A novel linguistic decision-making method based on the voting model for large-scale linguistic decision making

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Abstract

The notable characteristic of large-scale linguistic decision-making problems is that there are so many decision makers who provide linguistic assessments by using fuzzy linguistic representation models. In real-world applications, fuzzy linguistic terms mean different things for different people, and linguistic assessments based on different linguistic representation models may be simultaneous in the same large-scale linguistic decision-making problems. To this end, a novel linguistic decision-making method based on the voting model is proposed in the paper to deal with multi-linguistic assessments provided by decision makers. In large-scale linguistic decision process, evaluation-based voting is defined and multi-linguistic decision matrix is designed to represent multi-linguistic assessments provided by decision makers by using different linguistic representation models, and properties of the decision matrix are analyzed to show that linguistic assessments based on different linguistic representation models can be simultaneously represented. Based on multi-linguistic decision matrix, a new linguistic decision-making framework is developed to deal with large-scale linguistic decision-making problems with multi-linguistic assessments, in which normalization of multi-linguistic decision matrix and trust degrees of linguistic terms are contained, and more important, based on trust degrees of linguistic terms and 2-tuple fuzzy linguistic aggregation operators, an uniform fusion method of multi-linguistic assessments is proposed to aggregate multi-linguistic assessments of large-scale linguistic decision-making problems. Finally, user experiences of shared bikes, which are a large-scale linguistic decision-making problem in real-world applications, are employed to show the new decision-making framework and the uniform fusion method of multi-linguistic assessments, and furthermore, compared with existing linguistic decision-making methods analyzed in the example, it seems that multi-linguistic decision matrix and the uniform fusion method are useful and effective tools to deal with large-scale linguistic decision-making problems with multi-linguistic assessments.

Keywords Large-scale linguistic decision making · The voting model · Linguistic decision-making method · Linguistic aggregation operator

1 Introduction

For any decision-making problems, how to reasonably evaluate and express assessments of alternatives with respect to criteria is the first and important step. Because natural language is the expression way closest to one's cognitive process and expression habits in our daily lives, and more

important natural language or fuzzy linguistic terms provide a more direct way to represent imprecise or uncertain information, linguistic decision-making (LDM) problems have been paid close attention in recent years, in which assessments of alternatives provided by decision makers are represented by fuzzy linguistic terms or complex linguistic expressions (Li et al. 2020; Malhotra and Gupta 2020; Morente-Molinera et al. 2015; Rodríguez et al. 2016, 2013, 2014; Martínez and Herrera 2012). Up to now, many fuzzy linguistic representation models have been proposed to deal with complex LDM problems, such as considering the natural meanings ordering of natural languages and continuity of linguistic terms, 2-tuple fuzzy linguistic representation model (2-TLM) (Herrera and Martínez 2000) is utilized

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to represent aggregation of single-linguistic assessments of alternatives, and theoretical advantages of 2-TLM are computational simplicity, continuous linguistic information, no loss information, accuracy and understandability in dealing with LDM problems (Martínez and Herrera 2012; Martínez et al. 2015). To describe hesitancy of decision makers among linguistic terms, hesitant fuzzy linguistic term set (HFLTS) (Rodríguez et al. 2012) is proposed to overcome limitation of single-linguistic assessments and provide a great flexible representations of linguistic assessments in LDM problems. Considering probability distribution on linguistic terms, probabilistic linguistic term set (PLTS) (Pang et al. 2016) is presented to represent different importance or preference of linguistic terms in hesitant environment, which overcomes limitation of the same importance of linguistic terms in HFLTS. Combining the finite linguistic terms with discrete fuzzy number, the fuzzy linguistic model based on discrete fuzzy numbers (FLM-DFN) (Massanet et al. 2014) is developed to represent different importance or preference of linguistic terms in possible environment of LDM process.

Intuitively, 2-TLM, HFLTS, PLTS and FLM-DFN aim to express different imprecise or uncertain linguistic information, provide us tools to more reasonable express linguistic assessments of alternatives than using single-linguistic term in $S = \{s_0, \dots, s_g\}$ and obtain acceptable decision-making results. However, they also produce a significant challenge in decision-making process. For a LDM problem, because fuzzy linguistic terms mean different things for different decision makers, decision maker d_i maybe provide single-linguistic assessments of alternatives, but decision maker $d_{i'} (i' \neq i)$ maybe utilize 2-TLM, HFLTS, PLTS or FLM-DFN to assess alternatives according to his/her knowledge level (social context or experience); this means that multi-linguistic assessments based on these linguistic representation models are simultaneous in the LDM problem. In the framework of decision matrix, how to distinguish multi-linguistic assessments of alternatives is a significant challenge. In addition, aggregation of multi-linguistic assessments is a problem because no any linguistic aggregation operator can be utilized to simultaneously aggregate them. The situation is very common in large-scale linguistic decision environments, such as in social networks and e-democracy, huge amounts of users take part in decision making and provide linguistic assessments of alternatives based on fuzzy linguistic representation models (Ding and Palomares 2020; Zhong and Xu 2020; Zhang et al. 2017; Labella et al. 2018; Rodríguez et al. 2018), and multi-linguistic assessments instead of one kind of linguistic assessments may be more suitable to large-scale linguistic decision-making problems (LS-MLDM).

1.1 Related works

All the time, representation of linguistic assessments is an important issue in LDM problems. After computing with words proposed by Zadeh (1996), many basic linguistic representation models have been proposed to represent linguistic assessments of alternatives. In recent years, the increasing complexity of real-world LDM problems drives researchers to consider generalization linguistic representation model or complex linguistic expression (such as “between fair and good,” “at least good”) to naturally and freely assess alternatives with respect to criteria (Wang et al. 2018).

In Wu et al. (2020), based on a set whose elements are some subsets of a fixed linguistic term set $S = \{s_0, \dots, s_g\}$ and distribution information of the elements of the set, a flexible linguistic expression (FLE) is design to generalize or improve the construction of existing hesitant linguistic expressions. Formally, if linguistic terms of an element are consecutive in S and distribution information of the element is 1, then the FLE is reduced to HFLTS of S . If each element of the FLE is a single element subset of S , then the FLE may be reduced to PLTS, proportional HFLTS (Chen et al. 2016) or possibility distribution for HFLTS (Wu and Xu 2016) according to different distribution information. Till now, many kinds of hesitant linguistic expressions have been proposed to represent linguistic assessments of alternatives in hesitant linguistic decision environments (Wei et al. 2018; Liu et al. 2019; Gou et al. 2017), and it seems that FLE is an uniform representation of existing hesitant linguistic expressions with distribution information on linguistic terms (Wu et al. 2021; Chao et al. 2021).

