Crank–Nicolson method for solving uncertain heat equation

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Abstract

For usual uncertain heat equations, it is challenging to acquire their analytic solutions. A forward difference Euler method has been used to compute the uncertain heat equations' numerical solutions. Nevertheless, the Euler scheme is instability in some cases. This paper proposes an implicit task to overcome this disadvantage, namely the Crank–Nicolson method, which is unconditional stability. An example shows that the Crank–Nicolson scheme is more stable than the previous scheme (Euler scheme). Moreover, the Crank–Nicolson method is also applied to compute two characteristics of uncertain heat equation's solution—expected value and extreme value. Some examples of uncertain heat equations are designed to show the availability of the Crank–Nicolson method.

Keywords Crank–Nicolson method · Liu process · Numerical solution · Heat equation

1 Introduction

For dealing with the disturbance or white noise of a dynamic system, in theory, there are two mathematical tools: the differential of Winner process based on probability theory and the differential of Liu process ground on uncertainty theory. The essential difference between these two processes is that the former is a continuous and not Lipschitz continuous, whose increments are normal random variables. By comparison, Liu process is a Lipschitz continuous, whose increments are normal uncertain variables.

If we use the Liu process to model the disturbance in a differential system, then the differential equation evolves into an uncertain differential equation (UDE) (Liu 2008). Nowadays, UDE has been researched by many scholars and has been extensively applied in many areas like finance (Chen and Gao 2013, 2018; Yang and Zhu 2021), dynamic games (Yang and Gao 2016; Zhang et al. 2021) and COVID-19 spread (Lio 2021). In the field of a numerical method for UDE, many ways have been produced, such as the Milne

scheme in Gao (2016), Adams–Simpson scheme in Wang et al. (2015), Runge–Kutta scheme in Yang and Shen (2015), Adams scheme in Yang et al. (2015), Euler scheme in Yao and Chen (2013), and Hamming scheme in Zhang et al. (2017). For more other numerical studies, please consult (Kaya et al. 2020; Yokuş 2018; Yokuş and Yavuz 2021). Later, the thought to explain the disturbance with Liu process is spread in the other systems, such as higher-order UDEs in Jia et al. (2018) and Yao (2016), multifactor UDEs in Li et al. (2015), uncertain fractional differential equations in Zhu (2015), UDE with jumps in Yao (2012), uncertain partial differential equations in Yang and Yao (2017), and uncertain delay differential equations in Barbacioru (2010) and Ge and Zhu (2012).

Uncertain heat equation (UHE) pioneered by Yang and Yao (2017) represents the change of temperature with uncertain disturbance in a given area across time, which is a variety of uncertain partial differential equations. Yang and Yao (2017) supplied the analytic solution for a special type UHE, whose heat source and its diffusion term are functions to time and space but do not depend on its temperature. After that, under the linear growth and Lipschitz conditions, Yang and Ni (2017) showed an existence and uniqueness of the common UHE's solution; Yang (2018) declared that the UHE's solution could be demonstrated by a group of the ordinary heat equation's solutions via β -path way, and investigated a formula to calculate one characteristic of UHE's solution—expected value; Yang and Ni (2019) stud-

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ied another characteristic of UHE's solution—extreme value; besides, Yang (2019) introduced stability thought of UHE, called stability in measure.

It is needed to design a numerical approach to solve UHE since it is challenging to gain the analytic solution for most UHEs. By β -path of UHE, Yang (2018) first explored a forward difference Euler method to calculate the solution for a UHE. Nevertheless, this forward difference Euler scheme is conditional stability. We will propose a new numerical scheme named the Crank–Nicolson method to overcome the forward difference Euler scheme's instability limitation. The remainder of this paper is organized as below. Section 2 retrospects some fundamental notions and results in UHE. Section 3 proposes the Crank–Nicolson scheme to compute the UHE's solution. Section 4 designs two algorithms for obtaining two characteristics of the UHE's solution—expected value and extreme value. Lastly, Sect. 5 gives a brief conclusion.

