quasi-neighborhood systems in the area of rough sets and approximation operators.

Proposition 5.1 *Let $N : X \longrightarrow L^{L^X}$ be an L-quasi-neighborhood system on X. Then, for any $A, B \in L^X$, the lower approximation operator*

$$\underline{N}(A)(x) = \bigvee_{K \in L^X} \{N_x(K) \otimes S(K, A)\}$$

satisfies the following properties:

(1) *If $A \leq B$, then $\underline{N}(A) \leq \underline{N}(B)$;*
(2) *$\underline{N}(A) \leq A$;*
(3) *$\underline{N}(A) \leq \underline{N}(\underline{N}(A))$;*
(4) *$\underline{N}(A \otimes B) \geq \underline{N}(A) \otimes \underline{N}(B)$;*
(5) *$\underline{N}(\top) = \underline{\top}$, where $(L, \leq, \otimes)$ be a commutative unital quantal with $e = \top$.*

Proof We prove only item (4) and the proof of items (1), (2), (3) and (5) is the same as given in Propositions 3.8, 4.5 and 4.11 (Ref. Zhao et al. 2019). Let $N : X \longrightarrow L^{L^X}$ be an L-quasi-neighborhood system. For any $A, B \in L^X$ and any $x \in X$.

$$\begin{aligned}
&(\underline{N}(A) \otimes \underline{N}(B))(x)\\
&= \bigvee_{K \in L^X} \{N_x(K) \otimes S(K, A)\} \otimes \bigvee_{V \in L^X} \{N_x(V) \otimes S(V, B)\}\\
&= \bigvee_{K, V \in L^X} \{N_x(K) \otimes S(K, A) \otimes N_x(V) \otimes S(V, B)\}\\
&= \bigvee_{K, V \in L^X} \{N_x(K) \otimes N_x(V) \otimes S(K, A) \otimes S(V, B)\}\\
&\leq \bigvee_{K \otimes V \in L^X} \{N_x(K \otimes V) \otimes S(K \otimes V, A \otimes B)\}\\
&= \underline{N}(A \otimes B)(x).
\end{aligned}$$

□

Proposition 5.2 *Let $N : X \longrightarrow L^{L^X}$ be an L-quasi-neighborhood system on X. Then, the upper approximation operator*

$$\overline{N}(A)(x) = \bigwedge_{K \in L^X} \{N_x(K) \longrightarrow T(K, A)\},$$

satisfies the following properties:for any $A, B \in L^X$

(1) *If $A \leq B$, then $\overline{N}(A) \leq \overline{N}(B)$;*
(2) *$A \leq \overline{N}(A)$;*
If the quantale $(L, \leq, \otimes)$ satisfies the double negative law, then
(3) *$\overline{N}(A) \geq \overline{N}(\overline{N}(A))$,*
(4) *$\overline{N}(A \oplus B) \leq \overline{N}(A) \oplus \overline{N}(B)$.*

Proof The proof of items (1) and (2) is the same as the proof of Propositions 3.8 and 4.6 (Ref. Zhao et al. 2019). The proof of items (3) and (4) follows by Theorem 2.8 and Propositions 5.1. □

Theorem 5.3 *For an operator $g : L^X \longrightarrow L^X$, there exists an L-quasi-neighborhood system $N : X \longrightarrow L^{L^X}$ such that $g = \underline{N}$ if and only if g satisfies the following properties:*

(L_1) *$g(A) \leq g(B)$ whenever $A \leq B$;*
(L_2) *$g(A) \leq A$;*
(L_3) *$g(A) \leq g(g(A))$;*
(L_4) *$g(A \otimes B) \geq g(A) \otimes g(B)$;*
(L_5) *$g(\underline{\top}) = \underline{\top}$, where $(L, \leq, \otimes)$ be a commutative unital quantal with $e = \top$.*

Proof ($\Rightarrow$) Let $g = \underline{N}$, it follows immediately from Zhao et al. (2018) Theorem 3.1, we can get that $g : L^X \longrightarrow L^X$ satisfies the condition (L_1) and from Proposition 5.1 we can get that $g : L^X \longrightarrow L^X$ satisfies the conditions (L_2)–(L_5).

($\Leftarrow$) Let $g : L^X \longrightarrow L^X$ be an operator satisfies the conditions (L_1)–(L_5) and let the operator $N^g : X \longrightarrow L^{L^X}$ as defined in Zhao et al. (2018) as follows:

$$N^g_x(A) = \bigvee_{B \in L^X} \{g(B)(x) \otimes S(B, A)\}, \ \forall x \in X, \ A \in L^X.$$

As given in Theorem 3.1 of Zhao et al. (2018), we know that $\underline{N^g} = g$ holds. To complete the proof, we need to show that $N^g : X \longrightarrow L^{L^X}$ is an L-quasi-neighborhood system. To this end:

(N_1) Let $A \leq B$, then

$$\begin{aligned}
N^g_x(A) &= \bigvee_{K \in L^X} \{g(K) \otimes S(K, A)\}\\
&\overset{L_1}{\leq} \bigvee_{K \in L^X} \{g(K) \otimes S(K, B)\}\\
&= N^g_x(B)
\end{aligned}$$