In Dong et al. (2009), the numerical scale function from linguistic term set to real number set is designed and the numerical scale model of linguistic term set is proposed to extend 2-TLM under the numerical scale; formally, the numerical scale function transforms each linguistic term into the numerical index of linguistic term, and then operations, aggregations and preference relations on linguistic terms can be transformed into computations on numerical indexes of linguistic terms. By setting different numerical scale functions or interval numerical scale functions, it seems that the numerical scale model of linguistic term set provides a connection framework among different versions of 2-TLM, such as the proportional 2-TLM and the hesitant unbalanced linguistic model based on a linguistic hierarchy (Dong et al. 2016, 2013; Herrera et al. 2008; Wang and Hao 2006).

In Li et al. (2017), considering “fuzzy linguistic terms mean different things for different people,” a personalized individual semantics model of linguistic terms is developed, which aims to personalize individual semantics by means of an interval numerical scale and 2-TLM, *i.e.*, different people provides different numerical scale function or interval numerical scale function to transform each linguistic term

into numerical index. Intuitively, the personalized individual semantics model provides a connection framework among different versions of 2-TLM and the hesitant unbalanced linguistic model based on a linguistic hierarchy in large-scale linguistic decision environments (Li et al. 2018, 2019).

In Gou et al. (2017), double hierarchy linguistic term set is proposed to express the real thoughts of experts and handle complex linguistic information in large-scale linguistic decision environments. Formally, double hierarchy linguistic term set consists of two simple linguistic hierarchies, in which the first hierarchy linguistic term set is the main linguistic hierarchy and the second hierarchy is the linguistic feature or detailed supplementary of each linguistic term in the first hierarchy linguistic term set. By combining simple linguistic terms with linguistic feature, double hierarchy linguistic term set extends many linguistic representation and expresses complex linguistic information.

To our knowledge, many scholars concentrate to develop or generalize new linguistic representation models to represent and aggregate more complex linguistic assessments than classical representation models in large-scale linguistic decision environments. Theoretically, it seems that new linguistic representation model has more complex forms than classical model, and it always means more complex analysis, computation and difficult processing in large-scale linguistic decision making. In fact, 2-TLM, HFLTS, PLTS or FLM-DFN has many interesting and important advantages, such as simple forms, easy interpretation, accuracy, understandability, and computational simplicity; these advantages maybe lose in the new linguistic representation model due to adding new uncertain information.

1.2 Our contribution

In the paper, the voting model (also called as district-based election [37]) is employed in the decision process when evaluation information is provided by a huge amounts of decision makers, accordingly multi-linguistic decision matrix is designed to represent multi-linguistic assessments of alternatives. In a new LS-MLDM framework, an uniform fusion method is developed to simultaneously fuse multi-linguistic assessments of alternatives. Major contributions of the paper are summarized as follows:

1. The voting model is employed in large-scale decision process, which can help decision makers or users providing linguistic assessments of alternatives according to their expert knowledge and linguistic representation models. Then, multi-linguistic decision matrix is designed to represent different linguistic assessments of alternatives provided by different users, and properties show that multi-linguistic assessments of alternatives can be repre-

sented and distinguished in the multi-linguistic decision matrix;

2. A new decision-making framework to deal with LS-MLDM problems is developed, and the refined step can help decision makers providing linguistic assessments of alternatives and multi-linguistic decision matrix can uniformly represent multi-linguistic assessments of alternatives;
3. An uniform fusion method is proposed to aggregate multi-linguistic assessments, where trust degrees of linguistic terms can be computed in the multi-linguistic decision matrix and the uniform fusion method is presented by using trust degrees of linguistic terms. Compared with existing linguistic aggregation operators, the uniform fusion method is an useful and effective tool to deal with LS-MLDM problems.

The rest of the paper is structured as follows: In Sect. 2, basic fuzzy linguistic representation models and linguistic decision matrix are briefly reviewed. In Sect. 3, the voting model in large-scale linguistic decision-making environment is analyzed and multi-linguistic decision matrix is proposed to represent multi-linguistic assessments of alternatives, and then several interesting and important properties of the matrix are discussed. In Sect. 4, a new decision-making framework and main decision resolution scheme are presented to deal with LS-MLDM problems, and then based on normalization of multi-linguistic decision matrix and trust degrees of linguistic terms, an uniform fusion method is proposed to aggregate multi-linguistic assessments of alternatives. In Sect. 5, user experience of shared bikes as a LS-MLDM problem is employed to show the proposed method and make comparative analysis with existing linguistic decision-making methods. Sect. 6 represents conclusion and future works.

2 Preliminaries

In the section, basic fuzzy linguistic representation models are briefly reviewed and linguistic decision matrix of LDM problems is recalled.

2.1 Basic fuzzy linguistic representation model

Fuzzy linguistic representation model is the main issue of computing with words; in Yan et al. (2019), computing with words is divided into three categories, *i.e.*, via fuzzy sets of linguistic values, via fuzzy logic or algebra on the set of linguistic values and via an ordered structure of linguistic values. In LDM methods, computing with words via an ordered structure of linguistic values is also called as the symbolic model of fuzzy linguistic representation (Herrera and

Martínez 2000), 2-TLM is the foundation of the symbolic model; formally, 2-TLM is consisted by a pair of elements (s_j, α) , where linguistic term s_j is in an primary linguistic term set $S = \{s_0, \dots, s_g\}$ and $\alpha \in [-0.5, 0.5)$ is a numerical value that represents the value of the symbolic translation.

Definition 1 Herrera and Martínez (2000) Let $\beta \in [0, g]$ be the result of an aggregation of the indices of a set of linguistic terms assessed in an primary linguistic term set $S = \{s_0, s_1, \dots, s_g\}$, i.e., the result of a symbolic aggregation operation. Let $j = \text{round}(\beta)$ and $\alpha = \beta - j$ be two values such that $j \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.

where $\text{round}(\cdot)$ is the usual rounding operation. Theoretically, the symbolic translation α of linguistic term s_j supports “the difference of information” between a counting of information β and the closest linguistic term $s_j \in S$ ($j = \text{round}(\beta)$). Furthermore, 2-TLM provides transformation between numerical values of $[0, g]$ and 2-tuple linguistic terms (s_j, α) on S :

$$\Delta : [0, g] \longrightarrow S \times [-0.5, 0.5), \beta \mapsto \Delta(\beta) = (s_j, \alpha). \quad (1)$$

$$\Delta^{-1} : S \times [-0.5, 0.5) \longrightarrow [0, g],$$

$$(s_j, \alpha) \mapsto \Delta^{-1}(s_j, \alpha) = \beta = j + \alpha. \quad (2)$$

In the paper, all 2-tuple linguistic terms on S are denoted by $TS = \{(s_j, \alpha) | s_j \in S, \alpha \in [-0.5, 0.5)\}$. The new decision-making framework (Martínez et al. 2015) of LDM problems based on 2-TLM and its generalizations, such as multi-granular linguistic terms, the linguistic hierarchy and unbalanced linguistic assessments, have been widely studied (Malhotra and Gupta 2020; Morente-Molinera et al. 2015; Zhang et al. 2017; Dong et al. 2016, 2013).