2 Preliminaries

UHE, as a class of uncertain partial differential equations, is investigated by Yang and Yao (2017) to represent the change of temperature with white noise (described by the differential of Liu process) in a region $x \in R$ through time $t \geq 0$. A one-dimension UHE defined on a domain $x \times t = R \times [0, +\infty)$ is

$$\begin{cases} \frac{\partial H_{x,t}}{\partial t} - a^2 \frac{\partial^2 H_{x,t}}{\partial x^2} = p(x, t) + q(x, t) \frac{dC_t}{dt} \\ H_{x,0} = \mu(x) \end{cases} \quad (1)$$

where $H_{x,t}$ represents the temperature at node (x, t) , a^2 is the thermal diffusivity constant, $p(x, t)$ represents the heat source, $q(x, t)$ represents the diffusion term, C_t is a Liu process defined on a given uncertainty space, the differential dC_t/dt means the white noise with time t described by a Liu process, and $\mu(x)$ is an initial temperature at initial time $t = 0$. Yang and Yao (2017) gave that Eq. (1)'s solution is

$$\begin{aligned} H_{x,t} = & \int_{-\infty}^{+\infty} D(x-z, t) \mu(z) dz \\ & + \int_0^t \int_{-\infty}^{+\infty} D(x-z, t-s) p(z, s) dz ds \\ & + \int_0^t \int_{-\infty}^{+\infty} D(x-z, t-s) q(z, s) dz dC_s \end{aligned} \quad (2)$$

where

$$D(x, t) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right).$$

A general UHE is formulated as

$$\begin{cases} \frac{\partial H_{x,t}}{\partial t} - a^2 \frac{\partial^2 H_{x,t}}{\partial x^2} = p(x, t, H_{x,t}) + q(x, t, H_{x,t}) \frac{dC_t}{dt} \\ H_{x,0} = \mu(x). \end{cases} \quad (3)$$

If $\mu(x)$ is a bounded function, Yang and Ni (2017) declared the existence and uniqueness of Eq. (3)'s solution under linear growth and Lipschitz conditions.

Definition 1 (Yang 2018) Let $\beta \in (0, 1)$. Equation (3) has a β -path $H_{x,t}^\beta$ if it solves the corresponding ordinary heat equation

$$\begin{aligned} \frac{\partial H_{x,t}^\beta}{\partial t} - a^2 \frac{\partial^2 H_{x,t}^\beta}{\partial x^2} = & p(x, t, H_{x,t}^\beta) \\ & + \left| q(x, t, H_{x,t}^\beta) \right| \frac{\sqrt{3}}{\pi} \ln \frac{\beta}{1-\beta}. \end{aligned}$$

Theorem 1 (Yang 2018) If $H_{x,t}$ and $H_{x,t}^\beta$ are the unique solution and β -path of Eq. (3), respectively, then

$$\begin{aligned} \mathcal{M}\{H_{x,t} \leq H_{x,t}^\beta, \text{ for all } x \text{ and } t\} &= \beta, \\ \mathcal{M}\{H_{x,t} > H_{x,t}^\beta, \text{ for all } x \text{ and } t\} &= 1 - \beta. \end{aligned}$$

Yang (2018) proved that the β -path of Eq. (3) is the solution $H_{x,t}$'s inverse uncertainty distribution, that is,

$$\Upsilon_{x,t}^{-1}(\beta) = H_{x,t}^\beta.$$

For the common UHE (3), it isn't easy to know the analytic solution in most situations. Therefore, Yang (2018) created the forward difference Euler method to obtain β -path of Eq. (3). Nevertheless, the forward difference Euler method is unstable in some circumstances. This paper will suggest a new numerical scheme, namely the Crank–Nicolson method, which is unconditional stability.

3 Crank–Nicolson scheme for UHE

We first review the Crank–Nicolson scheme for ordinary heat equation, and then gives the corresponding Crank–Nicolson scheme for Eq. (3). Considering an initial value problem of an ordinary heat equation

$$\begin{cases} \frac{\partial H_{x,t}}{\partial t} - a^2 \frac{\partial^2 H_{x,t}}{\partial x^2} = p(x, t, H_{x,t}) \\ H_{x,0} = \mu(x). \end{cases} \quad (4)$$

