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ALGEBRAIC AND ANALYTICAL METHODS IN SOFT COMPUTING

δ-ideals of p-algebras

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Abstract

In p-algebras, the concepts of δ-ideals and principal δ-ideals are presented, and some of their respective properties are discussed. It is observed that the set $I^{\delta}(L)$ of all δ-ideals of a p-algebra L is a bounded lattice, and the class $I_p^{\delta}(L)$ of all principal δ-ideals forms a bounded sublattice of $I^{\delta}(L)$ and a Boolean algebra on its own. A characterization of a δ-ideal in terms of principal δ-ideals, in p-algebras, is given. Also, the concept of comaximality of δ-ideals is discussed in p-algebras. After that, a number of properties of the homomorphic image of δ-ideals are considered.

Keywords Pseudo-complemented lattices (p-algebras) · δ-ideals · Principal δ-ideals · Comaximality · $\sqcup$-Comaximality

1 Introduction

In distributive lattices and semi-lattices, Birkhoff (1948) and Frink (1962) presented the notion of pseudo-complements. Many authors later, such as Balbes and Horn (1970); Frink (1962) and Grätzer (1971), have characterized pseudo-complements in Stone algebras. In distributive p-algebras, Sambasiva Rao (2012) introduced and characterized the notion of δ-ideals which was extended by Badawy (2016a) to the concept of MS-algebras.

In this article, we continue studying δ-ideals (principal δ-ideals) in p-algebras and present a number of their main important properties. The contents of our work are organized as follows: in Sect. 2, we list some notions and notations that are needed for this topic. In Sect. 3, we introduce the notion of δ-ideals and establish some of their properties. It is proven that the set $I^{\delta}(L)$ of all δ-ideals of a p-algebra L forms a bounded lattice. In Sect. 4, we introduce the notion of principal δ-ideals. It is proven that the set $I_p^{\delta}(L)$ of all principal δ-ideals of a p-algebra L forms a bounded sublattice of $I^{\delta}(L)$ and a Boolean algebra on its own. A characterization of a δ-ideal via principal δ-ideals, in p-algebras, is given. Section 5 is devoted to the comaximality of δ-ideals in p-algebras and some related properties. Finally, in Sect. 6, some properties of the homomorphic images and inverse homomorphic images of δ-ideals are studied.

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2 Preliminaries

Here are some definitions and important results that we will use for the development of the paper.

Definition 2.1 (Davey and Priestley 2002) Let L be a lattice. Then,

(i) A nonempty subset J of L is called an ideal of L if

(1) $z, y \in J$ implies $z \vee y \in J$,
(2) $for\ a \in L,\ z \in J,\ a \leq z$ imply $a \in J$. Moreover, J is called a proper ideal of L if $J \neq L$. The set of all ideals of L is denoted by $I(L)$. Also, $(z] = \{c \in L : c \leq z\}$ is called the principal ideal generated by z.

(ii) Dually, a nonempty subset G of L is called a filter of L if
(1) $z, y \in G$ implies $z \wedge y \in G$,
(2) $for\ a \in L,\ z \in G,\ a \geq z$ imply $a \in G$.

Moreover, G is called a proper filter of L if $G \neq L$. $F(L)$ denotes to the set of all filters of L. Also, $[z) = \{c \in L : c \geq z\}$ is called the principal filter generated by z.

For a lattice L, $(I(L); \wedge, \vee)$ is the lattice of all ideals of L which is called the ideal lattice of L, where $J_1 \wedge J_2 = J_1 \cap J_2$ and

$$J_1 \vee J_2 = \{z \in L : z \leq j_1 \vee j_2 \ for\ some\ j_1 \in J_1, j_2 \in J_2\} \ \forall J_1, J_2 \in I(L).$$

Dually, $(F(L); \wedge, \vee)$ is the filter lattice of L(the lattice of all filters of L), where $G_1 \wedge G_2 = G_1 \cap G_2$ and

$$G_1 \vee G_2 = \{z \in L : z \geq g_1 \wedge g_2 \ for\ some\ g_1 \in G_1, g_2 \in G_2\}, \forall G_1, G_2 \in F(L).$$

Furthermore, $(I(L); \wedge, \vee)$ and $(F(L); \wedge, \vee)$ are distributive (modular) lattices if and only if L is a distributive (modular) lattice.

An ideal J_1 of a lattice L is called a prime ideal if $z \wedge y \in J_1$ implies $z \in J_1$ or $y \in J_1$.

Definition 2.2 (Blyth 2005; Grätzer 1971) The element a^* is the pseudo-complement of the element a of the lattice L with 0 if

$$a \wedge x = 0 \ iff \ x \leq a^*.$$

A lattice L with 0 is called a pseudo-complemented lattice (or a p-algebra) if each element of L has a pseudo-complement. A modular (distributive) p-algebra L is a p-algebra, whenever L is a modular (distributive) lattice. If L satisfies the Stone identity, $c^* \vee c^{**} = 1$ for all $c \in L$, then L is called an S-algebra. A Stone algebra L is an S-algebra, whenever L is a distributive lattice.

Theorem 2.3 (Blyth 2005; Katriňák and Mederly 1974) *Let L_1 be a p-algebra. For $z_1, y_1 \in L_1$. We have*

(1) $z_1 \leq y_1 \Rightarrow y_1^* \leq z_1^* \Rightarrow z_1^{**} \leq y_1^{**}$,
(2) $z_1^* = z_1^{***}$,
(3) $z_1 \wedge (z_1 \wedge y_1)^* = z_1 \wedge y_1^*$,
(4) $(z_1 \wedge y_1)^{**} = z_1^{**} \wedge y_1^{**}$,
(5) $(z_1 \vee y_1)^* = z_1^* \wedge y_1^*$. *Also, if L_1 is an S-algebra, then*
(6) $(z_1 \wedge y_1)^* = z_1^* \vee y_1^*$,
(7) $(z_1 \vee y_1)^{**} = z_1^{**} \vee y_1^{**}$.

A subset $B(L_1) = \{c \in L_1 : c = c^{**}\}$ of a p-algebra L_1 consists of all closed elements of L_1, and a subset $D(L_1) = \{z \in L_1 : z^* = 0\}$ consists of all dense elements of L_1.

Lemma 2.4 (Blyth 2005; Haviar 1995) *Let L_1 be a p-algebra. Then,*

(1) $(B(L_1); \wedge, \triangledown, ^*, 0, 1)$ *is a Boolean algebra, where* $z_1 \triangledown z_2 = (z_1 \vee z_2)^{**} = (z_1^* \wedge z_2^*)^*, \forall z_1, z_2 \in B(L_1)$.
(2) $D(L_1)$ *is a filter of* L_1.

A minimal prime ideal J_1 of a distributive p-algebra L is a prime ideal J_1 of L such that for $z \in J_1$ there exists $y \notin J_1$, $z \wedge y = 0$.

Definition 2.5 (Grätzer 1971) Let L and L_1 be two bounded lattices. A map $h : L \longrightarrow L_1$ is said to be a (0, 1)-lattice homomorphism if it preserves 0, 1, $\wedge$ and $\vee$.

Definition 2.6 Let $h : L \longrightarrow L_1$ be a (0, 1)-lattice homomorphism from a bounded lattice L into a bounded lattice L_1. The Kernel of h (briefly Ker h) and Cokernel of h (briefly Coker h) are defined by

Ker $h = \{z \in L : h(z) = 0\}$, and Coker $h = \{z \in L : h(z) = 1\}$, respectively.