Due to the complexity of decision-making problems, decision makers maybe hesitate to provide several linguistic terms instead of single-linguistic term in decision processes, HFLTS is proposed to describe hesitancy of decision makers among different linguistic terms, and up to now, it has become an important linguistic representation model and been widely utilized in LDM problems.

Definition 2 Rodríguez et al. (2012) Let $S = \{s_0, \dots, s_g\}$ be a linguistic term set, an HFLTS, H_S , is an ordered finite subset of the consecutive linguistic terms of S .

According to Definition 2, an HFLTS on S is formally expressed by $H_S = \{s_k, s_{k+1}, \dots, s_{k+t}\}$ such that $k \in \{0, 1, \dots, g\}$ and $k+t \leq g$. Compared with single-linguistic term, HFLTS provides different forms to represent linguistic assessments of alternatives. In addition, let H_S^1 and H_S^2 be two HFLTSs on S , the following basic operations on H_S^1 and H_S^2 are always used to manage hesitant fuzzy linguistic assessments in LDM problems: (1) lower bound:

$H_{S-} = \min(s_i) = s_j, s_i \in H_S$ and $s_i \geq s_j \forall i$; (2) upper bound: $H_{S+} = \max(s_i) = s_j, s_i \in H_S$ and $s_i \leq s_j \forall i$; (3) complement: $H_S^c = S - H_S = \{s_i | s_i \in S \text{ and } s_i \notin H_S\}$; (4) union: $H_S^1 \cup H_S^2 = \{s_i | s_i \in H_S^1 \text{ or } s_i \in H_S^2\}$; (5) intersection: $H_S^1 \cap H_S^2 = \{s_i | s_i \in H_S^1 \text{ and } s_i \in H_S^2\}$; (6) envelope: $\text{env}(H_S) = [H_{S-}, H_{S+}]$. In applications, HFLTS overcomes limitation of the number of linguistic terms and provides a flexible form to represent hesitancy among different linguistic terms. Now HFLTS decision methods have become hot in LDM (Liao and Xu 2015; Krishankumar et al. 2021; Wu et al. 2019; Wu and Xu 2018; Xu et al. 2017; Liao et al. 2015).

In hesitant fuzzy linguistic term set $H_S = \{s_k, s_{k+1}, \dots, s_{k+t}\}$, each linguistic term $s_{k+t'}$ ($t' \leq t$) has the same importance; however, the situation does not always happen in real-world LDM problems. In fact, decision makers may prefer some of linguistic terms in H_S , i.e., linguistic terms of H_S may be with probabilities, possibilities, belief degrees or membership degrees (Pei et al. 2012). PLTS is proposed to describe HFLTS with probability distribution.

Definition 3 Pang et al. (2016) Let $S = \{s_0, \dots, s_g\}$ be linguistic term set. Then, $L(p) = \{L^k(p^k) | L^k \in S, p^k \geq 0, k = 1, \dots, \sharp L(p), \sum_{k=1}^{\sharp L(p)} p^k \leq 1\}$ is used to indicate a probabilistic linguistic term set, where $L^k(p^k)$ is linguistic term L^k associated with probability p^k and $\sharp L(p)$ is the amount of all linguistic terms in $L(p)$.

In brief, a PLTS can be described as $L(p) = \{s_k(p_k), \dots, s_{k+t}(p_{k+t})\}$ such that each $p_{k+t'} \in [0, 1]$ and $\sum_{t'=0}^t p_{k+t'} \leq 1$. If $\sum_{t'=0}^t p_{k+t'} = 1$, then $L(p)$ is a PLTS with the complete information of probabilistic distribution. If $\sum_{t'=0}^t p_{k+t'} < 1$, then $L(p)$ is a PLTS with the non-complete information of probabilistic distribution. Now PLTS decision methods, including operations, aggregation operators and consensus-based decision, have been studied (Bai et al. 2017; Feng et al. 2020; Jiang and Liao 2020; Wu et al. 2019; Liu and Teng 2019; Mao et al. 2019; Chen et al. 2018, 2019, 2018; Chen and Chin 2017; Yang et al. 2019). By considering subjective evaluations of linguistic terms, FLM-DFN provides a flexible way to manage multi-granular linguistic terms in LDM.

Definition 4 Massanet et al. (2014) Let finite chain $L_n = \{0, \dots, n\}$. We call a subjective evaluation to discrete fuzzy number belonging to the partially ordered set $A_1^{L_n}$.

Formally, a fuzzy subset $A : R \longrightarrow [0, 1]$ is called as a discrete fuzzy number if its support is finite (Voxman 2001), i.e., there exist $x_1, \dots, x_n \in R$ with $x_1 \leq \dots \leq x_n$ such that $\text{supp}(A) = \{x_1, \dots, x_n\}$ and there are s and t with $1 \leq s \leq t \leq n$ such that 1) $A(x_i) = 1$ for any i with $s \leq i \leq t$ (core); 2) $A(x_i) \leq A(x_j)$ for each i, j with $1 \leq i \leq j \leq s$; 3) $A(x_i) \geq A(x_j)$ for each i, j with

$t \leq i \leq j \leq n$. According to Definition 4 and a bijective mapping between L_n and $S = \{s_0, \dots, s_n\}$, a FLM-DFN on S is described as $A = \{\mu_k/s_k, \dots, \mu_{k+t'}/s_{k+t'}, \dots, \mu_{k+t}/s_{k+t}\}$, in which each $\mu_{k+t'} \in [0, 1]$ and there exists t' such that $\mu_k \leq \mu_{k+1} \leq \dots, \mu_{k+t'} = 1$ and $\mu_{k+t'} \geq \dots \geq \mu_{k+t}$. In fact, FLM-DFN can be utilized to describe subjective evaluations of decision makers by membership degrees on linguistic terms, and FLM-DFN decision methods, property analysis or aggregation operators of FLM-DFNs have been studied (Riera et al. 2015; Riera and Torrens 2014, 2015; Ma et al. 2019; Roszkowska and Kacprzak 2016; Zhao et al. 2019).