Let space step be l , time step be τ , $x_j = jl$ ($j = 0, \pm 1, \pm 2, \dots$) and $t_i = i\tau$ ($i = 1, 2, \dots$), and $H_{j,i}$ and $p_{j,i}$ be represent the values of H_{x_j, t_i} and $p(x_j, t_i, H_{x_j, t_i})$ on

the node (x_j, t_i) , respectively. A forward difference Euler scheme is as follows,

$$\begin{cases} \frac{H_{j,i+1} - H_{j,i}}{\tau} - a^2 \frac{H_{j+1,i} + H_{j-1,i} - 2H_{j,i}}{l^2} = p_{j,i} \\ H_{j,0} = \mu(x_j), \end{cases} \quad (5)$$

that is,

$$H_{j,i+1} = (1 - 2r)H_{j,i} + rH_{j+1,i} + rH_{j-1,i} + \tau p_{j,i} \quad (6)$$

where $r = \tau a^2 / l^2$. Besides, for the forward difference (6), its truncation error and the stability condition are $O(\tau + l^2)$ and $r \leq 1/2$, respectively. The backward difference is as follows,

$$\begin{cases} \frac{H_{j,i} - H_{j,i-1}}{\tau} - a^2 \frac{H_{j+1,i} + H_{j-1,i} - 2H_{j,i}}{l^2} = p_{j,i} \\ H_{j,0} = \mu(x_j), \end{cases} \quad (7)$$

that is,

$$(1 + 2r)H_{j,i} - rH_{j+1,i} - rH_{j-1,i} - \tau p_{j,i} = H_{j,i-1}, \quad (8)$$

and the truncation error of (8) is $O(\tau + l^2)$, and it is unconditional stability.

We rewrite Eq. (5) as

$$\begin{cases} \frac{H_{j,i} - H_{j,i-1}}{\tau} - a^2 \frac{H_{j+1,i-1} + H_{j-1,i-1} - 2H_{j,i-1}}{l^2} = p_{j,i-1} \\ H_{j,0} = \mu(x_j), \end{cases} \quad (9)$$

Taking a convex combination with Eqs. (7) and (9) with coefficient $\theta \in [0, 1]$ and $1 - \theta$, respectively, we have

$$\begin{cases} \frac{H_{j,i} - H_{j,i-1}}{\tau} - a^2 \left[\theta \frac{H_{j+1,i} + H_{j-1,i} - 2H_{j,i}}{l^2} + (1 - \theta) \frac{H_{j+1,i-1} + H_{j-1,i-1} - 2H_{j,i-1}}{l^2} \right] \\ \quad = (1 - \theta)p_{j,i-1} + \theta p_{j,i} \\ H_{j,0} = \mu(x_j), \end{cases} \quad (10)$$

that is,

$$\begin{aligned} (1 + 2r\theta)H_{j,i} - r\theta H_{j+1,i} - r\theta H_{j-1,i} - \tau\theta p_{j,i} \\ = r(1 - \theta)H_{j+1,i-1} + [1 - 2r(1 - \theta)]H_{j,i-1} \\ + \tau(1 - \theta)p_{j,i-1} + r(1 - \theta)H_{j-1,i-1}, \end{aligned} \quad (11)$$

when $\theta \neq 1/2$, (11)'s truncation error is $O(l^2 + \tau)$, when $\theta = 1/2$, (11)'s truncation error is $O(\tau^2 + l^2)$. We note that the recurrence formula (11) is the backward difference when $\theta = 1$, and is the forward difference when $\theta = 0$.

When $\theta = 1/2$, the recurrence formula (11) is called Crank–Nicolson method as follows,

$$\begin{aligned} -rH_{j+1,i} + 2(1 + r)H_{j,i} - rH_{j-1,i} - \tau p_{j,i} \\ = 2(1 - r)H_{j,i-1} + rH_{j+1,i-1} + rH_{j-1,i-1} + \tau p_{j,i-1}. \end{aligned} \quad (12)$$

Note that the recurrence formula (11) is an implicit method. For each $i - 1$, if we want to get the “next” values of $H_{j,i}$ in time t_i for all x_j , a system of the algebraic equation needs to be resolved. Furthermore, the recurrence formula (11) is unconditional stability when $\theta \in [1/2, 1]$, and when $\theta \in [0, 1/2]$, its stability condition is

$$2r \leq \frac{1}{1 - 2\theta}.$$

In other words, Crank–Nicolson method (12) is unconditional stability.

