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FUZZY SYSTEMS AND THEIR MATHEMATICS

New horizon in fuzzy distributions: statistical distributions in continuous domains generated by Choquet integral

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Abstract

In this paper, some statistical properties of the Choquet integral are discussed. As an interesting application of Choquet integral and fuzzy measures, we introduce a new class of exponential-like distributions related to monotone set functions, called *Choquet exponential distributions*, by combining the properties of Choquet integral with the exponential distribution. We show some famous statistical distributions such as gamma, logistic, exponential, Rayleigh and other distributions are a special class of Choquet distributions. Then, we show that this new proposed Choquet exponential distribution is better on daily gold price data analysis. Also, a real dataset of the daily number of new infected people to coronavirus in the USA in the period of 2020/02/29 to 2020/10/19 is analyzed. The method presented in this article opens a new horizon for future research.

Keywords Fuzzy measures · Choquet integral · Statistical distribution · Gold price · Distorted probabilities · Fuzzy distributions

1 Introduction

Choquet integral is undoubtedly one of the most fundamental concepts in nonlinear theories (Choquet 1954; Sugeno 2013, 2015). In contrast, the theory and applications of the Choquet integral defined on a discrete set have also been the interesting topic of many studies, see (Beliakov et al. 2019; Faigle and Grabisch 2011; Horanská and Šipošová 2018; Narukawa and Torra 2005) and references cited therein.

This concept was first proposed by Choquet (Choquet 1954) and then very quickly was welcomed by researchers in various problems (Agahi et al. 2019; Mesiar et al. 2010; Ridaoui and Grabisch 2016; Torra and Narukawa 2016; Torra et al. 2016; Torra 2017). An interesting research topic for many researchers has been the related theory and applications of the Choquet integral on the discrete data sets (see (Faigle and Grabisch 2011; Grabisch and Labreuche 2016) and references therein). Many researchers have worked on different generalizations of the Choquet integral (Labreuche and Grabisch 2007; Grabisch 1996; Imaoka 1997; Horanská and Šipošová 2018; Karczmarek et al. 2018; Klement et al. 2009; Narukawa and Torra 2005; Zhang et al. 2022). In the continuous case, working on this concept can be complicated. So, Sugeno first discussed (Sugeno 2013, 2015) the Choquet integral on the real line and derivatives of functions with respect to monotone set functions. Recently, Torra (2017); Torra and Narukawa (2016); Torra et al. (2016) discussed some interesting applications of Choquet integral on the real line. The applications of these concepts in information fusion were discussed in Auephanwiriyakul et al. (2002); Šipošová et al. (2017); Torra et al. (2018); Torra and Narukawa (2016). The numerical and theoretical evaluation of discrete and continuous fuzzy measures, Choquet integral,

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and their applications have been recently studied by many researchers (Abu Arqub 2017; Abu Arqub et al. 2023a, b; Agahi et al. 2019; Agahi 2021; Alshammari et al. 2020; Beliakov et al. 2019; Faigle and Grabisch 2011; Grabisch 2016; Labreuche and Grabisch 2018; Sugeno 2013, 2015; Torra 2017; Torra and Narukawa 2016; Torra et al. 2016, 2020).

The Choquet integral has an interesting ability to model interacting criteria and decision theory (Grabisch 1996; Labreuche and Grabisch 2018). On the other side, the Choquet integral fails to represent its preferences in model interacting criteria, and it cannot model different decision strategies and needs some generalizations for this issue (Labreuche and Grabisch 2018, 2007).

Exponential distribution (also known as Boltzmann–Gibbs or thermal distributions (Collier 2004)), is an important and widely applied statistical distribution in statistical physics, mathematics, econophysics, finance and many other fields. In statistics, the exponential distribution is a commonly applied model in data analysis. The probability density function (pdf) of a random variable Z from the exponential distribution is given by

$$f_Z(z) = \theta \exp\{-\theta z\}, \; z > 0, \theta > 0, \tag{1}$$

with cumulative distribution function (cdf) $F(\cdot)$ such that $F(z) = \int_0^z f(t)\,\mathrm{d}t = 1 - e^{-\theta z}$ whenever $z > 0$.

An interesting subject in this field is the mixture of fuzzy measures and Choquet integral with continuous statistical distributions. In this paper, we introduce a new class of exponential-like distributions related to monotone set functions, called *Choquet exponential distributions*, by combining the properties of Choquet integral with the exponential distribution. In this paper, we work on the London Bullion Market (LBM) gold daily price in period of 1968/4/01 to 2018/05/15. Our observations show that our proposed Choquet exponential distributions are better in daily gold price data analysis. Intelligent data analysis and multiparameter optimization have also attracted many authors to study the real datasets (Al-Janabi and Alkaim 2020; Al-Janabi et al. 2021, 2020, ?). Our model selection using information criteria helps us to select a suitable statistical model from the list of our models and to find the best model for the data set (Wasserman 2000). We used two important tools in the model selection, the Akaike information criteria (AIC (Akaike 1973)) and the Bayesian information criteria (BIC (Schwarz 1978)). The related maximum likelihood estimation (MLE) of the proposed model has been obtained with the package "nlminb" in the software R, numerically, in our calculations.

The paper is organized as follows. In Sect. 2, we present some basic notations and related definitions for Choquet integral. Then, Choquet exponential distribution and its properties are introduced in Sect. 3. In Sect. 4, we show that our proposed Choquet exponential distribution is better in the daily LBM gold price data analysis in 1968/4/01 to 2018/05/15. Finally, a real dataset of the daily number of new infected people to coronavirus in the USA in the period of 2020/02/29 to 2020/10/19 is analyzed.