2.2 Linguistic decision matrix

In LDM methods, linguistic assessments of alternatives with respect to criteria are uniformly represented in alternatives-criteria decision matrix, *i.e.*, rows of the matrix are alternatives and columns are criteria, elements of matrix are linguistic assessments provided by the decision maker, *i.e.*, the decision maker employs a fuzzy linguistic representation model to represent linguistic assessments of alternatives. Formally, a LDM problem is that decision makers $M = \{d_1, \dots, d_m\}$ are asked to assess a set of alternatives $A = \{a_1, \dots, a_n\}$ with respect to $C = \{c_1, \dots, c_r\}$, where linguistic assessments of alternatives are selected from linguistic term set $S = \{s_0, \dots, s_g\}$, and then alternatives-criteria decision matrix provided by decision maker d_i ($i = 1, \dots, m$) is

$$D_i = (e_{jk}^i)_{n \times r} = \begin{matrix} & c_1 & \dots & c_r \\ \begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} & \begin{pmatrix} e_{11}^i & \dots & e_{1r}^i \\ \vdots & \vdots & \vdots \\ e_{n1}^i & \dots & e_{nr}^i \end{pmatrix} \end{matrix} \quad (3)$$

If $e_{jk}^i \in S$ ($j \in \{1, \dots, n\}, k \in \{1, \dots, r\}, i \in \{1, \dots, m\}$), then the decision maker d_i uses single-linguistic term to assess the alternative a_j with respect to the criterion c_k . If $e_{jk}^i \in TS$, then d_i uses 2-TLM on S to assess a_j with respect to c_k . If e_{jk}^i is an HFLTS (PLTS or FLM-DFN) on S , then d_i uses HFLTS (PLTS or FLM-DFN) to assess a_j with respect to c_k . Theoretically, limitation of alternatives-criteria decision matrix is higher-dimensional matrix and computational complexity. In addition, it is difficult to represent, distinguish and aggregate linguistic assessments when 2-TLM, HFLTS, PLTS and FLM-DFN are simultaneous in the alternatives-criteria decision matrix.

To overcome limitations of alternatives-criteria decision matrix, Pei et al proposed alternatives-linguistic terms decision matrix to represent linguistic assessments of alternatives with respect to each criterion (Pei et al. 2019), which is inspired by “every decision maker votes or assigns t (true or 1) or f (false or 0) on linguistic terms according to alternative with respect to the criterion,” *i.e.*,

$$D_{ik} = (m_{jl}^{ik})_{n \times (g+1)} = \begin{matrix} & s_0 & \dots & s_g \\ \begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} & \begin{pmatrix} m_{10}^{ik} & \dots & m_{1g}^{ik} \\ \vdots & \vdots & \vdots \\ m_{n0}^{ik} & \dots & m_{ng}^{ik} \end{pmatrix} \end{matrix} \quad (4)$$

where $m_{jl}^{ik} \in \{0, 1\}$ ($l \in \{0, \dots, g\}$), $m_{jl}^{ik} = 1$ means that d_i uses linguistic term $s_l \in S$ to assess alternative a_j with respect to the criterion c_k , $m_{jl}^{ik} = 0$ means that d_i does not use s_l to assess alternative a_j . Intuitively, alternatives-linguistic terms decision matrix D_{ik} is the refinement of alternatives-criteria decision matrix, and its advantages are reduction of dimension and distinction of linguistic assessments in distinct, partial unknown or hesitant linguistic environments. However, in many large-scale situations, huge amounts of decision makers employ different fuzzy linguistic representation models to represent linguistic assessments; according to their knowledge level (social context or experience), it is highly desirable to develop more effective linguistic decision matrix to represent, distinguish and handle multi-linguistic assessments of alternatives in LS-MLDM problems.

3 Multi-linguistic decision matrix derived by the voting model

In the section, the voting model is district-based election, which is used in large-scale linguistic decision-making environment. Then, evaluation-based voting is defined, multi-linguistic decision matrix is proposed, and its several interesting and important properties are discussed to show multi-linguistic assessments of alternatives can be represented in the decision matrix.

3.1 District-based election

A voting mechanism is analogous to a valuation in mathematical logic, in which a semantics is an assignment of ‘true’ or ‘false.’ By extending ‘true’ or ‘false’ into $[0, 1]$ representing the degree of confidence or scepticism of the agent, a voting mechanism for fuzzy logic has been proposed to computing with words (Lawry 1998). Recently, many researchers extensively focus on online social voting and many interesting new features of online social voting have been discussed, which are widely applied in recommendation systems, electoral systems, personalized advertising, market research or public opinion analysis (Wang et al. 2020; Yang and Liang 2017).

Definition 5 Filos-Ratsikas et al. (2020) A district-based election is defined as a tuple $\varepsilon = (\mathcal{M}, \mathcal{N}, D, W, V, f)$, where $\mathcal{M}$ is a set of m alternatives, $\mathcal{N}$ is a set of n voters, D is a set of k districts that define a partition of the set of vot-

ers, $W = (w_d)_{d \in D}$ is a weight vector consisting of a weight $w_d > 0$ for each district $d \in D$, $V = (v_1, \dots, v_n)$ is a valuation profile for the n voters such that $v_i = (v_{ij})_{j \in M}$ contains the valuation of voter i for all alternatives and f is a voting rule that maps a valuation (sub)profile to a single alternative in M .

Formally, a district-based election models a social electoral system, through which a set of voters are asked to vote over a set of alternatives; that is, the set of voters is partitioned into districts and each district holds a local election over all alternatives according to some voting rule. District $d \in D$ in Definition 5 contains n_d voters such that $\sum_{d \in D} n_d = n$, for each voter $i \in \mathcal{N}$, $d(i) \in D$ denotes i belongs to district d . The winner of each local election is awarded a weight that depends on the district, and the overall winner is chosen to be the alternative with the highest weight; that is, for every district $d \in D$, a local election between its members takes place, and the winner of this election is the alternative $j_d = f(v_d)$ that gets elected according to the voting rule f , where $v_d = (v_i)_{i: d(i)=d}$ denotes the valuation sub-profile of the voters that belong to district d so that $V = \bigcup_{d \in D} v_d$. The outcome of the district-based election ε is an alternative [37]

$$w(\varepsilon) \in \operatorname{argmax}_{j \in M} \left\{ \sum_{d \in D} w_d \cdot 1\{j = j_d\} \right\}.$$

Intuitively, the voting rule f has a local effect within each district d and a global effect over the whole district-based election, i.e., for a given valuation profile V , the social welfare of the voters for alternative $j \in M$ is defined as the total value that the voters have for alternative j ,

$$SW(j|V) = \sum_{i \in \mathcal{N}} v_{ij}.$$

In large-scale linguistic decision-making environment, a district-based election can be used to normalize the evaluation process provided by decision makers and obtain accurate linguistic assessments of alternatives.