Definition 2.7 A (0, 1)-lattice homomorphism $h : B_1 \longrightarrow B_2$, between Boolean algebras $B_1 = (B_1; \vee, \wedge, ', 0, 1)$ and $B_2 = (B_2; \vee, \wedge, ', 0, 1)$ is said to be a Boolean homomorphism if

$$h(z') = h(z)', \ \forall z \in B_1.$$

For more information about ideals, filters, intervals and p-algebras, we refer the readers to Badawy (2016a, b, 2017, 2018), Badawy and Atallah (2015, 2019), Badawy and Sambasiva Rao (2014), Badawy and Shum (2014), Sambasiva Rao (2012), Sambasiva Rao and Badawy (2014, 2017); Badawy and Helmy (2023); Badawy and Shum (2017).

3 δ-ideals of p-algebras

Definition 3.1 For any filter G of a p-algebra L, the set $\delta(G)$ is defined as follows:

$$\delta(G) = \{z \in L : z^* \in G\}.$$

Now, we study the properties of $\delta(G)$.

Lemma 3.2 *Let L be a p-algebra. Then for any filter G of L, $\delta(G)$ is an ideal of L.*

Proof Since $0^* = 1 \in G$, then $0 \in \delta(G)$. Let $a, b \in \delta(G)$. Then, $a^*, b^* \in G$. Hence, $(a \vee b)^* = a^* \wedge b^* \in G$. Thus, $a \vee b \in \delta(G)$. Now, let $x \in L$, $x \leq a$ for $a \in \delta(G)$. Then, $a^* \in G$. Hence $x^* \geq a^* \in G$. Thus $x \in \delta(G)$. Then, we conclude that $\delta(G)$ is an ideal of L. □

Lemma 3.3 *For any two filters F, G of a p-algebra L, we have*

(1) $(G$ *is a proper filter of* $L) \Rightarrow G \cap \delta(G) = \varnothing$,

(2) $x \in \delta(G) \Rightarrow x^{**} \in \delta(G)$,
(3) $x \in G \Rightarrow x^{*} \in \delta(G)$,
(4) $G \subseteq F \Rightarrow \delta(G) \subseteq \delta(F)$,
(5) $\delta(D(L)) = \{0\}$,
(6) $G \subseteq D(L) \Rightarrow \delta(G) = \{0\}$,
(7) $G = L \Leftrightarrow \delta(G) = L$.

Proof (1) Assume that $y \in G \cap \delta(G)$. Then, $y \in G$ and $y^* \in G$. Thus, $0 = y \wedge y^* \in G$ and hence $G = L$, which is a contradiction. Therefore, $G \cap \delta(G) = \phi$.
(2) Let $y \in \delta(G)$. Since $y^{***} = y^* \in G$, then $y^{**} \in \delta(G)$.
(3) Let $y \in G$. Then, $y^{**} \in G$. Thus, $y^* \in \delta(G)$ by definition of $\delta(G)$.
(4) Let $F \subseteq G$ and $z \in \delta(F)$. Then, $z^* \in F \subseteq G$ and hence $z^* \in G$. Thus, $z \in \delta(G)$. Therefore, $\delta(F) \subseteq \delta(G)$.
(5) $\delta(D(L)) = \{z \in L : z^* \in D(L)\} = \{z \in L : z^{**} = 0\} = \{z \in L : z^* = 1\} = \{0\}$.
(6) Let $G \subseteq D(L)$. By (4) and (5), we get $\delta(G) \subseteq \delta(D(L)) = \{0\}$. Thus, $\delta(G) = \{0\}$.
(7) Let $L = G$. Then,

$$L = G \Leftrightarrow 0^{**} = 0 \in G \Leftrightarrow 1 = 0^* \in \delta(G) \Leftrightarrow \delta(G) = L.$$

□

Definition 3.4 Assume that L is a p-algebra. An ideal J of L is said to be a δ-ideal if $J = \delta(G)$ for some filter G of L.

Lemma 3.5 *In a p-algebra, every prime ideal without a dense element is a δ-ideal.*

Proof Let J be a prime ideal without a dense element. Then, $(L - J)$ is a prime filter. Let $z \in J$. Clearly, $z \wedge z^* = 0 \in J$, and $z \vee z^*$ is a dense element. Hence, $z \vee z^* \notin J$. Since J is an ideal of L and $z \in J$, we get $z^* \notin J$ and hence $z^* \in L - J$. Thus, $z \in \delta(L - J)$. Therefore, $J \subseteq \delta(L - J)$.

Conversely, let $z \in \delta(L - J)$. Then, $z^* \in L - J$. Thus, $z^* \notin J$. Since $0 = z \wedge z^* \in J$, and J is a prime ideal, we get $z \in J$. Hence, $\delta(L - J) \subseteq J$. Thus, $J = \delta(L - J)$. Therefore, J is a δ-ideal. □

Lemma 3.6 *A proper δ-ideal J of a p-algebra L contains no dense element.*

Proof Assume that J is a proper δ-ideal of L and $z \in J$. Suppose that $z \in D(L)$, since $z \in J = \delta(G_1)$ for some filter G_1 of L. Hence, $0 = z^* \in G_1$, which is a contradiction. Thus, $J \cap D(L) = \phi$. □

Let $I^\delta(L)$ denote the set of all δ-ideals of L. The following example shows that $I^\delta(L)$ is not a sublattice of $I(L)$.

Example 3.7 Consider a p-algebra $L = B_4 \oplus M_3$ in Fig. 1, where $B_4 = \{0 < x, y < d\}$ is the four Boolean lattice and $M_3 = \{z, a, b, c, 1\}$ is the diamond lattice, and ⊕ stands for the ordinal sum. Let $J_1 = \{0, x\}$, $J_2 = \{0, y\}$ be two ideals of L, and $G_1 = \{y, d, z, a, b, c, 1\}$, $G_2 = \{x, d, z, a, b, c, 1\}$ two filters of L. Clearly, $\delta(G_1) = J_1$ and $\delta(G_2) = J_2$. Therefore, J_1 and J_2 are δ-ideals of L. Now, we observe that $\delta(G_1) \vee \delta(G_2) = \{0, x, y, d\}$, which is not a δ-ideal of L but $\delta(G_1) \cap \delta(G_2) = J_1 \cap J_2 = \{0\} = \delta(D(L))$ is a δ-ideal of L. Consequently, $(I^\delta(L); \cap, \vee)$ is not a sublattice of $I(L)$, but $(I^\delta(L); \cap)$ is a ∧-subsemilattice of the semi-lattice $(I(L); \cap)$.

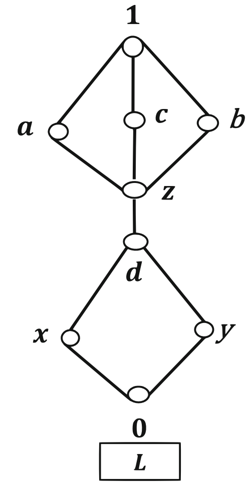

Fig. 1 $L = B_4 \oplus M_5$ is a modular p-algebra

Theorem 3.8 *For a p-algebra L,* $(I^\delta(L); \cap, \sqcup)$ *forms a bounded lattice, where*
$\delta(G_1) \cap \delta(G_2) = \delta(G_1 \cap G_2)$ *and* $\delta(G_1) \sqcup \delta(G_2) = \delta(G_1 \vee G_2)$.