2 Preliminaries and notations

Some basic notations and related definitions for the nonlinear Choquet integral (Choquet 1954; Sugeno 2013, 2015; Torra 2017; Torra and Narukawa 2016; Torra et al. 2016) are presented in this section.

2.1 Choquet integral

Definition 1 The set function $\gamma : \mathcal{F} \to [0, \infty]$ on a measurable space $(\Omega, \mathcal{F})$ is called a monotone set function if the following properties are satisfied:
(i) $\gamma(\emptyset) = 0$;
(ii) If $W, S \in \mathcal{F}$ and $W \subseteq S$, then $\gamma(W) \leq \gamma(S)$.

Definition 2 A *fuzzy measure* is a monotone set function γ with $\gamma(\Omega) = 1$.

Definition 3 Let $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a Borel measurable space. If $\mathfrak{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function with $\mathfrak{m}(0) = 0$, then a monotone set function $\gamma_{\mathfrak{m}}$ acting on $\mathcal{B}(\mathbb{R})$ is called the distorted Lebesgue measure on $\mathcal{B}(\mathbb{R})$, whenever

$$\gamma_{\mathfrak{m}}(\cdot) = \mathfrak{m}(\eta(\cdot)), \tag{2}$$

where η is the Lebesgue measure.

Definition 4 Choquet (1954); Denneberg (1994); Pap (1995) Let γ be a monotone set function on a measurable space $(\Omega, \mathcal{F})$. The Choquet integral of a nonnegative measurable function Z with respect to γ on $H \in \mathcal{F}$ is evaluated as

$$(\mathcal{C}) \int_H Z d\gamma = \int_0^{+\infty} \gamma(H \cap \{\omega | Z(\omega) \geq t\})\,\mathrm{d}t. \tag{3}$$

The right-hand side of the above formula is the improper Riemann integral.

3 Choquet distributions

In this section, we introduce a new class of exponential-like distributions related to monotone set functions, called *Choquet exponential distributions*, by combining the properties of Choquet integral with the exponential distribution.

3.1 Choquet exponential distributions

Assume $\mathcal{B}(\mathbb{R}^+)$ as the σ-algebra of all Borel subsets of $\mathbb{R}^+$. Let f be a continuous, differentiable, nonnegative and decreasing function. One can easily obtain the following result (Agahi et al. 2019; Sugeno 2013; Torra 2017):

$$\begin{aligned}(\mathcal{C})\int_0^z f\,d\gamma &= \int_0^{f(0)} \gamma\left([0,z]\cap\{x|f(x)\geq r\}\right)\mathrm{d}r\\ &= \int_0^{f(z)} \gamma\left([0,z]\right)\mathrm{d}r + \int_{f(z)}^{f(0)} \gamma\left(\left[0, f^{-1}(r)\right]\right)\mathrm{d}r\\ &= f(z)\,\gamma[0,z] + \int_{f(z)}^{f(0)} \gamma\left(\left[0, f^{-1}(r)\right]\right)\mathrm{d}r\\ &= f(z)\,\gamma[0,z] + \int_z^0 f'(t)\gamma\left([0,t]\right)\mathrm{d}t\\ &= f(z)\,\gamma[0,z] - \int_0^z f'(t)\gamma\left([0,t]\right)\mathrm{d}t \end{aligned} \tag{4}$$

where $\gamma : \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ is a monotone set function. We also have changed the integration variable r to $t = f^{-1}(r)$ to obtain the fourth line of the above equation. If

$$f(z) = \theta e^{-\theta z}, z > 0, \theta > 0, \tag{5}$$

is the probability density function of the exponential distribution, then we have

$$(\mathcal{C})\int_0^z f\,d\gamma = \theta e^{-\theta z}\gamma[0,z] + \theta^2 \int_0^z e^{-\theta t}\gamma[0,t]\mathrm{d}t. \tag{6}$$

Then, the cumulative distribution function of the Choquet distribution with respect to γ is given by

$$F_{(\mathcal{C})}(z) = \frac{1}{\zeta} \times (\mathcal{C})\int_0^z f\,d\gamma, \quad z > 0, \tag{7}$$

where

$$\zeta = \lim_{z\to\infty} (\mathcal{C})\int_0^z f\,d\gamma. \tag{8}$$

For $\zeta < +\infty$, the Choquet distribution is well defined. Consider a function Ψ given by $\Psi(z) = \gamma([0,z])$, where γ is the distorted Lebesgue measure. Then, $\gamma'([0,z]) = \Psi'(z)$ if this derivative exists and is positive, and $\gamma'([0,z]) = 0$, otherwise. Hence,

$$\begin{aligned} f_{\mathcal{C}}(z) &= \frac{dF_{\mathcal{C}}(z)}{\mathrm{d}z} = \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}z}\left[(\mathcal{C})\int_0^z f\,d\gamma\right]\\ &= \frac{\theta e^{-\theta z}}{\zeta}\left\{-\theta\gamma[0,z] + \gamma'[0,z] + \theta\gamma[0,z]\right\}\\ &= \frac{\theta e^{-\theta z}}{\zeta}\gamma'[0,z]. \end{aligned} \tag{9}$$

Here, we used Eq. (6) to obtain the second line of the above equation.

Definition 5 A continuous random variable Z is said to have a Choquet exponential distribution with respect to a distorted Lebesgue measure γ, if its probability distribution function of Z is given by

$$f_{\mathcal{C}}(z) = \frac{\theta e^{-\theta z}}{\zeta}\gamma'[0,z], z > 0, \theta > 0, \tag{10}$$

where ζ is defined in (8) and $\gamma'([0,z])$ exists and is positive, and $\gamma'([0,z]) = 0$, otherwise.