Following, we produce the Crank–Nicolson approach to solve Eq. (3). If $p(\cdot)$ and $q(\cdot)$ meet the linear growth and Lipschitz conditions, and the initial value $\mu(x)$ is bounded, those hypotheses mean that the solution of Eq. (3) is unique. Then, the β -path of Eq. (3) is a solution of the below equation

$$\begin{cases} \frac{\partial H_{x,t}^\beta}{\partial t} - a^2 \frac{\partial^2 H_{x,t}^\beta}{\partial x^2} = p(x, t, H_{x,t}^\beta) \\ \quad + |q(x, t, H_{x,t}^\beta)| \Upsilon^{-1}(\beta) \\ H_{x,0}^\beta = \mu(x) \end{cases} \quad (13)$$

where

$$\Upsilon^{-1}(\beta) = \frac{\sqrt{3}}{\pi} \ln \frac{\beta}{1 - \beta}.$$

For the sake of convenience, we write

$$h(x, t, H_{x,t}^\beta) = p(x, t, H_{x,t}^\beta) + |q(x, t, H_{x,t}^\beta)| \Upsilon^{-1}(\beta).$$

Then, Eq. (13) can be reduced to

$$\begin{cases} \frac{\partial H_{x,t}^\beta}{\partial t} - a^2 \frac{\partial^2 H_{x,t}^\beta}{\partial x^2} = h(x, t, H_{x,t}^\beta) \\ H_{x,0}^\beta = \mu(x). \end{cases} \quad (14)$$

To obtain the inverse uncertainty distribution or β -path of UHE's solution, we design the Crank–Nicolson algorithm as below.

Step 0 Domain $O = \{(x, t) | 0 \leq t \leq T, 0 \leq x \leq L\}$.

Step 1 Give two positive integers n and m , let the space step be $l = L/n$ and the time step be $\tau = T/m$. Write $x_j := jl$ ($j = 0, 1, \dots, n$), $t_i := i\tau$ ($i = 0, 1, \dots, m$), $H_{ji}^\beta := H_{x_j, t_i}^\beta$, and $h_{ji}^\beta := h(x_j, t_i, H_{ji}^\beta)$.

Fig. 1 Node diagram

Step 2 For a $\beta \in (0, 1)$.

Step 3 Compute $H_{j,i+1}^\beta$ ($j = 0, 1, \dots, n$, $i = 0, 1, \dots, m-1$) via

$$\begin{cases} H_{0,i+1}^\beta = (1-r)H_{0,i}^\beta + rH_{1,i}^\beta + \tau h_{0,i}^\beta \\ -rH_{j+1,i+1}^\beta + 2(1+r)H_{j,i+1}^\beta - \tau h_{j,i+1}^\beta - rH_{j-1,i+1}^\beta \\ = rH_{j+1,i}^\beta + 2(1-r)H_{j,i}^\beta + rH_{j-1,i}^\beta + \tau h_{j,i}^\beta, \\ j = 1, 2, \dots, n-1 \\ H_{n,i+1}^\beta = (1-r)H_{n,i}^\beta + rH_{n-1,i}^\beta + \tau h_{n,i}^\beta \end{cases}$$

where $r = a^2\tau/l^2$, $H_{j,0}^\beta = \mu(x_j)$ (Fig. 1 shows the node diagram of the Crank–Nicolson algorithm).

Step 4 For different β , repeat Steps 2 and 3.

Step 5 The β -path $H_{x,t}^\beta$ is obtained.

For each i , if we want to get the “next” values of $H_{j,i+1}$ at time t_{i+1} for all x_j , a system of $n-1$ -dimensional algebraic equation must be solved. In others words, the recursion formula in Step 3 is equal to solve an $n-1$ -dimensional algebraic equation, and the coefficient matrix of this algebraic equation is tridiagonal.