Proof Let G_1 and G_2 be two filters of a p-algebra L. We prove that the infimum and supremum of both $\delta(G_1)$ and $\delta(G_2)$ in $I^\delta(L)$ are $\delta(G_1 \cap G_2)$ and $\delta(G_1 \vee G_2)$, respectively, that is,

$$\delta(G_1) \cap \delta(G_2) = \delta(G_1 \cap G_2) \;\; and \;\; \delta(G_1) \sqcup \delta(G_2) = \delta(G_1 \vee G_2).$$

Since $G_1 \cap G_2 \subseteq G_1, G_2$, $\delta(G_1 \cap G_2) \subseteq \delta(G_1), \delta(G_2)$. Thus, $\delta(G_1 \cap G_2)$ is a lower bound of $\delta(G_1)$ and $\delta(G_2)$. Assume $\delta(H)$ is another lower bound of $\delta(G_1)$ and $\delta(G_2)$. Then, $\delta(H) \subseteq \delta(G_1), \delta(G_2)$. Let $z \in \delta(H)$. Then, $z \in \delta(G_1), \delta(G_2)$ and hence $z^* \in G_1 \cap G_2$. Thus, $z \in \delta(G_1 \cap G_2)$. So $\delta(H) \subseteq \delta(G_1 \cap G_2)$ and hence $\delta(G_1 \cap G_2)$ is the greatest lower bound of both $\delta(G_1)$ and $\delta(G_2)$ in $I^\delta(L)$.
Clearly, $\delta(G_1 \vee G_2)$ is an ideal. Now, we prove that $\delta(G_1 \vee G_2)$ is the least upper bound of $\delta(G_1)$ and $\delta(G_2)$ in $I^\delta(L)$. Since $G_1, G_2 \subseteq G_1 \vee G_2$, then $\delta(G_1), \delta(G_2) \subseteq \delta(G_1 \vee G_2)$. Thus, $\delta(G_1 \vee G_2)$ is an upper bound of $\delta(G_1)$ and $\delta(G_2)$. Let $\delta(H)$ be another upper bound of $\delta(G_1)$ and $\delta(G_2)$. Then, $\delta(G_1), \delta(G_2) \subseteq \delta(H)$. Let $z \in \delta(G_1 \vee G_2)$. Then, $z^* \in G_1 \vee G_2$ and hence $z^* \geq g_1 \wedge g_2$ for some $g_1 \in G_1$ and $g_2 \in G_2$. This implies $g_1^* \in \delta(G_1)$ and $g_2^* \in \delta(G_2)$. Since $\delta(G_1), \delta(G_2) \subseteq \delta(H)$, then $g_1^*, g_2^* \in \delta(H)$. Now,

$$
\begin{aligned}
&g_1^* \in \delta(H), g_2^* \in \delta(H) \Rightarrow g_1^* \vee g_2^* \in \delta(H)\\
&\quad \Rightarrow (g_1^* \vee g_2^*)^{**} \in \delta(H)\\
&\quad \Rightarrow (g_1^{**} \wedge g_2^{**})^* \in \delta(H)\\
&\quad \Rightarrow z^{**} \leq (g_1 \wedge g_2)^* = (g_1 \wedge g_2)^{***} = (g_1^{**} \wedge g_2^{**})^* \in \delta(H)\\
&\quad \Rightarrow z^{**} \in \delta(H)\\
&\quad \Rightarrow z \in \delta(H).
\end{aligned}
$$

Thus, $\delta(G_1 \vee G_2) \subseteq \delta(H)$. Therefore, $\delta(G_1 \vee G_2)$ is the least upper bound of $\delta(G_1)$ and $\delta(G_2)$ in $I^\delta(L)$. Then, sup $\{\delta(G_1), \delta(G_2)\} = \delta(G_1 \vee G_2)$. It is clear that $\delta(L) = L$ and $\delta([1)) = \{0\} = (0]$ are the greatest and smallest members of $I^\delta(L)$, respectively. Thus, $(I^\delta(L); \cap, \sqcup)$ is a bounded lattice. □

4 Principal δ-ideals of a p-algebra

In this section, we introduce and investigate the basic properties of principal δ-ideals of a p-algebra L.

Lemma 4.1 *Let L be a p-algebra. Then for each $z \in L$, $(z^*]$ is a δ-ideal of L.*

Proof We prove that $(z^*] = \delta([z))$. To this end, let $a \in (z^*]$. Then, $a \leq z^*$ and hence $a^* \geq z^{**} \geq z \in [z)$. So $a \in \delta([z))$. Therefore, $(z^*] \subseteq \delta([z))$. On the other hand, let $a \in \delta([z))$. Then, $a^* \in [z)$. Thus, $a^* \geq z$. Since $a \leq a^{**} \leq z^* \in (z^*]$, we get $a \in (z^*]$. Thus, $\delta([z)) \subseteq (z^*]$. Therefore, $(z^*] = \delta([z))$. □

Definition 4.2 A δ-ideal of the form $\delta([z)) = (z^*]$ for $z \in L$ is called a principal δ-ideal of L.

Theorem 4.3 *For a p-algebra L, we have the following statements:*

(1) $\delta([z)) = \delta([z^{**}))$, $\forall\, z \in L$.
(2) $\delta([d)) = \{0\}$, $\forall\, d \in D(L)$.
(3) *if $z \leq y$, then $\delta([y)) \subseteq \delta([z))$, but the converse is not hold.*
(4) *for a filter G of L , $\delta([z)) \subseteq \delta(G)$ for all $z \in G$.*

Proof (1) From Lemma 4.1, $\delta([z)) = (z^*] = (z^{***}] = \delta([z^{**}))$.
(2) for all $d \in D(L)$, $\delta([d)) = (d^*] = (0] = \{0\}$.
(3) Let $z \leq y$. Then, $[y) \subseteq [z)$. By (4) of Lemma 3.3, we get $\delta([y)) \subseteq \delta([z))$. For the converse, consider the p-algebra in Fig. 2. Clearly, $\{0\} = \delta([c)) \subseteq \delta([e)) = \{0, a\}$, but $c \nleq e$ and $e \nleq c$.
(4) For all $z \in G$, suppose that $a \in \delta([z))$. Now,

$$
\begin{aligned}
&a \in \delta([z)) \Rightarrow a^* \in [z)\\
&\quad \Rightarrow a^* \geq z \in G\\
&\quad \Rightarrow a^* \in G\\
&\quad \Rightarrow a \in \delta(G).
\end{aligned}
$$

Thus, $\delta([z)) \subseteq \delta(G)$. □

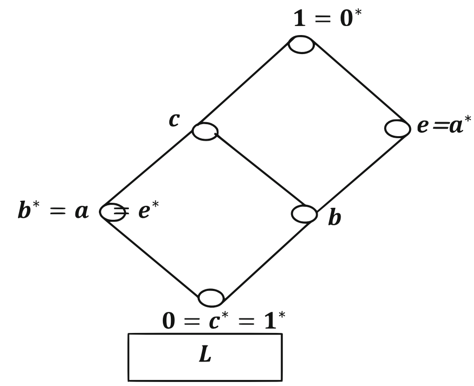

Fig. 2 L is a stone algebra

Theorem 4.4 *Let L be a p-algebra. Then, the following are equivalent:*

(1) *L is an S-algebra.*
(2) *For any $z_1, z_2 \in L$, $(z_1 \wedge z_2)^* = z_1^* \vee z_2^*$.*
(3) *For any two filters G_1, G_2 of L, $\delta(G_1) \vee \delta(G_2) = \delta(G_1 \vee G_2)$.*
(4) *$I^\delta(L)$ is a bounded sublattice of $I(L)$.*

Proof (1) $\Rightarrow$ (2): It is clear from (6) of Theorem 2.3. (2) $\Rightarrow$ (3): Assume (2) holds and $G_1, G_2 \in L$. Since $G_1, G_2 \subseteq G_1 \vee G_2$, then $\delta(G_1), \delta(G_2) \subseteq \delta(G_1 \vee G_2)$. Thus, $\delta(G_1) \vee \delta(G_2) \subseteq \delta(G_1 \vee G_2)$.