One can evaluate the related monotone increasing measures γ to a Choquet exponential distribution $f_{\mathcal{C}}(z)$ as follows. Let $f_{\mathcal{C}}(z) = \frac{\theta e^{-\theta z}}{\zeta}\gamma'[0,z]$. Then, $\gamma'[0,z] = \frac{\zeta f_{\mathcal{C}}(z)}{\theta e^{-\theta z}}$. So,

$$\gamma[0,x] = \int_0^x \frac{\zeta f_{\mathcal{C}}(z)}{\theta e^{-\theta z}}\mathrm{d}z. \tag{11}$$

Using some monotone increasing distortion functions $\mathfrak{m}(x)$ with $\mathfrak{m}(0) = 0$, it is possible to find different Choquet exponential distributions. The main idea of generating any new element of the huge set of Choquet exponential distributions is given in Fig. 1. Using this flowchart, one easily produces a wide class of Choquet distributions. Table 1 gives us some special examples of the Choquet exponential distribution (10), and their related leading distortion function $\mathfrak{m}(x)$.

Here, we would like to focus on one of the proposed Choquet exponential distributions in Table 1, which is more suitable to study our real dataset in the next section. For example, the distortion function $\mathfrak{m}(t) = \beta\sinh[\alpha t] + (1-\beta)(\cosh[\alpha t] - 1)$ is a monotone increasing function with $\mathfrak{m}(0) = 0$. Applying the related distorted Lebesgue measure leads us to a Choquet exponential distribution, which is a new generalized exponential distribution, denoted by $X \sim GCE(\alpha,\beta,\lambda)$, with probability density function given as follows

$$\begin{aligned} &f^{\mathcal{C}}(x,\lambda,\alpha,\beta)\\ &= \frac{e^{-\lambda x}\left(\lambda^2-\alpha^2\right)\left(\beta\cosh[\alpha x] + (1-\beta)\sinh[\alpha x]\right)}{\left(\beta\lambda + (1-\beta)\alpha\right)},\\ &0 \leq \alpha < \lambda,\ 0 \leq \beta \leq 1. \end{aligned} \tag{12}$$

If $\beta = 1$, then a new distribution, denoted by $X \sim GCE(\lambda,\alpha)$, with probability density function is given by

$$f^{\mathcal{C}}(x,\lambda,\alpha,1) = \frac{e^{-\lambda x}\left(\lambda^2-\alpha^2\right)\cosh[\alpha x]}{\lambda}, 0 \leq \alpha < \lambda. \tag{13}$$

Table 1 Some different examples of the Choquet distribution $f_{\mathcal{C}}(x)$ in the class of distorted Lebesgue measures with different distorted functions $\mathfrak{m}(x)$

$f_{\mathcal{C}}(x)$	Name/Conditions	$\mathfrak{m}(x)$
$\theta e^{-\theta x}$	Exponential, $\theta > 0$	x
$\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$	Rayleigh, $\sigma > 0$	$\sqrt{\frac{\pi}{2}}\sigma e^{\frac{1}{2}\sigma^2\theta^2}\left(\mathrm{erf}[\frac{\theta\sigma}{\sqrt{2}}] - \mathrm{erf}[\frac{\sigma^2\theta - x}{\sigma\sqrt{2}}]\right) + \frac{1}{\theta}\left(1 - e^{-\frac{x^2}{2\sigma^2}+\theta x}\right)$
$\frac{1}{\beta^\alpha \Gamma[\alpha]} x^{\alpha-1} e^{-\frac{x}{\beta}}$	Gamma, $\beta > 0, \alpha > 0$	$\frac{x^\alpha\left(\Gamma[\alpha]-\Gamma\left[\alpha, x\left(\frac{1}{\beta}-\theta\right)\right]\right)}{\theta\left(\beta x\left(\frac{1}{\beta}-\theta\right)\right)^\alpha \Gamma[\alpha]}$
$\frac{\theta^\alpha}{\Gamma[\alpha]} x^{\alpha-1} e^{-\theta x}$	Gamma, $\theta > 0, \alpha > 0$	x^α
$(\theta - \alpha) e^{-(\theta-\alpha)x}$	Exponential, $\theta > \alpha > 0$	$e^{\alpha x} - 1$
$\frac{x^{\beta-1} e^{-\theta x}\{e^{\alpha x}(\beta+\alpha x)-\beta\}}{\beta\Gamma[\beta]\left(-\theta^{-\beta}+\theta(\theta-\alpha)^{-\beta-1}\right)}$	$\theta > \alpha > 0$, $\beta > -1$	$x^\beta\left(e^{\alpha x} - 1\right)$
$(\theta + \alpha) e^{-(\theta+\alpha)x}$	Exponential, $\theta + \alpha > 0$	$1 - e^{-\alpha x}$
$\frac{x^{\beta-1} e^{-(\theta+\alpha)x}\{\alpha x+\beta(e^{\alpha x}-1)\}}{\Gamma[1+\beta]\left(\theta^{-\beta}-\theta(\theta+\alpha)^{-\beta-1}\right)}$	$\theta > 0, \beta > -1$, $\theta + \alpha > 0$	$x^\beta\left(1 - e^{-\alpha x}\right)$
$\frac{x^{\beta-1}(\beta+\alpha x)e^{(\alpha-\theta)x}}{\theta(\theta-\alpha)^{-1-\beta}\Gamma[\beta+1]}$	$\theta > \alpha > 0$, $\beta > 0$	$x^\beta e^{\alpha x}$
$\frac{\theta^2-\alpha^2}{\theta}\cosh[\alpha x]\, e^{-\theta x}$	$\theta > \alpha \geq 0$	$\sinh \alpha x$
$\frac{2x^{\beta-1}\left(\theta^2-\alpha^2\right)^{1+\beta}(\alpha x\cosh[\alpha x]+\beta\sinh[\alpha x])e^{-\theta x}}{\theta\left((\theta+\alpha)^{1+\beta}-(\theta-\alpha)^{1+\beta}\right)\Gamma[\beta+1]}$	$\theta > \alpha > 0$, $\beta > 0$	$x^\beta \sinh[\alpha x]$
$\frac{2\alpha e^{-\theta x}\left(\theta^2-\alpha^2\right)^{1+\beta}\cosh[\alpha x]\sinh^{\beta-1}[\alpha x]}{\theta\Gamma[\beta]\left((\theta+\alpha)^{1+\beta}-(\theta-\alpha)^{1+\beta}\right)}$	$\theta > \alpha > 0$, $\beta > 0$	$(\sinh[\alpha x])^\beta$
$\frac{\theta^2-\alpha^2}{\alpha}\sinh[\alpha x]\, e^{-\theta x}$,	$\theta > \alpha \geq 0$	$\cosh[\alpha x] - 1$
$\frac{2\theta^\beta\left(\theta^2-\alpha^2\right)^{1+\beta} e^{-\theta x} x^{\beta-1}\{(\cosh[\alpha x]-1)\beta+\alpha x\sinh[\alpha x]\}}{\left\{(\theta(\theta+\alpha))^{\beta+1}+(\theta(\theta-\alpha))^{\beta+1}-2\left(\theta^2-\alpha^2\right)^{1+\beta}\right\}\Gamma[\beta+1]}$	$\theta > \alpha \geq 0$, $\beta > 0$	$x^\beta(\cosh[\alpha x] - 1)$
$\frac{\left(\theta^2-\alpha^2\right)e^{-\theta x}(\beta\cosh[\alpha x]+(1-\beta)\sinh[\alpha x])}{\beta\theta+(1-\beta)\alpha}$	$\theta > \alpha \geq 0$, $0 \leq \beta \leq 1$	$\beta\sinh[\alpha x] + (1-\beta)(\cosh[\alpha x] - 1)$
$\frac{\theta e^{-\theta(x-\gamma)}}{\left(1+e^{-\theta(x-\gamma)}\right)^2}$	Logistic, $\theta > 0, \gamma \in \mathbb{R}$	$\frac{e^{\theta\gamma}}{\theta}\left(\frac{1}{1+e^{\theta\gamma}} + \frac{1}{1+e^{\theta(x-\gamma)}} - 1\right) + \frac{e^{\theta\gamma}}{\theta}\left(\ln\left(e^{\theta x} + e^{\theta\gamma}\right) - \ln\left(1 + e^{\theta\gamma}\right)\right)$