Example 1 We consider an UHE with $p(x, t, H_{x,t}) = 0$, $q(x, t, H_{x,t}) = -e^{-t}$, and $\mu(x) = 0$. Reference Yang and Yao (2017) got its analytic solution

$$H_{x,t} = - \int_0^t e^{-s} ds,$$

whose inverse uncertainty distribution or β -path is

$$H_{x,t}^\beta = \frac{\sqrt{3}}{\pi} (1 - e^{-t}) \ln \frac{\beta}{1-\beta}.$$

Firstly, we use the Crank–Nicolson method to get the numerical solution and compare it with the analytical solu-

(a) β -path

(b) Error surface

Fig. 2 Numerical solution with Crank–Nicolson method

tion. Set $O = \{[0, 1] \times [0, 1]\}$ and

$$\beta = 0.6, m = 150, n = 10.$$

Figure 2a shows the β -path via the Crank–Nicolson method; Fig. 2b gives the error surface, which shows this numerical solution is pretty close to the analytical solution. Secondly, summing the absolute difference between this numerical solution and analytical solution on each point (x_j, t_i) , we get that the Crank–Nicolson scheme’s error is 0.0836, the Euler scheme’s error is $8.9007 \times 10^{+12}$. Besides, Table 1 shows the results according to the different parameters, we can notice that the Euler scheme is instability in some cases, and the Crank–Nicolson method is stable in the given cases. The reason is that the forward difference’s stability condition is $r \leq 1/2$, and the Crank–Nicolson scheme is unconditional stability.

Example 2 We consider an UHE with $p(x, t, H_{x,t}) = e^{-t} \sin x$, $q(x, t, H_{x,t}) = 1 + \cos x$, and $\mu(x) = \sin x$. Let

Table 1 Error analysis in Example 1

L	T	β	m	n	r	Error	
						Forward difference	Crank–Nicolson
1	1	0.1	800	10	1/8	2.4542	0.4470
1	1	0.2	600	10	1/6	1.5491	0.2823
1	1	0.3	400	10	1/4	0.9476	0.1729
1	1	0.4	200	10	1/2	0.4546	0.0833
1	1	0.6	150	10	2/3	$8.9007 * 10^{+12}$	0.0836
1	1	0.7	125	10	4/5	$2.0497 * 10^{+22}$	0.1753
1	1	0.8	100	10	1	$1.7211 * 10^{+27}$	0.2882
1	1	0.9	80	10	5/4	$3.4490 * 10^{+28}$	0.4595

$O = \{(x, t) | 0 \leq t \leq 1, 0 \leq x \leq 10\}$, and

$m = 200, n = 200$.

By Crank–Nicolson method, we have the β -path or inverse uncertainty distribution of numerical solution. Figure 3 shows the 0.7-path and 0.3-path of numerical solution in Example 2.

Example 3 We consider an UHE with $p(x, t, H_{x,t}) = 2 \sin(3x)$, $q(x, t, H_{x,t}) = e^{-t} \sin x$, and $\mu(x) = e^x \sin x$. Let $O = \{(x, t) | 0 \leq x \leq 1, 0 \leq t \leq 1\}$, and

$m = 200, n = 10$.

By Crank–Nicolson method, we obtain the β -path of this numerical solution. Figure 4 shows the 0.7-path and 0.3-path of numerical solution in Example 3.

4 Expected value and extreme value

In this section, we will use the Crank–Nicolson approach to obtain two characteristics of solution in Eq. (3)—the expected value and the extreme value, and designs the corresponding two algorithms.

4.1 Crank–Nicolson method for expected value

Let the function $J(\cdot)$ be monotone and $H_{x,t}$ be a solution of (3). This subsection provides an approach to calculate $E[J(H_{x,T})]$ on a terminal time T .

Step 0 Give an iteration number M , a domain $O = \{0 \leq x \leq L, 0 \leq t \leq T\}$, two positive integers n and m . Set $\tau = T/m$, $l = L/n$, $\beta = 0$, and $k = 0$.

Step 1 Set $k \leftarrow k + 1$ and $\beta = k/M$.

Step 2 Compute $H_{j,i+1}^\beta$ ($i = 0, 1, \dots, m-1, j = 0, 1, \dots, n$) by

(a) $\beta = 0.7$

(b) $\beta = 0.3$

Fig. 3 β -path in Example 2

$$\begin{cases} H_{0,i+1}^\beta = (1-r)H_{0,i}^\beta + rH_{1,i}^\beta + \tau h_{0,i}^\beta \\ -rH_{j+1,i+1}^\beta + 2(1+r)H_{j,i+1}^\beta - \tau h_{j,i+1}^\beta - rH_{j-1,i+1}^\beta \\ = rH_{j+1,i}^\beta + 2(1-r)H_{j,i}^\beta + rH_{j-1,i}^\beta + \tau h_{j,i}^\beta, \\ j = 1, 2, \dots, n-1 \\ H_{n,i+1}^\beta = rH_{n-1,i}^\beta + (1-r)H_{n,i}^\beta + \tau h_{n,i}^\beta \end{cases}$$

(a) $\beta = 0.7$ (b) $\beta = 0.3$ Fig. 4 β -path in Example 3

where $r = a^2\tau/l^2$, $H_{j,0}^\beta = \mu(x_j)$.