- In district-based election, partitions of the set of voters $\mathcal{N}$ aim to distinguish the local voters, which have the same voting rule. In large-scale linguistic decision-making environment, districts D can be used to cluster decision makers or users which have the similar knowledge level (social context or experience) or use the same fuzzy linguistic representation model to evaluate alternatives with respect to criteria;
- In district-based election, the overall winner is dependent on the local valuation $j_d = f(v_d)$ as well as the overall valuation $w(\varepsilon)$, in which the local valuation $j_d = f(v_d)$

is limited in district $d \in D$ such that the unit-sum representation $\sum_{j \in M} v_{ij} = 1$ for every voter $i \in \mathcal{N}$ and the total value of every alternative $j \in M$ is $SW(j|V)$. In large-scale linguistic decision-making environment, the local valuation and the total value of every alternative can be used to aggregate multi-linguistic assessments of alternatives based on linguistic representation models in LS-MLDM problems.

Accordingly, an evaluation mechanism can be formally defined to model the evaluation process provided by decision makers.

Definition 6 An evaluation-based voting is defined as a tuple $\varepsilon = (A, M, \overline{M}, W, V, f)$, where

- $A = \{a_1, \dots, a_n\}$ is a set of n alternatives.
- $M = \{d_1, \dots, d_m\}$ is a set of m decision makers or users.
- $\overline{M} = \{M_1, \dots, M_p\}$ is partitions of the set of M . Each decision maker or user belongs to a partition $M_{p'}$, every partition $M_{p'}$ contains $m_{p'}$ decision makers or users such that $\sum_{M_{p'} \in \overline{M}} m_{p'} = m$.
- $W = (w_{p'})_{M_{p'} \in \overline{M}}$ is a weight vector consisting of a weight $w_{p'} > 0$ for every partition $M_{p'} \in \overline{M}$ and $\sum_{M_{p'} \in \overline{M}} w_{p'} = 1$.
- $V = (v_1, \dots, v_m)$ is an evaluation profile for the m decision makers or users such that $v_{m'} = (v_{m'n'})_{a_{n'} \in A}$ contains the evaluation information of the decision maker or user $d_{m'}$ for all alternatives.
- f is a voting rule that maps an evaluation (sub)profile to a single alternative in A . In decision-making environment, the voting rule f consists of a set $C = \{c_1, \dots, c_r\}$ of r criteria which are utilized by decision makers or users to evaluate alternatives, the domain and constraint of evaluation of decision makers or users and so on.

In Definition 6, different evaluation profiles and voting rules correspond to different evaluation information of alternatives represented by different fuzzy linguistic model.

1. In classical fuzzy decision methods, partitions of the set of decision makers or users M are not considered, in other words, $p = 1$ and $w_{p'} = 1$ in the evaluation-based voting for classical fuzzy decision making. In addition, the domain of evaluation is $[0, 1]$ and each evaluation with respect to $c_{r'} \in C$ is fuzzy subset on $[0, 1]$, $v_{m'n'}$ has the form

$$\begin{aligned} v_{m'} &= (v_{m'n'})_{a_{n'} \in A} \\ &= ((\mu_{11}^{m'}, \dots, \mu_{1r}^{m'}); \dots; (\mu_{n1}^{m'}, \dots, \mu_{nr}^{m'})), \end{aligned}$$

and the evaluation information of the decision maker or user $d_{m'}$ for alternative $a_{n'}$ with respect to criteria $C = \{c_1, \dots, c_r\}$ is $a_{n'} = f(v_{m'}) = (\mu_{n'1}^{m'}, \dots, \mu_{n'r}^{m'})$. In classical fuzzy decision-making methods, $v_{m'}$ is constructed by fuzzy decision matrix, i.e.,

$$f(v_{m'}) = D_{m'} = \begin{matrix} & c_1 & \cdots & c_r \\ \begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} & \begin{pmatrix} \mu_{11}^{m'} & \cdots & \mu_{1r}^{m'} \\ \vdots & \ddots & \vdots \\ \mu_{n1}^{m'} & \cdots & \mu_{nr}^{m'} \end{pmatrix} \end{matrix}.$$

2. In LDM methods, $p = 1$ and $w_{p'} = 1$ in the evaluation-based voting, the domain of evaluation is linguistic term set $S = \{s_0, \dots, s_g\}$ and each evaluation with respect to $c_{r'} \in C$ is a linguistic term, $v_{m'n'}$ has the form

$$v_{m'} = (v_{m'n'})_{a_{n'} \in A} = ((e_{11}^{m'}, \dots, e_{1r}^{m'}); \dots; (e_{n1}^{m'}, \dots, e_{nr}^{m'})).$$

the linguistic evaluation $v_{m'}$ provided by decision maker $d_{m'}$ is represented by linguistic decision matrix Eq. (3).

3. In fuzzy linguistic multiset decision method (Pei et al. 2019), the domain of evaluation is the space $\{0, 1\}^{g+1}$ with respect to linguistic term set $S = \{s_0, \dots, s_g\}$ and each evaluation with respect to every $c_{r'} \in C$ is an vector in the space $\{0, 1\}^{g+1}$, then $v_{m'n'}$ has the form

$$v_{m'} = (v_{m'n'})_{a_{n'} \in A} = (((e_{10}^{m'}, \dots, e_{1g}^{m'}), \dots, (e_{10}^{m'r}, \dots, e_{1g}^{m'r})); \dots; ((e_{n0}^{m'}, \dots, e_{ng}^{m'}), \dots, (e_{n0}^{m'r}, \dots, e_{ng}^{m'r}))). \quad (5)$$

which is represented by alternatives-linguistic terms decision matrix.