error function: $\mathrm{erf}[w] = \frac{2}{\sqrt{\pi}}\int_0^w e^{-s^2} ds$

Figure 2 presents $f^{\mathcal{C}}(x, \lambda, \alpha, 1)$ for some values of λ and α. Also, Fig. 3 shows $f^{\mathcal{C}}(x, \lambda, \alpha, 0)$ for different λ and α.

The moment generating function of $X \sim GCE(\lambda, \alpha, \beta)$ is given by

$$
\begin{aligned}
\mathcal{M}_X(t) &= \int_0^\infty e^{tx} f^{\mathcal{C}}(x, \lambda, \alpha, \beta)\, dx \\
&= \frac{\left(\lambda^2 - \alpha^2\right)}{\beta\lambda + (1-\beta)\alpha}\Bigg((1-\beta)\int_0^\infty e^{(t-\lambda)x}\sinh[\alpha x]\, dx \\
&\qquad + \beta\int_0^\infty e^{(t-\lambda)x}\cosh[\alpha x]\, dx\Bigg) \\
&= \frac{\left(\lambda^2 - \alpha^2\right)[\alpha(\beta - 1) + \beta(t - \lambda)]}{\left(\alpha^2 - (\lambda - t)^2\right)(\beta\lambda + (1-\beta)\alpha)}, \quad t < \lambda - \alpha. \qquad (14)
\end{aligned}
$$

The expected value of $X \sim GCE(\lambda, \alpha, \beta)$ is given by

$$
E(X) = \frac{1}{\lambda - \alpha} + \frac{1}{\lambda + \alpha} - \frac{\beta}{\alpha - \alpha\beta + \beta\lambda}. \qquad (15)
$$

Also, $E\left(X^2\right)$ for $X \sim GCE(\alpha, \beta, \lambda)$ is given by

$$
E\left(X^2\right) = \frac{\left(\lambda^2 - \alpha^2\right)\left(\frac{1}{(\lambda-\alpha)^3} + \frac{2\beta-1}{(\lambda+\alpha)^3}\right)}{\alpha - \alpha\beta + \beta\lambda}. \qquad (16)
$$

Similarly, $E(X^r), r = 1, 2, 3, \cdots, n$, of $X \sim GCE(\alpha, \beta, \lambda)$ can be found as

$$
E\left(X^r\right) = \frac{\left(\lambda^2 - \alpha^2\right)\Gamma(r + 1)}{2(\alpha - \alpha\beta + \beta\lambda)}
$$