(N_2)

$$\begin{aligned}
&N^g_x(K) \otimes N^g_x(V)\\
&= \bigvee_{B \in L^X} \{g(B)(x) \otimes S(B, K)\}\\
&\otimes \bigvee_{C \in L^X} \{g(C)(x) \otimes S(C, V)\}\\
&= \bigvee_{B, C \in L^X} \{g(B)(x) \otimes S(B, K)\\
&\otimes g(C)(x) \otimes S(C, V)\}\\
&= \bigvee_{B, C \in L^X} \{g(B)(x) \otimes g(C)(x)\\
&\otimes S(B, K) \otimes S(C, V)\}\\
&\overset{S4}{\leq} \bigvee_{B, C \in L^X} \{g(B)(x) \otimes g(C)(x)\\
&\otimes S(B \otimes C, K \otimes V)\}\\
&\overset{L5}{\leq} \bigvee_{B \otimes C \in L^X} \{g(B \otimes C)(x)\\
&\otimes S(B \otimes C, K \otimes V)\}\\
&\leq N^g_x(K \otimes V).
\end{aligned}$$

(N_3) The proof of the condition $N^g_x(K) \leq K(x)$ is the same as presented in the second part of Theorem 3.4 in Zhao et al. (2018).
(N_4) The proof of this item is the same as presented in the second part of Theorem 3.6 in Zhao et al. (2018). □

Theorem 5.4 *For a commutative $(L, \leq, \otimes) \in |\boldsymbol{Quant}|$ with the law of double negative and an operator $g : L^X \longrightarrow$*

L^X, then there exists an L-quasi-neighborhood system $N : X \longrightarrow L^{L^X}$ such that $g = \overline{N}$ if and only if g satisfies the following properties:

(U_1) *$g(A) \leq g(B)$ whenever $A \leq B$;*
(U_2) *$A \leq g(A)$;*
(U_3) *$g(g(A)) \leq g(A)$;*
(U_4) *$g(A \oplus B) \leq g(A) \oplus g(B)$.*

Proof $(\Rightarrow)$ Let $g = \overline{N}$, it follows immediately from Zhao et al. (2018) Theorem 3.7 we can get that $g : L^X \longrightarrow L^X$ satisfies the condition (U_1) and from Proposition 5.2 we can get that $g : L^X \longrightarrow L^X$ satisfies the conditions (U_2)–(U_4).

$(\Leftarrow)$ Let $g : L^X \longrightarrow L^X$ be an operator satisfies the conditions (U_1)–(U_4) and let the operator $N^g : X \longrightarrow L^{L^X}$ as defined in Zhao et al. (2018) as follows:

$$N^g_x(A) = \bigvee_{K \in L^X} \{\neg g(\neg K)(x) \otimes S(K, A)\}, \ \forall x \in X,\ A \in L^X.$$

As given in Theorem 3.7 of Zhao et al. (2018) we know that $\overline{N}^g = g$ holds. To complete the proof, we need to show that $N^g : X \longrightarrow L^{L^X}$ is an L-quasi-neighborhood system. To this end:

(N_1) Let $A \leq B$, then

$$\begin{aligned} N^g_x(A) &= \bigvee_{K \in L^X} \{\neg g(\neg K) \otimes S(K, A)\} \\ &\overset{L_1}{\leq} \bigvee_{K \in L^X} \{\neg g(\neg K) \otimes S(K, B)\} \\ &= N^g_x(B) \end{aligned}$$

(N_2)

$$\begin{aligned} & N^g_x(K) \otimes N^g_x(V) \\ &= \bigvee_{B \in L^X} \{\neg g(\neg B)(x) \otimes S(B, K)\} \\ &\otimes \bigvee_{C \in L^X} \{\neg g(\neg C)(x) \otimes S(C, V)\} \\ &= \bigvee_{B, C \in L^X} \{\neg g(\neg B)(x) \otimes S(B, K) \\ &\otimes \neg g(\neg C)(x) \otimes S(C, V)\} \\ &= \bigvee_{B, C \in L^X} \{\neg g(\neg B)(x) \otimes \neg g(\neg C)(x) \\ &\otimes S(B, K) \otimes S(C, V)\} \\ &\overset{S4}{\leq} \bigvee_{B, C \in L^X} \{\neg g(\neg B)(x) \otimes \neg g(\neg C)(x) \\ &\otimes S(B \otimes C, K \otimes V)\} \\ &= \bigvee_{B, C \in L^X} \{\neg[g(\neg B)(x) \oplus g(\neg C)(x) \\ &\otimes S(B \otimes C, K \otimes V)\} \\ &\overset{U4}{\leq} \bigvee_{B, C \in L^X} \{\neg g(\neg B) \oplus \neg C)(x) \\ &\otimes S(B \otimes C, K \otimes V)\} \\ &= \bigvee_{B, C \in L^X} \{\neg g(\neg(B \otimes C))(x) \\ &\otimes S(B \otimes C, K \otimes V)\} \\ &\leq N^g_x(K \otimes V). \end{aligned}$$

(N_3) The proof of the condition $N^g_x(K) \leq K(x)$ is the same as presented in the second part of Theorem 3.9 in Zhao et al. (2018).

(N_4) The proof of this item is the same as presented in the second part of of Theorem 3.11 in Zhao et al. (2018). □

6 Conclusion

In this paper, based on the concept of semi-quantales as a theoretical basis, we proposed a theory of quantale-valued quasi-neighborhood systems as a generalized form of the resent Höhle–Šostak's L-neighborhood systems and investigate its basic properties. Then, the relation between quantale-valued quasi-neighborhood systems and other many valued topological notions such as L-quasi-topologies, L-quasi-interior operators, L-generalized neighborhood operator systems and L-quasi-uniform structures. In addition, the relationship between quantale-valued quasi-neighborhood systems and L-fuzzy rough approximation operators is investigated. In future work, we plan to combine the ideas of quantale-valued quasi-neighborhood systems, multigranulation rough set, variable precision rough set and fuzzy rough set. In addition, we plan to introduce optimistic, pessimistic, and compromise models for multigranulation variable precision fuzzy rough set based on quantale-valued quasi-neighborhood systems.

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Declarations

Conflict of interest The author declare that he have no conflict of interest.

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