3.2 Multi-linguistic decision matrix

Here, a new voting rule f is employed to obtain linguistic evaluation information in large-scale linguistic decision-making environment, i.e., the domain of evaluation is the space $[0, 1]^{g+1}$ with respect to linguistic term set $S = \{s_0, \dots, s_g\}$ and each evaluation is an vector on the space $[0, 1]^{g+1}$, compared with Eq. (5), elements in $v_{m'}$ are extended from $\{0, 1\}$ to $[0, 1]$ and the linguistic evaluation $v_{m'}$ of all alternatives provided by the decision maker or user $d_{m'}$ is represented by multi-linguistic decision matrix, i.e.,

$$D_{ik} = (m_{jl}^{ik})_{n \times (g+1)} = \begin{matrix} & s_0 & \cdots & s_g \\ \begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} & \begin{pmatrix} m_{10}^{ik} & \cdots & m_{1g}^{ik} \\ \vdots & \ddots & \vdots \\ m_{n0}^{ik} & \cdots & m_{ng}^{ik} \end{pmatrix} \end{matrix}, \quad (6)$$

where $m_{jl}^{ik} \in [0, 1]$ ($l = 0, \dots, g$), $m_{jl}^{ik} = 1$ means that the decision maker d_i confidently use linguistic term $s_l \in S$ to assess alternative a_j with respect to the criterion c_k , $m_{jl}^{ik} \in (0, 1)$ means that the decision maker d_i with degree of confidence m_{jl}^{ik} use linguistic term $s_l \in S$ to assess alternative a_j with respect to the criterion c_k and $m_{jl}^{ik} = 0$ means that linguistic term s_l is not suitable to assess alternative a_j . The following properties show that multi-linguistic decision matrix is an extension of alternatives-linguistic terms decision matrix, and PLTS and FLM-DFN assessments of alternatives can also be represented by multi-linguistic decision matrix D_{ik} .

Proposition 1 In multi-linguistic decision matrix D_{ik} Eq. (6), let $D_{ik}^j = \{m_{jl}^{ik} \mid m_{jl}^{ik} > 0, l = 0, 1, \dots, g\}$. Then, for decision maker d_i and the criterion c_k ,

1. linguistic assessment of alternative a_j is distinct or single-linguistic term if and only if $|D_{ik}^j| = 1$ and $\sum_{l=0}^g m_{jl}^{ik} = 1$;
2. linguistic assessment of alternative a_j is partial unknown if and only if $D_{ik}^j = \emptyset$ or $\sum_{l=0}^g m_{jl}^{ik} = 0$;
3. linguistic assessment of alternative a_j is an HFLTS if and only if there exists natural numbers l and t such that $l + t \leq g$ and $\sum_{t'=0}^t m_{j(l+t')}^{ik} = |D_{ik}^j| > 1$.

Proof According to explanation of multi-linguistic decision matrix Eq. (6), the decision maker d_i with degree of confidence m_{jl}^{ik} uses linguistic term $s_l \in S$ to assess alternative a_j with respect to the criterion c_k if and only if $m_{jl}^{ik} \in D_{ik}^j$.

- (1) $|D_{ik}^j| = 1$ means that linguistic assessment of alternative a_j is a single-linguistic term $s_l \in S$ with degree of confidence m_{jl}^{ik} , $\sum_{l=0}^g m_{jl}^{ik} = 1$ means $m_{jl}^{ik} = 1$, i.e., linguistic assessment of alternative a_j is distinct or single-linguistic term if and only if $|D_{ik}^j| = 1$ and $\sum_{l=0}^g m_{jl}^{ik} = 1$.
- (2) $D_{ik}^j = \emptyset$ or $\sum_{l=0}^g m_{jl}^{ik} = 0$ means each $m_{jl}^{ik} = 0$ and vice versa, i.e., each linguistic term $s_l \in S$ is not suitable to assess alternative a_j or linguistic assessment of alternative a_j is partial unknown.
- (3) $\sum_{t'=0}^t m_{j(l+t')}^{ik} = |D_{ik}^j| > 1$ means that the set of consecutive linguistic terms $\{s_l, s_{l+1}, \dots, s_{l+t}\}$ with degree of confidence $\{m_{j(l)}^{ik}, m_{j(l+1)}^{ik}, \dots, m_{j(l+t)}^{ik}\}$ is the linguistic assessment of alternative a_j with respect to the criterion c_k , $\sum_{t'=0}^t m_{j(l+t')}^{ik} = |D_{ik}^j|$ means each $m_{j(l+t')}^{ik} = 1$ and vice versa, i.e., HFLTS $\{s_l, s_{l+1}, \dots, s_{l+t}\}$ is used to assess alternative a_j with respect to c_k . $\square$

Proposition 2 In multi-linguistic decision matrix Eq. (6), let $D_{ik}^j = \{m_{jl}^{ik} \mid m_{jl}^{ik} > 0, l = 0, 1, \dots, g\}$. Then for decision maker d_i and the criterion c_k ,

1. linguistic assessment of alternative a_j is a PLTS if and only if $|D_{ik}^j| > 1$ and $\sum_{l=0}^g m_{jl}^{ik} \leq 1$;
2. linguistic assessment of alternative a_j is a FLM-DFN if and only if $|D_{ik}^j| > 1$, $\sum_{l=0}^g m_{jl}^{ik} \in (1, g)$ and there exists $m_{jl}^{ik} = 1$ such that $m_{j0}^{ik} \leq m_{j1}^{ik} \leq \dots \leq m_{jl}^{ik}$ and $m_{jl}^{ik} \geq m_{j(l+1)}^{ik} \dots \geq m_{jg}^{ik}$.

Proof (1) $|D_{ik}^j| > 1$ means that more than one linguistic term is utilized to assess alternative a_j with respect to the criterion c_k and vice versa, $\sum_{l=0}^g m_{jl}^{ik} \leq 1$ means each $m_{jl}^{ik} \in [0, 1)$. According to Definition 3, all degrees of confidence in D_{ik}^j satisfy probability distribution on linguistic terms, i.e., degree of confidence can be considered as probability of linguistic term and linguistic assessment of alternative a_j is a PLTS.

(2) $\sum_{l=0}^g m_{jl}^{ik} \in (1, g)$ means that there exists m_{jl}^{ik} such that $0 < m_{jl}^{ik} < 1$, i.e., all degrees of confidence in D_{ik}^j satisfy membership degrees of linguistic terms. In addition, there exists $m_{jl}^{ik} = 1$ such that $m_{j0}^{ik} \leq m_{j1}^{ik} \leq \dots \leq m_{jl}^{ik}$ and $m_{jl}^{ik} \geq m_{j(l+1)}^{ik} \dots \geq m_{jg}^{ik}$ satisfies Definition 4; hence, degree of confidence can be considered as membership degree of linguistic term and D_{ik}^j is a FLM-DFN. $\square$

Properties 1 and 2 show that multi-linguistic decision matrix D_{ik} is a flexible and effective tool to represent linguistic assessments of alternatives when decision makers utilize several fuzzy linguistic representation models to evaluate alternatives.