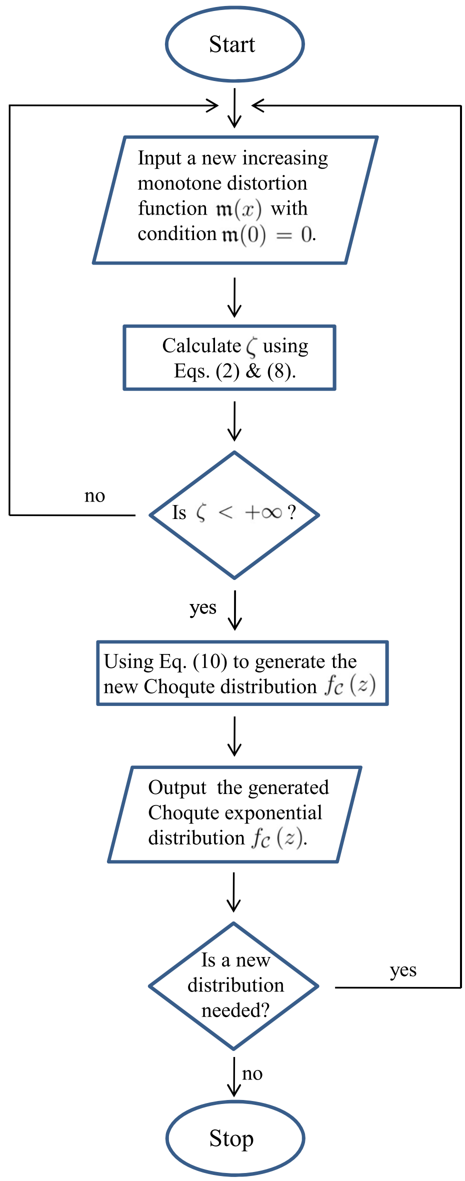

Fig. 1 The diagram of generating the class of Choquet distributions

$$\left\{\beta\left[(\lambda-\alpha)^{-r-1}+(\lambda+\alpha)^{-r-1}\right] + (1-\beta)\left(\lambda^2-\alpha^2\right)^{-r-1} \left[(\lambda+\alpha)^{r+1}-(\lambda-\alpha)^{r+1}\right]\right\}. \tag{17}$$

By using Eqs. (15-16), we have

$$\mathrm{Var}\,(X) = \frac{\left(\lambda^2-\alpha^2\right)\left(\frac{1}{(\lambda-\alpha)^3}+\frac{2\beta-1}{(\lambda+\alpha)^3}\right)}{\alpha-\alpha\beta+\beta\lambda} - \left(\frac{1}{\lambda-\alpha}+\frac{1}{\lambda+\alpha}-\frac{\beta}{\alpha-\alpha\beta+\beta\lambda}\right)^2$$

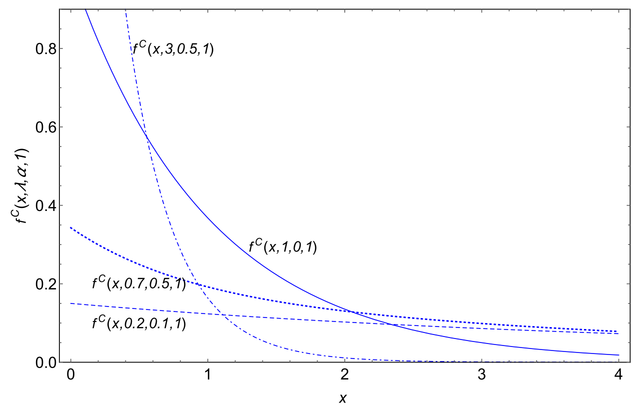

Fig. 2 Plots of the probability density function $f^C(x, \lambda, \alpha, 1)$ for different values of λ and α. The full curve presents $f^C(x, 1, 0, 1)$, the dotted curve presents $f^C(x, 0.7, 0.5, 1)$, the dashed curve presents $f^C(x, 0.2, 0.1, 1)$, and the dot-dashed curve presents $f^C(x, 3, 0.5, 1)$. The relating Choquet exponential probability distribution function $f^C(x, \lambda, \alpha, 1)$ is given by Eq. (13)

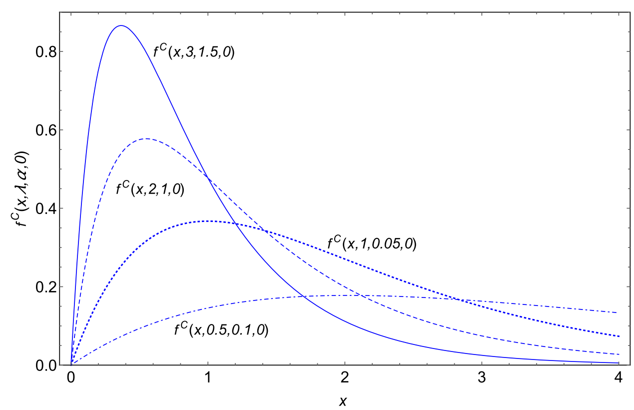

Fig. 3 Plots of the probability density function $f^C(x, \lambda, \alpha, 0)$ for different values of λ and α. The full curve presents $f^C(x, 3, 1.5, 0)$, the dashed curve presents $f^C(x, 2, 1, 0)$, the dotted curve presents $f^C(x, 1, 0.05, 0)$, and the dot-dashed curve presents $f^C(x, 0.5, 0.1, 0)$. The relating Choquet exponential probability distribution function $f^C(x, \lambda, \alpha, \beta)$ is given by Eq. (12)

$$= \frac{1}{(\lambda-\alpha)^2} + \frac{1}{(\lambda+\alpha)^2} - \frac{\beta^2}{(\alpha-\alpha\beta+\beta\lambda)^2}. \tag{18}$$

3.2 Properties of the proposed method and comparison with previous works.

Let us recall the advantages of our proposed new generalized exponential distribution. Firstly, it is an interesting application Choquet integral to generate a large class of distributions (a zoo of distributions!). Secondly, generating new statistical distributions is easy with this method (see Fig. 1 and Table 1). Thirdly, this class of distributions includes lots of famous statistical distributions such as gamma, logistic, exponential, Rayleigh and other distributions, see Table 1. Fourthly, many