Step 3 Obtain $H_{x_j,T}^\beta$ ($j = 0, 1, \dots, n$). When $k < M$, then go back to Step 1.

Step 4 Get

$$E[J(H_{x_j,T})] = \frac{1}{M-1} \sum_{k=1}^{M-1} J(H_{x_j,T}^{k/M})$$

at time T and point x_j ($j = 0, 1, \dots, n$).

Example 4 We reconsider the UHE in Example 1 with $O = \{x \in [0, 20], t \in [0, 1]\}$, and compute $E[H_{x,t}]$. Let the parameters be

$$m = 200, n = 200.$$

By the algorithm in Subsection 4.1, we have the expected value $E[H_{x,1}]$, whose curve is shown in Fig. 5.

Fig. 5 Expected value $E[H_{x,1}]$ in Example 4 Fig. 6 Expected value $E[H_{x,1}]$ in Example 5

Example 5 We reconsider the UHE in Example 2 with $O = \{x \in [0, 20], t \in [0, 1]\}$, and compute $E[H_{x,t}]$. Let the parameters be

$$m = 200, n = 200.$$

By the algorithm in Subsection 4.1, we have the expected value $E[H_{x,1}]$, whose curve is presented in Fig. 6.

Example 6 We reconsider the UHE on Example 3 with $O = \{x \in [0, 2], t \in [0, 2]\}$, and compute $E[H_{x,t}]$. Let the parameters be

$$m = 400, n = 20.$$

By the algorithm in Subsection 4.1, we have the expected value $E[H_{x,2}]$, whose curve is shown in Fig. 7.

Fig. 7 Expected value $E[H_{x,2}]$ in Example 6

4.2 Crank–Nicolson method for extreme value

Let $J_1(\cdot)$ or $J_2(\cdot)$ be a strictly increasing or decreasing function and $H_{x,t}$ be a UHE's solution under a domain $O = \{x \in [0, L], t \in [0, T]\}$. This subsection proposes the method to obtain the following four extreme values' inverse uncertainty distributions

$$F_1 = \sup_{(x,t) \in O} J_1(H_{x,t}),$$

$$G_1 = \inf_{(x,t) \in O} J_1(H_{x,t}),$$

$$F_2 = \sup_{(x,t) \in O} J_2(H_{x,t}),$$

and

$$G_2 = \inf_{(x,t) \in O} J_2(H_{x,t}).$$

Step 1 Give two positive integers n and m , let $l = L/n$ and $\tau = T/m$.

Step 2 For a $0 < \beta < 1$. Set $i = 0$,

$$F_1 = \max\{J_1(\mu(x_j)), j = 0, 1, \dots, n\},$$

$$G_1 = \min\{J_1(\mu(x_j)), j = 0, 1, \dots, n\},$$

$$F_2 = \max\{J_2(\mu(x_j)), j = 0, 1, \dots, n\},$$

and

$$G_2 = \min\{J_2(\mu(x_j)), j = 0, 1, \dots, n\}.$$

Step 3 Compute $H_{j,i+1}^\beta$ ($= 0, 1, \dots, n, i = 0, 1, \dots, m - 1, j$) via

$$\begin{cases} H_{0,i+1}^\beta = (1-r)H_{0,i}^\beta + rH_{1,i}^\beta + \tau h_{0,i}^\beta \\ -rH_{j+1,i+1}^\beta + 2(1+r)H_{j,i+1}^\beta - \tau h_{j,i+1}^\beta - rH_{j-1,i+1}^\beta \\ = rH_{j+1,i}^\beta + 2(1-r)H_{j,i}^\beta + rH_{j-1,i}^\beta + \tau h_{j,i}^\beta, \\ j = 1, 2, \dots, n-1 \\ H_{n,i+1}^\beta = rH_{n-1,i}^\beta + (1-r)H_{n,i}^\beta + \tau h_{n,i}^\beta \end{cases}$$

where $r = a^2\tau/l^2$, $H_{j,0}^\beta = \mu(x_j)$.