Example 1 A customer d_i evaluates quality (c_k) of five clothes by online product reviews, clothes are jacket (J), trousers (T), shirt (S), trousers suit (TS) and underdress (U), the primary linguistic term set is $S = \{s_0$ (extremely poor), s_1 (very poor), s_2 (poor), s_3 (slightly poor), s_4 (fair), s_5 (slightly good), s_6 (good), s_7 (very good), s_8 (extremely good)). According to life habit and experience, d_i is familiar with jacket and trousers, but partly knows or does not know about shirt, trousers suit and underdress, hence multi-linguistic decision matrix provided by d_i with respect to c_k may be as follows:

$$D_{ik} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ \begin{matrix} J \\ T \\ S \\ TS \\ U \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.7 & 1 & 0.8 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}.$$

According to Properties 1 and 2, due to $|D_{ik}^1| = 1$ and $\sum_{l=0}^8 m_{il}^{ik} = 1$, linguistic assessment of J provided by d_i with respect to c_k is a single-linguistic term s_6 . Due

to $\sum_{l=0}^8 m_{2l}^{ik} = |D_{ik}^2| = 2$, linguistic assessment of T provided by d_i with respect to c_k is HFLTS $\{s_4, s_5\}$. Due to $\sum_{l=0}^8 m_{5l}^{ik} = 0$, i.e., linguistic assessment of U provided by d_i with respect to c_k is partial unknown. Due to $|D_{ik}^3| = 3 > 1$ and $\sum_{l=0}^8 m_{3l}^{ik} = 0.9 < 1$, linguistic assessment of S provided by d_i with respect to c_k is PLTS $\{s_5(0.5), s_6(0.2), s_7(0.2)\}$. Due to $|D_{ik}^4| = 5 > 1$, $\sum_{l=0}^8 m_{4l}^{ik} = 3.6 \in (1, 8)$ and $m_{46}^{ik} = 1$, i.e., linguistic assessment of TS provided by d_i with respect to c_k is FLM-DFN $\{0.5/s_4, 0.7/s_5, 1/s_6, 0.8/s_7, 0.6/s_8\}$.

4 The linguistic decision-making framework of LS-MLDM problems

In the section, a new linguistic decision-making framework is designed to deal with LS-MLDM problems, in which evaluation-based voting and the multi-linguistic decision matrix are utilized to normalize the decision process and represent linguistic assessments provided by decision makers in large-scale linguistic decision-making environments. In addition, main decision resolution scheme of the framework is analyzed and trust degrees of linguistic terms in the multi-linguistic decision matrix are developed; more important a uniform fusion method of multi-linguistic assessments is proposed according to trust degrees of linguistic terms.

4.1 The linguistic decision-making phases of LS-MLDM problems

Formally, classical linguistic decision-making framework of LDM problems is mainly consisted of four phases (Martínez et al. 2015):

- The choice of the linguistic term set with its semantics, which provides linguistic expression domain for decision makers to evaluate alternatives, and linguistic assessments based on a fuzzy linguistic representation model are uniformly represented in alternatives-criteria linguistic decision matrix;
- The choice of linguistic aggregation operators, which is chosen for aggregating linguistic assessments based on linguistic representation model. Theoretically, different linguistic aggregation operators obtain different results;
- Aggregation phase, which obtains linguistic collective assessments by using the chosen linguistic aggregation operator;
- Exploitation phase, which obtains a ranking on linguistic aggregation results to choose the best alternatives. In real-world applications, different fuzzy linguistic representation models are corresponding to different rankings and obtain the different best solution of LDM problem.

Fig. 1 The linguistic decision-making phases of classical LDM problems

The classical linguistic decision-making framework of LDM problems is pictured in Fig. 1 (Labella et al. 2018). Compared with LS-MLDM problems, because there are a huge amounts of decision makers and fuzzy linguistic terms mean different things for different decision makers, in decision process different fuzzy linguistic representation models may be employed by decision makers to assess alternatives with respect to criteria, a new linguistic decision-making framework of LS-MLDM problems is designed in the paper, which is pictured in Fig. 2 and mainly consisted of the following five phases:

1. The choice of the linguistic term set with its semantics, which is the primary linguistic expression domain and helps decision makers to provide linguistic assessments of alternatives;
2. The choice of voting rule f in evaluation-based voting (Definition 6) and partition the set of decision makers according to knowledge level (social context or experience), then multi-linguistic decision matrix can be constructed, in which all multi-linguistic assessments of alternatives provided by the decision maker are uniformly represented;
3. Normalization of multi-linguistic decision matrix, which aims to transform multi-linguistic assessments into the same linguistic representation and avoid undesirable consequences flowing from different fuzzy linguistic representation models;
4. Aggregation phase, in which linguistic aggregation operators are chosen for aggregating the normalized linguistic assessments;
5. Exploitation phase, in which a ranking on linguistic aggregation results is obtained and selection of the best alternatives can be finished.

Comparing Fig. 1 with Fig. 2, it can be noticed that the first phase in the classical linguistic decision-making framework is refined by (1) and (2) in the new linguistic decision-making

framework. In fact, considering a huge amounts of decision makers with different knowledge levels (social context or experience), in evaluation-based voting, they maybe use different voting rule f to provide and obtain more suitable or accurate linguistic assessments of alternatives than one kind of linguistic representation model. According to Properties 1 and 2, the following one-to-one mappings can be designed to formally describe linguistic assessments of alternative provided by decision makers with respect to criterion and distinguish the function relationship between fuzzy linguistic representations and multi-linguistic decision matrices:

- Single assessment of alternative and the function relationship between linguistic term set and multi-linguistic decision matrix:

$$F_H : M \times C \times S \times A \longrightarrow \{0, 1\},$$

$$(d_i, c_k, s_l, a_j) \longmapsto m_{j1}^{ik} \in \{0, 1\}, \quad (7)$$

in which $F_H(d_i, c_k, s_l, a_j) = 0$ means that decision maker d_i with respect to criterion c_k does not utilize linguistic term s_l to assess alternative a_j , $F_H(d_i, c_k, s_l, a_j) = 1$ means that decision maker d_i utilizes linguistic term s_l to assess a_j with respect to criterion c_k . Single-linguistic decision matrix provided by decision maker d_i can be constructed by using values $\{F_H(d_i, c_k, s_l, a_j) | s_l \in S, a_j \in A\}$ and single-linguistic assessment of alternative a_j is equal to $\sum_{l=0}^g F_H(d_i, c_k, s_l, a_j) = 1$;

- HFLTS assessment of alternative and the function relationship between HFLTSs and multi-linguistic decision matrix:

$$F_H : M \times C \times S \times A \longrightarrow \{0, 1\},$$

$$(d_i, c_k, s_l, a_j) \longmapsto m_{j1}^{ik} \in \{0, 1\}, \quad (8)$$

in which $F_H(d_i, c_k, s_l, a_j) = 0$ means that decision maker d_i does not utilize linguistic term s_l to assess alternative a_j , $F_H(d_i, c_k, s_l, a_j) = 1$ means