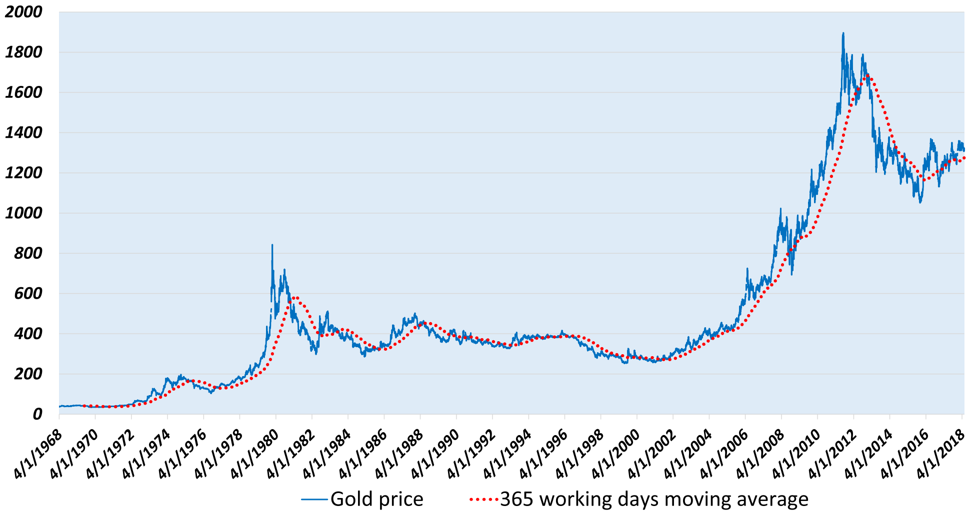

Fig. 4 The LBM gold daily price in USD per troy ounce (full curve) and its last 365 working days moving average (dotted curve), in the period of 1968/4/01 to 2018/05/15. To obtain 365 working days moving average of the gold price, one can imply $m = 365$ in Eq. (21) for the gold daily price dataset

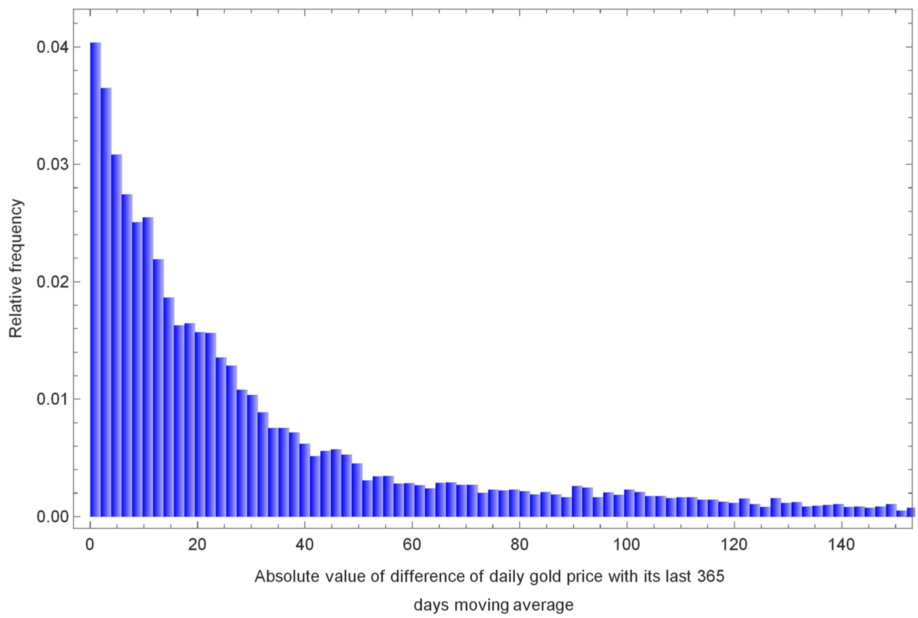

Fig. 5 Histogram for the absolute value of the difference of daily LBM gold price in USD per troy ounce with its last 365 working days moving average for period 1968/4/01 to 2018/05/15. The 365 working days moving average can be obtained by implying $m = 365$ in Eq. (21) for the gold daily price dataset

Table 2 Summary statistics from the absolute value of the difference of LBM gold daily price with its last 365 working days moving average in the period of 1968/4/01 to 2018/05/15

n	Minimum	Maximum	Mean	Median	Standard deviation
12309	0.000410959	547.4185	54.58823	29.84452	67.15968

of produced distributions by our method could be strong tools for studying real data sets. Section 4 contains two examples of applications of these new distributions. In this work, we used the exponential distribution to Eqs. (7-9) to find the generalized class Choquet exponential distribution.

Another advantage of our method is the possibility of replacing exponential distribution with other distributions to find generalized Choquet distributions. It should be mentioned that calculating the Choquet integral of non-monotone distributions is usually a very hard obstacle (Torra and Narukawa 2016). Then, a disadvantage of our method is that it cannot generalize to non-monotone initial distributions easily.

Reference Mehri-Dehnavi et al. (2019) has implied the nonlinear integrals to generate the generalized form of non-monotone distributions by using a new nonlinear integral which can be evaluated easier than Choquet integral and can lead to a generalized class of new distributions. It should be mentioned that the current method is a much stronger tool for creating a wider class of new distributions (see Fig. 1 and Table 1).

4 Application to some real datasets

4.1 Model selection using information criteria

Model selection helps us to select a suitable model from the list of our models and to find the best model to the data (Wasserman 2000). Two important tools in the model selection are the Akaike information criteria (AIC (Akaike 1973)) and the Bayesian information criteria (BIC (Schwarz 1978)):

$$\text{AIC} := 2\kappa - 2\ln\ \mathcal{L}, \tag{19}$$

$$\text{BIC} := -2\ln\ \mathcal{L} + \kappa\ln n, \tag{20}$$

which $\mathcal{L}$ is the maximum value of the likelihood function, κ is the number of parameters, and n is the total number of data.