Conversely, let $z \in \delta(G_1 \vee G_2)$. Then, $z^* \in G_1 \vee G_2$ and hence $z^* \geq g_1 \wedge g_2$ for some $g_1 \in G_1$ and $g_2 \in G_2$ implies $g_1^* \in \delta(G_1)$ and $g_2^* \in \delta(G_2)$. Now,

$$
\begin{aligned}
&z^* \geq g_1 \wedge g_2 \Rightarrow z^{**} \leq (g_1 \wedge g_2)^* = g_1^* \vee g_2^* \in \delta(G_1) \vee \delta(G_2)\\
&\quad \Rightarrow z^{**} \in \delta(G_1) \vee \delta(G_2)\\
&\quad \Rightarrow z \in \delta(G_1) \vee \delta(G_2) \qquad \text{(as } z \leq z^{**}\text{)}
\end{aligned}
$$

Thus, $\delta(G_1 \vee G_2) \subseteq \delta(G_1) \vee \delta(G_2)$. Therefore, $\delta(G_1) \vee \delta(G_2) = \delta(G_1 \vee G_2)$.
(3) $\Rightarrow$ (4): Assume (3) holds. It is clear that $\delta(L)$ and $\delta([1))$ are the greatest and smallest members of $I^\delta(L)$, respectively. Let $\delta(G_1)$, $\delta(G_2) \in I^\delta(L)$. Then, $\delta(G_1) \vee \delta(G_2) = \delta(G_1 \vee G_2) \in I^\delta(L)$ and $\delta(G_1) \cap \delta(G_2) = \delta(G_1 \cap G_2) \in I^\delta(L)$. Thus, $I^\delta(L)$ is a bounded sublattice of $I(L)$.

(4) $\Rightarrow$ (1): Assume (4). Since $(z^*]$ and $(z^{**}]$ are δ-ideals of L, we have

$$\begin{aligned}(z^* \vee z^{**}] &= (z^*] \vee (z^{**}] \\ &= \delta([z)) \vee \delta([z^*)) \\ &= \delta([z) \vee [z^*)) \qquad \text{(by (3))} \\ &= \delta([z \wedge z^*)) \\ &= \delta([0)) \\ &= (0^*] \\ &= (1].\end{aligned}$$

Thus, $z^* \vee z^{**} = 1$. Therefore, L is an S-algebra. □

Now, we characterize the concept of δ-ideals in terms of principal δ-ideals.

Theorem 4.5 *Let J be a δ-ideal of a p-algebra L. Then,*

(1) $j \in J \Leftrightarrow j^{**} \in J$.
(2) $\delta([j^*)) \subseteq J, \forall j \in J$.
(3) $J = \bigcup_{j \in J} \delta([j^*))$.

Proof (1) Let $j \in J$. Since J is a δ-ideal of L, then $j \in J = \delta(G)$ for some filter G of L. Hence, $j^{***} = j^* \in G$. Thus, $j^{**} \in \delta(G) = J$. The converse implication follows from the fact that $j \leq j^{**}$.

(2) Let j be an element of J and let $x \in \delta([j^*))$. Then, $x^* \in [j^*)$ and hence $x \leq x^{**} \leq j^{**} \in J$ (by (1)). Thus, $x \in J$. Therefore, $\delta([j^*)) \subseteq J$.

(3) Since J is a δ-ideal, we get $J = \delta(G)$ for some filter G of L. Let $z \in J$. Then,

$$\begin{aligned} z \in J = \delta(G) &\Rightarrow z^* \in G \\ &\Rightarrow z^* \in [z^*) \subseteq G \\ &\Rightarrow z^{**} \in \delta([z^*)) \subseteq \delta(G) \\ &\Rightarrow z \leq z^{**} \in \delta([z^*)) \\ &\Rightarrow z \in \delta([z^*)) \subseteq \bigcup_{j \in J} \delta([j^*)) \qquad \text{(as } z \in J) \end{aligned}$$

Then, $J \subseteq \bigcup_{j \in J} \delta([j^*))$.
Conversely, let $z \in \bigcup_{j \in J} \delta([j^*))$. Then,

$$\begin{aligned} z \in \bigcup_{j \in J} \delta([j^*)) &\Rightarrow z \in \delta([k^*)),\ k \in J \\ &\Rightarrow z \in \delta([k^*)) = (k^{**}] \subseteq J \qquad \text{(as } k^{**} \in J) \\ &\Rightarrow z \in J. \end{aligned}$$

Thus, $\bigcup_{j \in J} \delta([j^*)) \subseteq J$. Therefore, $J = \bigcup_{j \in J} \delta([j^*))$. □

Now, a characterization of δ-ideals is given.

Theorem 4.6 *Let L be a p-algebra. Then, for an ideal J of L, the following statements are equivalent:*

(1) *J is a δ-ideal.*
(2) *For any $a, b \in L$, $\delta([a^*)) = \delta([b^*))$ and $a \in J$ imply $b \in J$.*
(3) *$J = \bigcup_{c \in J} \delta([c^*))$.*

Proof (1) $\Rightarrow$ (2): Let J be a δ-ideal of L. Suppose that $\delta([a^*)) = \delta([b^*))$ and $a \in J$. Then, $a \in J = \delta(G)$ for some filter G of L. Hence, $a^* \in G$. $(a^{**}] = \delta([a^*)) = \delta([b^*)) = (b^{**}]$ implies $a^{**} = b^{**}$. Then, $b^* = a^* \in G$. Therefore, $b \in \delta(G) = J$.
(2) $\Rightarrow$ (3): Assume (2) holds and $a \in J$. Since $a^{**} \in (a^{**}]$, we get $a \leq a^{**} \in (a^{**}] = \delta([a^*))$. Thus, $a \in \delta([a^*)) \subseteq \bigcup_{c \in J} \delta([c^*))$ (as $a \in J$). Therefore, $J \subseteq \bigcup_{c \in J} \delta([c^*))$.