Step 4 Set

$$F_1 \leftarrow \max\{F_1, J_1(H_{j,i+1}^\beta), j = 0, 1, \dots, n\}$$

$$G_1 \leftarrow \min\{G_1, J_1(H_{j,i+1}^\beta), j = 0, 1, \dots, n\}$$

$$F_2 \leftarrow \max\{F_2, J_2(H_{j,i+1}^{1-\beta}), j = 0, 1, \dots, n\}$$

$$G_2 \leftarrow \min\{G_2, J_2(H_{j,i+1}^{1-\beta}), j = 0, 1, \dots, n\}$$

and $i \leftarrow i + 1$.

Step 5 Repeat Steps 3–5 on n times.

Step 6 The supremum and infimum

$$\sup_{(x,t) \in O} J_1(H_{x,t}), \quad \inf_{(x,t) \in O} J_1(H_{x,t}),$$

$$\sup_{(x,t) \in O} J_2(H_{x,t}), \quad \inf_{(x,t) \in O} J_2(H_{x,t})$$

are determined by

$$F_1, G_1, F_2, G_2,$$

respectively.

Example 7 We reconsider the UHE in Example 1 with $O = \{x \in [0, 20], t \in [0, 10]\}$ and compute the supremum

$$\sup_{(x,t) \in O} H_{x,t}$$

and infimum

$$\sup_{(x,t) \in O} H_{x,t}.$$

Let the parameters be

$$m = 1000, n = 200.$$

By the algorithm in Subsection 4.2, we have the supremum and infimum of solution, whose curves are shown in Fig. 8.

Fig. 8 Extreme values' inverse uncertainty distributions in Example 7

Example 8 We reconsider the UHE in Example 2 with $O = \{x \in [0, 20], t \in [0, 1]\}$, and compute the supremum

$$\sup_{(x,t) \in O} H_{x,t}$$

and infimum

$$\sup_{(x,t) \in O} H_{x,t}.$$

Let the parameters be

$$m = 200, n = 20.$$

By the algorithm in Subsection 4.2, we have the inverse uncertainty distributions of supremum and infimum of solution, whose curves are presented in Fig. 9.

Example 9 We reconsider the UHE on Example 3 with $O = \{x \in [0, 2], t \in [0, 1]\}$, and compute the supremum

$$\sup_{(x,t) \in O} H_{x,t}$$

and infimum

$$\sup_{(x,t) \in O} H_{x,t}.$$

Let the parameters be

$$m = 400, n = 20.$$

By the algorithm in Subsection 4.2, we have the inverse uncertainty distributions of supremum and infimum of solution, whose curves are shown in Fig. 10.

Fig. 9 Extreme values' inverse uncertainty distributions in Example 8

Fig. 10 Extreme values' inverse uncertainty distributions in Example 9

5 Conclusion

This paper explored a new approach, namely the Crank–Nicolson method, to compute the numerical solution of UHE. Compared to this method and the forward difference Euler method proposed in Yang (2018), this method's advantage is unconditional stability. In contrast, the forward difference Euler scheme is conditional stability under $r \leq 1/2$. To explain this advantage, we applied the Crank–Nicolson method and the Euler method to Example 1, in which UHE has an analytical solution. Figure 1 shows the β -path and error surface via the Crank–Nicolson method. Table 1 shows that the Euler scheme is instability in some cases, and the Crank–Nicolson scheme is stable in the given situations. Examples 2 and 3 gave the other UHEs, whose β -paths are shown in Figs. 2 and 3 via the Crank–Nicolson method.

Besides, the two algorithms are investigated to compute two characteristics of UHE's solution—the expected value and the extreme value. Examples 4–9 were designed to compute the expected value and extreme value of some UHEs, and the corresponding curves are shown in Figs. 5, 6, 7, 8, 9, and 10 via the Crank–Nicolson method.

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Declarations

Conflict of interest The authors claim that they have no conflict of interest.

Ethical approval This study does not include any studies with human participants or animals performed by any of the authors.

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