4.2 Application to LBM gold price data

In this subsection, we work on the London Bullion Market (LBM) gold daily price in USD per troy ounce in period of 1968/4/01 to 2018/05/15. The gold data are available at https://fred.stlouisfed.org Figure 4 shows the LBM gold daily price (full curve) and its last 365 working days moving average (dotted curve), for all working days, in period of 1968/4/01 to 2018/05/15.

Definition 6 For a daily time series $\{a_i,\ i = 1, 2, \ldots, n\}$, the m-days moving average time series $\{MA_t,\ t = m, m+1, \ldots, n\}$ is defined as

$$MA_t = \frac{1}{m}\sum_{j=t-m+1}^{t} a_j. \tag{21}$$

Figure 5 shows the histogram for the absolute value of the difference of daily LBM gold price with its last 365 working days moving average in 1968/4/01 to 2018/05/15. The summary of Fig. 5 related to the LBM gold price data is given in Table 2.

The maximum likelihood estimation (MLE) of the parameters λ and α for $GCE(\lambda, \alpha)$ is obtained by

$$\ln\ \mathcal{L}(\lambda, \alpha) = \sum\nolimits_{i=1}^{n}\left[\ln\left(\lambda^2 - \alpha^2\right) - \lambda x_i + \ln\left(\cosh\left[\alpha x_i\right]\right) - \ln(\lambda)\right]. \tag{22}$$

The above relation is solved with the package "nlminb" in the software R, numerically.

Table 3 Some distributions on absolute difference of LBM gold daily price with its last 365 working days moving and their comparison by AIC and BIC. The first column lists the name of the candidate distributions and the (their) best-fitting parameter(s) leading to the maximum likelihood of the data and the proposed distribution

Distribution	$-\log\mathcal{L}$	AIC	BIC
Exponential $\lambda = 0.01831897$	61542.76	123087.5	123094.9
Our proposed $GCE(\lambda, \alpha)$ $\lambda = 0.03144573$, $\alpha = 0.01615027$	**61244.94**	**122493.9**	**122509**
Weibull $\alpha = 0.8569414$, $\beta = 50.1980887$	61269.1	122542.2	122557
Normal $\mu = 54.58823$, $\sigma = 67.15695$	69250.08	138504.2	138519
Gamma $\alpha = 0.80939385$, $\beta = 0.01482732$	61346.12	122696.2	122711.1
Log normal $\mu = 3.267278$, $\sigma = 1.396034$	61789.35	123582.7	123597.5
Logistic $\mu = 41.60666$, $s = 30.88459$	67662.49	135329	135343.8

The obtained values in Table 3 show that the $GCE(\lambda = 0.03144573, \alpha = 0.01615027)$ is better for the daily LBM gold price data in 1968/4/01 to 2018/05/15.

4.3 Application to coronavirus data in the USA

To illustrate the applicability of the proposed distributions, we recall another data set of the daily number of newly infected people to coronavirus in the USA in period of 2020/02/29 to 2020/10/19 [21]. Figure 6 shows the the daily number of newly infected cases in the USA (full curve) and its last 7 days moving average (dotted curve) in the period of 2020/02/29 to 2020/10/19.

Now, we work on the difference of the daily number of newly infected people with its last 7 days moving average. The summary of Fig. 7 related to the absolute value of the daily number of new infected people in the USA with its last 7 days moving average in the period of 2020/02/29 to 2020/10/19 is given in Table 4.

Due to AIC and BIC, Table 5 indicates that the proposed distribution $GCE(\lambda, \alpha)$ with $\lambda = 0.009742356$, $\alpha = 0.009509922$ performs better than other distributions to describe the difference of the daily number of newly infected people with coronavirus with its last 7 days moving average in the USA.

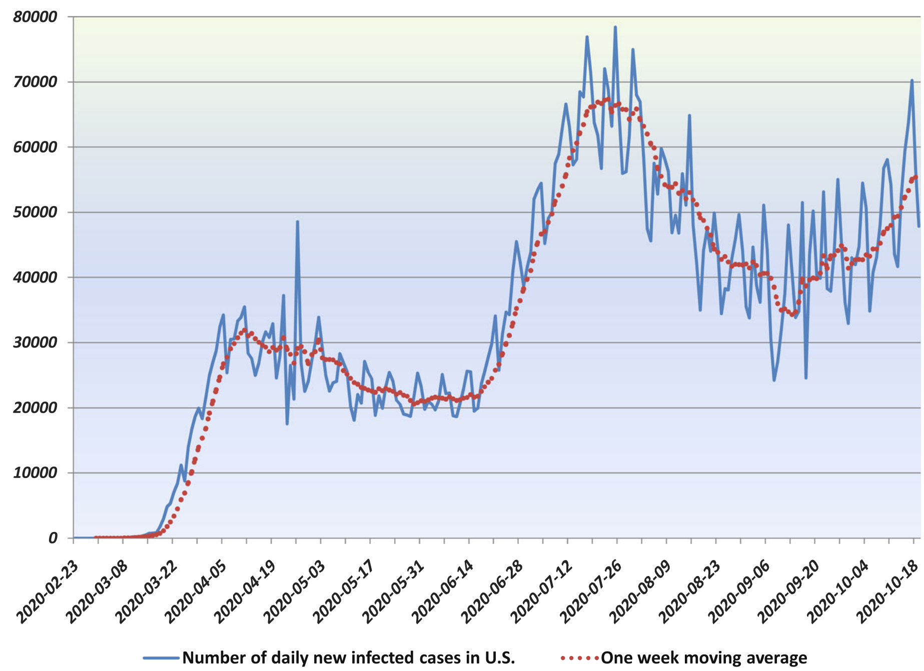

Fig. 6 The daily number of newly infected cases in the USA (full curve) and its last 7 days moving average (dotted curve) in the period of 2020/02/29 to 2020/10/19. The 7 days moving average of this daily dataset is obtained by implying $m = 7$ in Eq. (21) for this dataset