Conversely, let $a \in \bigcup_{c \in J} \delta([c^*))$. Then, $a \in \delta([b^*))$ for some $b \in J$. Since $a \in \delta([b^*)) = \delta([b^{***}))$ and $b \in J$, then $b^{**} \in J$ by (2). Since $a \in \delta([b^*))$, we get $a^* \in [b^*)$ and hence $a^* \geq b^*$ which implies $a \leq a^{**} \leq b^{**} \in J$. Thus, $a \in J$. Therefore, $\bigcup_{c \in J} \delta([c^*)) \subseteq J$. So $J = \bigcup_{c \in J} \delta([c^*))$.
(3) $\Rightarrow$ (1): At first, we need to prove that $\bigcup_{c \in J} [c^*)$ is a filter of L. Since $1 = 0^* \in [0^*) \subseteq \bigcup_{c \in J} [c^*)$, we get $1 \in \bigcup_{c \in J} [c^*)$.Let $a, b \in \bigcup_{c \in J} [c^*)$. Then, $a \in [x^*), b \in [y^*)$ for some $x, y \in J$ and hence $a \wedge b \geq x^* \wedge y^* = (x \vee y)^*$. Thus, $a \wedge b \in [(x \vee y)^*) \subseteq \bigcup_{c \in J} [c^*)$ (as $x \vee y \in J$). Now, let $z \geq b \in \bigcup_{c \in J} [c^*)$. Then, $z \geq b \in [x^*)$ for some $x \in J$. Thus, $z \in [x^*) \subseteq \bigcup_{c \in J} [c^*)$. Therefore, $\bigcup_{c \in J} [c^*)$ is a filter of L.

Secondly, we prove that $\delta(\bigcup_{c \in J} [c^*)) = \bigcup_{c \in J} \delta([c^*))$. Since $[c^*) \subseteq \bigcup_{c \in J} [c^*)$, then $\delta([c^*)) \subseteq \delta(\bigcup_{c \in J} [c^*))$. Thus, $\bigcup_{c \in J} \delta([c^*)) \subseteq \delta(\bigcup_{c \in J} [c^*))$.

Conversely, let $a \in \delta(\bigcup_{c \in J} [c^*))$. Then,

$$\begin{aligned} a \in \delta(\bigcup_{c \in J} [c^*)) &\Rightarrow a^* \in \bigcup_{c \in J} [c^*) \\ &\Rightarrow a^* \in [x^*) \qquad \text{(for some } x \in J) \\ &\Rightarrow a \in \delta([x^*)) \subseteq \bigcup_{c \in J} \delta([c^*)) \qquad \text{(as } x \in J) \end{aligned}$$

Then, $\delta(\bigcup_{c \in J} [c^*)) \subseteq \bigcup_{c \in J} \delta([c^*))$. Thus, $\delta(\bigcup_{c \in J} [c^*)) = \bigcup_{c \in J} \delta([c^*))$. Now, we prove (3) $\Rightarrow$ (1). Assume that $J = \bigcup_{c \in J} \delta([c^*))$. Then, $J = \bigcup_{c \in J} \delta([c^*)) = \delta(\bigcup_{c \in J} [c^*))$. Hence, J is a δ-ideal of L. □

Let $I_p^\delta(L) = \{(z^*] : z \in L\} = \{\delta([z)) : z \in L\}$ be the set of all principal δ-ideals of L.

Theorem 4.7 *Let L be a p-algebra. Then,*

(1) *$(I_p^\delta(L); \wedge, \sqcup, (0], L)$ is a bounded sublattice of $I^\delta(L)$.*
(2) *$I_p^\delta(L)$ is a Boolean algebra.*
(3) *$I_p^\delta(L)$ is a homomorphic image of L.*

(4) *$B(L)$ is isomorphic of $I_p^\delta(L)$.*

Proof (1) Let $(x^*], (y^*] \in I_p^\delta(L)$. Then,

$$(x^*] \wedge (y^*] = (x^* \wedge y^*] = ((x \vee y)^*] \in I_p^\delta(L)$$

and

$$\begin{aligned}(x^*] \sqcup (y^*] &= \delta([x)) \sqcup \delta([y)) = \delta([x) \vee [y)) \\ &= \delta([x \wedge y)) = ((x \wedge y)^*] \in I_p^\delta(L).\end{aligned}$$

We observe that $L, (0] \in I_p^\delta(L)$ which are the greatest and least elements of $I_p^\delta(L)$, respectively. Thus, $I_p^\delta(L)$ is a bounded sublattice of $I^\delta(L)$.

(2) Let $(x^*], (y^*], (z^*] \in I_p^\delta(L)$. We observe that

$$\begin{aligned}(x^*] \sqcup (y^*] &= \delta([x)) \sqcup \delta([y)) \\ &= \delta([x) \vee [y)) \\ &= \delta([x \wedge y)) \\ &= ((x \wedge y)^*] \\ &= ((x \wedge y)^{***}] \\ &= ((x^{**} \wedge y^{**})^*] \\ &= (x^* \triangledown y^*].\end{aligned}$$

Now, we prove that $I^\delta(L)$ is a distributive lattice.

$$\begin{aligned}(x^*] \cap ((y^*] \sqcup (z^*]) &= (x^*] \cap (y^* \triangledown z^*] \\ &= (x^* \wedge (y^* \triangledown z^*)] \\ &= ((x^* \wedge y^*) \triangledown (x^* \wedge z^*)] \qquad \text{(as } x^*, y^*, z^* \in B(L)) \\ &= ((x \vee y)^* \triangledown (x \vee z)^*] \\ &= ((x \vee y)^*] \sqcup ((x \vee z)^*] \\ &= (x^* \wedge y^*] \sqcup (x^* \wedge z^*] \\ &= ((x^*] \cap (y^*]) \sqcup ((x^*] \cap (z^*]).\end{aligned}$$

Thus, $I_p^\delta(L)$ is a bounded distributive sublattice of $I^\delta(L)$. We have $(z^*] \wedge (z^{**}] = (z^* \wedge z^{**}] = (0]$ and $(z^*] \sqcup (z^{**}] = \delta([z)) \sqcup \delta([z^*)) = \delta([z) \vee [z^*)) = \delta([z \wedge z^*)) = \delta([0)) = L$. Thus, $(z^{**}]$ is the complement of $(z^*]$, and we can write $((z^*])' = (z^{**}]$. Hence, $(I_p^\delta(L); \wedge, \sqcup, ', (0], L)$ is a Boolean algebra.

(3) Define $\alpha : L \longrightarrow I_p^\delta(L)$ by $\alpha(z) = (z^{**}]$. Clearly, α is a well-defined map, and $\alpha(0) = (0]$, $\alpha(1) = L$. Let $z, y \in L$. Then,

$$\begin{aligned}\alpha(z \wedge y) &= ((z \wedge y)^{**}] \\ &= (z^{**} \wedge y^{**}] \\ &= (z^{**}] \wedge (y^{**}] \\ &= \alpha(z) \wedge \alpha(y)\end{aligned}$$

and

$$\begin{aligned}\alpha(z \vee y) &= ((z \vee y)^{**}] \\ &= ((z^* \wedge y^*)^*] \\ &= \delta([z^* \wedge y^*)) \\ &= \delta([z^*) \vee [y^*)) \\ &= \delta([z^*)) \sqcup \delta([y^*)) \\ &= (z^{**}] \sqcup (y^{**}] \\ &= \alpha(z) \sqcup \alpha(y).\end{aligned}$$

Now, $\alpha(z^*) = (z^{***}] = ((z^{**}])' = (\alpha(z))'$. Thus, α is a homomorphism of L into $I_p^\delta(L)$. Now, for every $(z^{**}] \in I_p^\delta(L)$, there exists $z \in L$ such that $\alpha(z) = (z^{**}]$. Thus, α is an onto map. Moreover, α is not a one-to-one map, because of $\delta([a)) = \delta([b))$ implies $a^* = b^*$ and $a \neq b$.