Table 4 Summary statistics of the difference of the daily number of new infected people in the USA with its last 7 days moving average in the period of 2020/02/29 to 2020/10/19

n	Minimum	Median	Mean	Maximum	Standard deviation
234	1.571429	3050.857	4251.589	19390.857	3691.711

Table 5 Modeling of the difference of daily number of newly infected people in the USA with its last 7 days moving average, in the period of 2020/02/27 to 2020/07/09. The first column lists the name of the candidate distributions and the (their) best-fitting parameter(s) leading to the maximum likelihood of the data and the proposed distribution

Distribution	$-\log \mathcal{L}$	AIC	BIC
Exponential $\lambda = 0.0002352062$	2189.081	4380.162	4383.617
Our proposed $GCE(\lambda, \alpha)$ $\lambda = 0.009742356$, $\alpha = 0.009509922$	**2186.453**	**4376.906**	**4383.817**
Weibull $\alpha = 1.02399$, $\beta = 4288.53797$	2188.985	4381.97	4388.881
Normal $\mu = 4251.589$, $\sigma = 3683.815$	2253.57	4511.14	4518.051
Log normal $\mu = 7.750885$, $\sigma = 1.511089$	2242.341	4488.682	4495.593
Logistic $\mu = 3771.262$, $s = 2014.330$	2249.276	4502.552	4509.463
Gamma $\alpha = 0.9600334204$, $\lambda = 0.0002258058$	2188.953	4381.906	4388.817

5 Conclusion

In this paper, we have introduced the Choquet exponential distribution. We have also obtained some statistical attributes of a new class of Choquet exponential distribution $GCE(\lambda, \alpha)$. Furthermore, the LBM gold daily price in the period of 1968/4/01 to 2018/05/15 has been analyzed. For this dataset, it has been obtained that the $GCE(\lambda, \alpha)$ distribution is better for the daily LBM gold price data. Our example illustrates the potential of newly introduced Choquet exponential distributions, enlarging the buffer of distribution functions appropriate for fitting probabilistic models to data with nearly exponential behavior, covered till now by exponential, gamma, Raleigh, logistic and similar distributions. Also, our results on a real dataset of the daily number of newly infected people to coronavirus in the USA in the period of 2020/02/29 to 2020/10/19 have been analyzed. In fact, in this paper, we have generated the class of Choquet exponential distributions by using the Choquet integral and the exponential distribution. This class is applicable to study of the positive real data sets. The following are the advantages of our method:

- Some statistical properties of the Choquet integral are discussed. As a special case of Choquet statistical distribu-

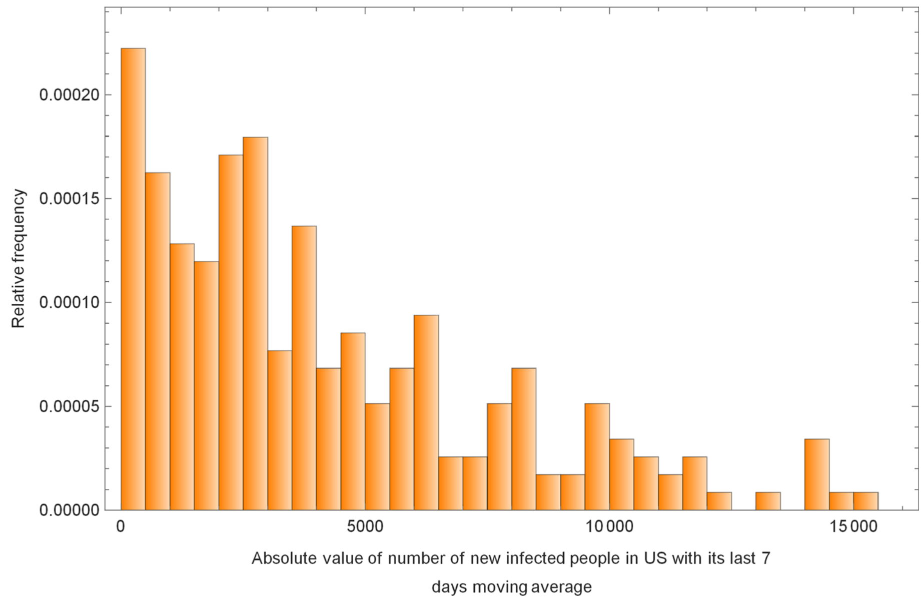

Fig. 7 Histogram for the absolute value of the daily number of new infected people in the USA with its last 7 days moving average in the period of 2020/02/29 to 2020/10/19. The 7 days moving average of the daily number of new infected people is obtained by implying $m = 7$ in Eq. (21) for this dataset

tions, we show that some famous statistical distributions such as gamma, logistic, exponential, Rayleigh and other distributions are a special class of Choquet distributions (see Table 1).

- Our results are applicable in mathematical finance, physics, statistical physics and econophysics (see Sects. 4 and 5).
- Our results are an interesting application of Choquet integral and monotone set functions.

As an interesting subject for further research, we can extend the application of our method into other statistical distributions using Choquet integral to obtain the other classes of Choquet distributions that could be useful to study different types of real datasets.

Author Contributions HM-D contributed to conceptualization, methodology, the investigation, formal analysis and writing—review and editing, and provided software. HA was involved in conceptualization, methodology, investigation, formal analysis and writing—review and editing. RM contributed to conceptualization and writing—review and editing.

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Data availability There is not any data availability for paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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