(4) Define $h : B(L) \longrightarrow I_p^\delta(L)$ by $h(z) = (z]$. Clearly, h is a well-defined map. Let $z, y \in B(L)$. Then,

$$\begin{aligned}h(z \wedge y) &= (z \wedge y] \\ &= (z] \wedge (y] \\ &= h(z) \wedge h(y)\end{aligned}$$

and

$$\begin{aligned}h(z \triangledown y) &= (z \triangledown y] \\ &= ((z \vee y)^{**}] \\ &= ((z^* \wedge y^*)^*] \\ &= \delta([z^* \wedge y^*)) \\ &= \delta([z^*) \vee [y^*)) \\ &= \delta([z^*)) \sqcup \delta([y^*)) \\ &= (z^{**}] \sqcup (y^{**}] \\ &= (z] \sqcup (y] \qquad \text{(as } z, y \in B(L)) \\ &= h(z) \sqcup h(y).\end{aligned}$$

We have $h(0) = (0]$, $h(1) = L$, and $h(z^*) = (z^*] = (z^{***}] = ((z^{**}])' = ((z])' = (h(z))'$. For every $(z^{**}] \in I_p^\delta(L)$, there exists $a \in B(L)$ such that $a = z^{**}$. Then, $h(a) = (a] = (z^{**}] = (z]$. Also, let $h(z) = h(y)$. Then, $(z] = (y]$ and hence $z = y$. Thus, h is one to one. Therefore, h is an isomorphism from $B(L)$ into $I_p^\delta(L)$. □

Corollary 4.8 *Assume that L is an S-algebra. Then, $(I_p^\delta(L); \wedge, \vee, (0], L)$ is a bounded sublattice of $I(L)$.*

5 Comaximality of δ-ideals

This section is devoted to introducing the notion of comaximality of δ-ideals and studying some related properties.

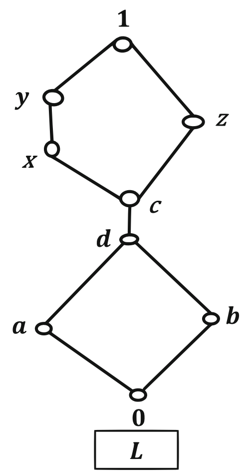

Fig. 3 $L = B_4 \oplus N_5$ is a non-modular p-algebra

Let us recall that two ideals J_1 and J_2 of a p-algebra L are called comaximal if $J_1 \vee J_2 = L$.
Now, we introduce the $\sqcup$-comaximality of δ-ideals of a p-algebra L.

Definition 5.1 Two δ-ideals J_1 and J_2 of a p-algebra L are called $\sqcup$-comaximal if $J_1 \sqcup J_2 = L$.

Lemma 5.2 *Any two comaximal δ-ideals of a p-algebra L are $\sqcup$-comaximal.*

Proof Let J_1 and J_2 be two comaximal δ-ideals of a p-algebra L. Then, $J_1 \vee J_2 = L$. Since $J_1 = \delta(G_1)$ and $J_2 = \delta(G_2)$ for some filters G_1 and G_2 of L, we get $L = \delta(G_1) \vee \delta(G_2)$. Now,

$$L = \delta(G_1) \vee \delta(G_2) \subseteq \delta(G_1 \vee G_2) = \delta(G_1) \sqcup \delta(G_2) = J_1 \sqcup J_2.$$

Thus, J_1 and J_2 are $\sqcup$-comaximal. □

The following example shows that the converse of the above lemma is not true.

Example 5.3 Consider a p-algebra $L = B_4 \oplus N_5$ in Fig. 3, where $B_4 = \{0 < a, b < d\}$ is the four Boolean lattice and $N_5 = \{c, x, y, z, 1\}$ is the Pentagon lattice. We observe that $(a]$ and $(b]$ are $\sqcup$-comaximal δ-ideals, but they are not comaximal as $(a] \sqcup (b] = (b^*] \sqcup (a^*] = \delta([b)) \sqcup \delta([a)) = \delta([b) \vee [a)) = \delta([b \wedge a)) = \delta([0)) = L$, and $(a] \vee (b] = (a \vee b] = (d] \neq L$.

The converse of the above lemma holds in the following special case:

Corollary 5.4 *Any two $\sqcup$-comaximal δ-ideals of an S-algebra L are comaximal.*

Lemma 5.5 *Let L be a p-algebra. If $z, y \in L$ such that $z \wedge y = 0$, then $\delta([z))$ and $\delta([y))$ are $\sqcup$-comaximal in L.*

Proof Let $z, y \in L$ with $z \wedge y = 0$. Then,

$$\delta([z)) \sqcup \delta([y)) = \delta([z) \vee [y)) = \delta([z \wedge y)) = \delta([0)) = L.$$

Therefore, $\delta([z))$ and $\delta([y))$ are $\sqcup$-comaximal in L. □

Theorem 5.6 *Let L be a distributive p-algebra. Then,*

(1) *Every prime δ-ideal of L is a minimal prime ideal.*
(2) *Any two distinct prime δ-ideals of L are $\sqcup$-comaximal.*

Proof (1) Let J_1 be a prime δ-ideal of L. Then, $J_1 = \delta(G_1)$ for some filter G_1 of L, let $z \in J_1 = \delta(G_1)$. Then, $z^* \in G_1$. We have $0 = z \wedge z^* \in J_1$. Suppose that $z^* \in J_1$. Then, $z^* \in G_1 \cap \delta(G_1) \neq \phi$ which contradicts with **(1)** of Lemma 3.3. Hence, $z^* \notin J_1$, that is, for $z \in J_1$, there exists $y = z^* \notin J_1$ such that $z \wedge y = 0$. Thus, J_1 is a minimal prime ideal.

(2) Let J_1 and J_2 be two distinct prime δ-ideals of L. Then by (1), J_1 and J_2 are minimal prime ideals. Let $a \in J_1 - J_2$ and $b \in J_2 - J_1$. Since J_1 and J_2 are two minimal prime ideals, then there exist $x \notin J_1$ and $y \notin J_2$ such that $a \wedge x = 0 = b \wedge y$. Since $x \notin J_1$ and $b \notin J_1$, then $b \wedge x \notin J_1$(as J_1 is a prime ideal), similarly $a \wedge y \notin J_2$. By definition of pseudo-complement and the fact that J_1 is a prime ideal, we get $(b \wedge x)^* \in J_1$. Thus, $\delta([b \wedge x)) = ((b \wedge x)^*] \subseteq J_1$. Similarly, $\delta([a \wedge y)) \subseteq J_2$. Now, $(b \wedge x) \wedge (a \wedge y) = (a \wedge x) \wedge (b \wedge y) = 0 \wedge 0 = 0$. Then by Lemma 5.5, we get $\delta([a \wedge y))$ and $\delta([b \wedge x))$ are $\sqcup$-comaximal. Hence,

$$L = \delta([a \wedge y)) \sqcup \delta([b \wedge x)) \subseteq J_1 \sqcup J_2.$$

Thus, $J_1 \sqcup J_2 = L$. Therefore, J_1 and J_2 are $\sqcup$-comaximal.

□

Let J be an ideal of a p-algebra L. For any $z \in L$, consider

$$\delta_J([z)) = \{j \in J : j^* \in [z)\}.$$

Lemma 5.7 *Let J be an ideal of p-algebra L. Then,*

(1) *$\delta_J([z)) = J \cap \delta([z))$ is an ideal of J.*
(2) *$\delta_J([z))$ is a δ-ideal of L, whenever J is a δ-ideal of L.*

Proof (1) It is clear that $\delta_J([z)) = \{j \in J : j \in \delta([z))\} = J \cap \delta([z))$, and hence, $\delta_J([z))$ is an ideal of J.

(2) Since $\delta_J([z)) = J \cap \delta([z))$ and $J, \delta([z))$ are two δ-ideals of L, then $\delta_J([z))$ is a δ-ideal of L.

□

Theorem 5.8 (1) *Let J be a principal δ-ideal of a p-algebra L. Then for any $z, y \in L$ with $z \wedge y = 0$, $\delta_J([z))$ and $\delta_J([y))$ are $\sqcup$-comaximal in J.*

(2) *Let J be a δ-ideal of a distributive p-algebra L. Then for any $z, y \in L$ with $z \wedge y = 0$, $\delta_J([z))$ and $\delta_J([y))$ are $\sqcup$-comaximal in J.*

Proof (1) Let $J = \delta([a))$ be a principal δ-ideal of a p-algebra L and $z, y \in L$ with $z \wedge y = 0$. Then by Lemma 5.5, we have $\delta([z)) \sqcup \delta([y)) = L$. Now,

$$\begin{aligned}
\delta_J([z)) \sqcup \delta_J([y)) &= (J \cap \delta([z))) \sqcup (J \cap \delta([y))) \\
&= (\delta([a)) \cap \delta([z))) \sqcup (\delta([a)) \cap \delta([y))) \quad (\text{as } J = \delta([a))) \\
&= ((a^*] \cap (z^*]) \sqcup ((a^*] \cap (y^*]) \\
&= (a^*] \cap ((z^*] \sqcup (y^*]) \quad (\text{as } I_p^{\delta}(L) \text{ is distributive}) \\
&= \delta([a)) \cap (\delta([z)) \sqcup \delta([y))) \\
&= \delta([a)) \cap L \quad (\text{as } \delta([z)) \sqcup \delta([y)) = L) \\
&= J \cap L \\
&= J
\end{aligned}$$

Therefore, $\delta_J([z))$ and $\delta_J([y))$ are $\sqcup$-comaximal in J.

(2) Using a similar way of (1), one can prove that $\delta_J([z))$ and $\delta_J([y))$ are $\sqcup$-comaximal in J.

□

6 Homomorphic images of δ-ideals

This section discusses the properties of images and the inverse images of δ-ideals (principal δ-ideals) with respect to a homomorphism of two p-algebras. By a homomorphism on a p-algebra L, we mean a lattice homomorphism h satisfying $(h(x))^* = h(x^*)$ for all $x \in L$.

Theorem 6.1 *Assume that $h : L_1 \longrightarrow L_2$ is an onto homomorphism of a p-algebra L_1 to a p-algebra L_2. Then,*

(1) *The image of a principal δ-ideal is a principal δ-ideal, that is, for any $z \in L_1$, $h(\delta[z)) = \delta([h(z)))$.*
(2) *for any filter G_1 of L_1, $h(\delta(G_1)) = \delta(h(G_1))$.*
(3) *for any δ-ideal J_1 of L_1, $h(J_1)$ is a δ-ideal of L_2.*
(4) *for any δ-ideal J of L_1, $h(J) = \bigcup_{j \in J} \delta([(h(j))^*))$.*

Proof (1) For all $z \in L_1$, we get

$$\begin{aligned}
h(\delta([z))) = h((z^*]) = h\{c \in L_1 : c \leq z^*\} &= \{h(c) \in L_2 : h(c) \leq h(z^*)\} \\
&= \{h(c) \in L_2 : h(c) \leq h(z)^*\} \\
&= (h(z)^*] = \delta([h(z))).
\end{aligned}$$

(2) For any filter G_1 of L_1, let $z \in \delta(h(G_1))$. Then,

$$\begin{aligned}
z \in \delta(h(G_1)) &\Rightarrow z^* \in h(G_1) \\
&\Rightarrow z^* = h(j_1) \ for\ j_1 \in G_1 \\
&\Rightarrow z \leq z^{**} = (h(j_1))^* = h(j_1^*) \in h(\delta(G_1)) \\
&\Rightarrow z \in h(\delta(G_1)) \quad (\text{as } h(\delta(G_1)) \text{is an ideal}) \\
&\Rightarrow \delta(h(G_1)) \subseteq h(\delta(G_1)).
\end{aligned}$$

Conversely, let $z \in h(\delta(G_1))$. Then,

$$\begin{aligned}
z \in h(\delta(G_1)) &\Rightarrow z = h(y) \ for\ y \in \delta(G_1) \\
&\Rightarrow y^* \in G_1 \\
&\Rightarrow z^* = (h(y))^* = h(y^*) \in h(G_1) \\
&\Rightarrow z \in \delta(h(G_1)) \\
&\Rightarrow h(\delta(G_1)) \subseteq \delta(h(G_1))
\end{aligned}$$

Thus, $h(\delta(G_1)) = \delta(h(G_1))$.

(3) Let J_1 be a δ-ideal of L_1. Then, $J_1 = \delta(G_1)$ for some filter G_1 of L_1. Now,

$$\begin{aligned}
h(J_1) &= h(\delta(G_1)) \\
&= h\{z \in L_1 : z^* \in G_1\} \\
&= \{h(z) \in L_2 : h(z^*) \in h(G_1)\} \\
&= \{h(z) \in L_2 : h(z)^* \in h(G_1)\} \\
&= \delta(h(G_1)).
\end{aligned}$$

Thus, $h(J_1)$ is a δ-ideal of L_2.

(4) For any δ-ideal J of L_1, $J = \bigcup_{j \in J} \delta([j^*))$ from (3) Theorem 4.6. Let $z \in h(J)$. Then, $z = h(j)$ for some $j \in J$. Then, $z \leq z^{**} \in (z^{**}] = \delta([z^*)) = \delta([(h(j))^*)) \subseteq \bigcup_{j \in J} \delta([(h(j))^*))$. Thus, $h(J) \subseteq \bigcup_{j \in J} \delta([(h(j))^*))$. Conversely, let $z \in \bigcup_{j \in J} \delta([(h(j))^*))$. Now,

$$\begin{aligned}
z \in \bigcup_{j \in J} \delta([(h(j))^*)) &\Rightarrow z \in \delta([(h(c))^*)),\ c \in J \\
&\Rightarrow z \in ((h(c))^{**}] \\
&\Rightarrow z \leq ((h(c))^{**} = h(c^{**}) \in h(J) \quad (\text{as } c^{**} \in J) \\
&\Rightarrow z \in h(J) \quad (\text{as } h(J) \text{is an ideal})
\end{aligned}$$

Thus, $\bigcup_{j \in J} \delta([(h(j))^*)) \subseteq h(J)$. Therefore, $h(J) = \bigcup_{j \in J} \delta([(h(j))^*))$.

□

Theorem 6.2 *Let $h : L_1 \to L_2$ be a homomorphism of a p-algebra L_1 into a p-algebra L_2. Then,*

(1) *Ker h is a δ-ideal of L_1.*
(2) *For any δ-ideal K of L_2, $h^{-1}(K)$ is a δ-ideal of L_1 containing Ker h.*

Proof (1) Since h is a homomorphism of a p-algebra L_1 into a p-algebra L_2, then Ker $h = \{z \in L_1 : h(z) = 0\}$ and Coker $h = \{z \in L_1 : h(z) = 1\}$ are an ideal and a filter of L_1, respectively. We show that Ker $h = \delta(\text{Coker } h)$. Let $z \in$ Ker h. Then,

$$\begin{aligned} z \in\ Ker\ h &\Leftrightarrow h(z) = 0 \\ &\Leftrightarrow (h(z))^* = h(z^*) = 1 \\ &\Leftrightarrow z^* \in Coker\ h \\ &\Leftrightarrow z \in \delta(Coker\ h) \end{aligned}$$

Thus, Ker $h = \delta(\text{Coker } h)$.

(2) Let K be a δ-ideal of L_2. Then, $K = \delta(G_1)$ for some filter G_1 of L_2. Since $h^{-1}(K)$ is an ideal of L_1. We prove that $h^{-1}(K) = \delta(h^{-1}(G_1))$. So let $z \in h^{-1}(K)$. Then,

$$\begin{aligned} z \in h^{-1}(K) &\Leftrightarrow h(z) = c, c \in K = \delta(G_1) \\ &\Leftrightarrow h(z)^* = h(z^*) = c^* \in G_1 \qquad (\text{as } c \in \delta(G_1) \Rightarrow c^* \in G_1) \\ &\Leftrightarrow z^* \in h^{-1}(\{c^*\}) \subseteq h^{-1}(G_1) \\ &\Leftrightarrow z \in \delta(h^{-1}(G_1)) \end{aligned}$$

Thus, $h^{-1}(K) = \delta(h^{-1}(G_1))$. Therefore, $h^{-1}(K)$ is a δ-ideal of L_1. Let $z \in Ker\ h$. Then, $h(z) = 0 \in K = \delta(G_1)$, and hence, $h(z^*) = h(z)^* = 1 \in G_1$ implies $z^* \in h^{-1}(G_1)$. Thus, $z \in \delta(h^{-1}(G_1))$. Therefore, Ker $h \subseteq \delta(h^{-1}(G_1))$.

□

Theorem 6.3 *Let $h : L_1 \longrightarrow L_2$ be an onto homomorphism of a p-algebra L_1 to a p-algebra L_2. Then,*

(1) $\alpha : I_p^\delta(L_1) \longrightarrow I_p^\delta(L_2)$, $\alpha(\delta([z))) = \delta([h(z)))$, $\forall \delta([z)) \in I_p^\delta(L_1)$ *is a Boolean homomorphism.*

(2) $\alpha : I^\delta(L_1) \longrightarrow I^\delta(L_2)$, $\alpha(\delta(G)) = \delta(h(G))$, $\forall \delta(G) \in I^\delta(L_1)$ *is a homomorphism.*

Proof (1) Define $\alpha : I_p^\delta(L_1) \longrightarrow I_p^\delta(L_2)$ by $\alpha(\delta([z))) = \delta([h(z)))$. Clearly, α is well-defined. $\alpha(L_1) = \alpha(\delta([0_{L_1})))$ $= \delta([h(0_{L_1}))) = \delta([0_{L_2})) = \delta(L_2) = L_2$ and $\alpha\{0_{L_1}\} = \alpha(\delta([1_{L_1}))) = \delta([h(1_{L_1}))) = \delta([1_{L_2})) = \{0_{L_2}\}$. Let $\delta([a)), \delta([y)) \in I_p^\delta(L_1)$. Then,

$$\begin{aligned} \alpha(\delta([z)) \wedge \delta([y))) &= \alpha(\delta([z) \wedge [y))) \\ &= \alpha(\delta([z \vee y))) \\ &= \delta([h(z \vee y)) \\ &= \delta([h(z) \vee h(y))) \\ &= \delta([h(z)) \wedge [h(y))) \\ &= \delta([h(z))) \wedge \delta([h(y))) \\ &= \alpha(\delta([z))) \wedge \alpha(\delta([y))) \end{aligned}$$

and

$$\begin{aligned} \alpha(\delta([z)) \sqcup \delta([y))) &= \alpha(\delta([z) \vee [y))) \\ &= \alpha(\delta([z \wedge y))) \\ &= \delta([h(z \wedge y)) \\ &= \delta([h(z) \wedge h(y))) \\ &= \delta([h(z)) \vee [h(y))) \\ &= \delta([h(z))) \sqcup \delta([h(y))) \\ &= \alpha(\delta([z))) \sqcup \alpha(\delta([y))). \end{aligned}$$

Since $I_p^\delta(L_1)$ and $I_p^\delta(L_2)$ are Boolean algebras, we get $(\alpha(\delta([z)))^* = (\alpha((z^*]))^* = (\alpha((z^*]))'$. Thus,

$$\begin{aligned} \alpha((\delta([z)))^*) &= \alpha(((z^*])^*) \\ &= \alpha((z^{**}]) \\ &= \alpha(\delta([z^*))) \\ &= \delta([h(z^*)) \\ &= \delta([(h(z))^*)) \\ &= ((h(z))^{**}] \\ &= (((h(z))^*])^* \\ &= (\delta([h(z))))^* \\ &= (\alpha(\delta([z))))^*. \end{aligned}$$

Therefore, α is a Boolean homomorphism of $I_p^\delta(L_1)$ into $I_p^\delta(L_2)$.

(2) Define $\alpha : I^\delta(L_1) \longrightarrow I^\delta(L_2)$ by $\alpha(J_1) = \delta(h(G))$ where $J_1 = \delta(G)$. Clearly, α is well-defined and $\alpha\{0_{L_1}\} = \{0_{L_2}\}$, $\alpha(L_1) = L_2$. Let $J_1, J_2 \in I^\delta(L_1)$. Then, $J_1 = \delta(G_1)$ and $J_2 = \delta(G_2)$ for some filters G_1 and G_2 of L_1. Then, we get

$$\begin{aligned} \alpha(J_1 \wedge J_2) &= \alpha(\delta(G_1) \wedge \delta(G_2)) \\ &= \alpha(\delta(G_1 \wedge G_2)) \\ &= \delta(h(G_1 \wedge G_2)) \\ &= \delta(h(G_1)) \wedge \delta(h(G_2)) \\ &= \alpha(J_1) \wedge \alpha(J_2) \end{aligned}$$

and

$$\begin{aligned} \alpha(J_1 \sqcup J_2) &= \alpha(\delta(G_1) \sqcup \delta(G_2)) \\ &= \alpha(\delta(G_1 \vee G_2)) \\ &= \delta(h(G_1 \vee G_2)) \\ &= \delta(h(G_1)) \sqcup \delta(h(G_2)) \\ &= \alpha(J_1) \sqcup \alpha(J_2). \end{aligned}$$

Therefore, α is a homomorphism of $I^\delta(L_1)$ into $I^\delta(L_2)$.

□

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