Fig. 2 The new linguistic decision-making phases of LS-MLDM problems

that decision maker d_i utilizes linguistic term s_l to assess alternative a_j . HFLTS linguistic decision matrix provided by decision maker d_i can be constructed by using values $\{F_H(d_i, c_k, s_l, a_j) | s_l \in S, a_j \in A\}$ and HFLTS assessment of alternative a_j is equal to $\sum_{l=0}^g F_H(d_i, c_k, s_l, a_j) = N$, which is a natural number in $(1, g)$;

- PLTS assessment of alternative and the function relationship between PLTSs and multi-linguistic decision matrix:

$$F_H : M \times C \times S \times A \longrightarrow [0, 1],$$

$$(d_i, c_k, s_l, a_j) \longmapsto m_{jl}^{ik} \in [0, 1], \quad (9)$$

in which $F_H(d_i, c_k, s_l, a_j) = m_{jl}^{ik}$ means that decision maker d_i utilize linguistic term s_l to assess alternative a_j with probability m_{jl}^{ik} . PLTS linguistic decision matrix provided by decision maker d_i can be constructed by using values $\{F_H(d_i, c_k, s_l, a_j) | s_l \in S, a_j \in A\}$ and PLTS assessment of alternative a_j is equal to $0 < \sum_{l=0}^g F_H(d_i, c_k, s_l, a_j) \leq 1$;

- FLM-DFN assessment of alternative and the function relationship between FLM-DFNs and multi-linguistic decision matrix:

$$F_H : M \times C \times S \times A \longrightarrow [0, 1],$$

$$(d_i, c_k, s_l, a_j) \longmapsto m_{jl}^{ik} \in [0, 1], \quad (10)$$

in which $F_H(d_i, c_k, s_l, a_j) = m_{jl}^{ik}$ means that decision maker d_i utilize linguistic term s_l to assess alternative a_j with membership degree m_{jl}^{ik} . FLM-DFN linguistic decision matrix provided by decision maker d_i can be constructed by using values $\{F_H(d_i, c_k, s_l, a_j) | s_l \in S, a_j \in A\}$ and FLM-DFN assessment of alternative a_j is equal to $1 < \sum_{l=0}^g F_H(d_i, c_k, s_l, a_j) \leq g$;

- Multi-linguistic assessment of alternative and the function relationship between multi-linguistic representations and multi-linguistic decision matrix:

$$F_H : M \times C \times S \times A \longrightarrow [0, 1],$$

$$(d_i, c_k, s_l, a_j) \longmapsto m_{jl}^{ik} \in [0, 1], \quad (11)$$

in which $F_H(d_i, c_k, s_l, a_j) = m_{jl}^{ik}$ means that decision maker d_i utilize linguistic term s_l to assess alternative a_j with degree of confidence m_{jl}^{ik} . Multi-linguistic decision matrix provided by decision maker d_i can be constructed by using values $\{F_H(d_i, c_k, s_l, a_j) | s_l \in S, a_j \in A\}$ and multi-linguistic assessment of alternative a_j is equal to $0 \leq \sum_{l=0}^g F_H(d_i, c_k, s_l, a_j) \leq g$.

4.2 Normalization of multi-linguistic decision matrix

Theoretically, normalization of multi-linguistic decision matrix is equal to normalization of multi-linguistic assessments. Here, normalization of multi-linguistic assessments is to transform degree of confidence on linguistic terms into trust degrees of linguistic terms in multi-linguistic assessments of alternatives.

Definition 7 In multi-linguistic decision matrix D_{ik} , for alternative a_j ($j = 1, \dots, n$), trust degree of linguistic term s_l ($l = 0, \dots, g$) provided by decision maker d_i with respect to criterion c_k is defined as

$$u_{jl}^{ik} = \begin{cases} \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}}, & \text{if } \sum_{l=0}^g m_{jl}^{ik} > 0, \\ m_{jl}^{ik} = 0, & \text{if } \sum_{l=0}^g m_{jl}^{ik} = 0. \end{cases} \quad (12)$$

Based on trust degree u_{jl}^{ik} of linguistic term s_l ($l = 0, \dots, g$), multi-linguistic decision matrix D_{ik} can be converted into linguistic decision matrix D'_{ik} with trust degrees of linguistic terms.

$$D'_{ik} = \begin{pmatrix} u_{10}^{ik} & \cdots & u_{1g}^{ik} \\ \vdots & & \vdots \\ u_{n0}^{ik} & \cdots & u_{ng}^{ik} \end{pmatrix} = \begin{pmatrix} \frac{m_{10}^{ik}}{\sum_{l=0}^g m_{1l}^{ik}} & \cdots & \frac{m_{1g}^{ik}}{\sum_{l=0}^g m_{1l}^{ik}} \\ \vdots & & \vdots \\ \frac{m_{n0}^{ik}}{\sum_{l=0}^g m_{nl}^{ik}} & \cdots & \frac{m_{ng}^{ik}}{\sum_{l=0}^g m_{nl}^{ik}} \end{pmatrix} \quad (13)$$

Proposition 3 In trust degree linguistic decision matrix D'_{ik} , trust degrees of each alternative a_j ($j = 1, \dots, n$) are such that $0 \leq u_{jl}^{ik} \leq 1$ and $\sum_{l=0}^g u_{jl}^{ik} = 1$.

Proof For each trust degree u_{jl}^{ik} of linguistic term s_l ($l = 0, \dots, g$), according to multi-linguistic decision matrix Eq. 6, we have

$$0 \leq u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} \leq \frac{m_{jl}^{ik}}{m_{jl}^{ik}} = 1.$$

For each alternative a_j of linguistic decision matrix D'_{ik} , we have

$$\sum_{l=0}^g u_{jl}^{ik} = \sum_{l=0}^g \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} = \frac{\sum_{l=0}^g m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} = 1.$$

□

In the following, the relationship between degree of confidence m_{jl}^{ik} in D_{ik} Eq. (6) and trust degree u_{jl}^{ik} of linguistic term s_l in D'_{ik} Eq. (13) is analyzed.

Proposition 4 For any D_{ik} , D'_{ik} and $D_{ik}^j = \{m_{jl}^{ik} | m_{jl}^{ik} > 0, l = 0, 1, \dots, g\}$.

1. If linguistic assessment of alternative a_j is distinct or single-linguistic term, then $u_{jl}^{ik} = m_{jl}^{ik}$ for any $l = 0, \dots, g$;
2. If linguistic assessment of alternative a_j is hesitant linguistic term set, then $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{|D_{ik}^j|} \leq m_{jl}^{ik}$;
3. If linguistic assessment of alternative a_j is probabilistic linguistic term set and $m_{jl}^{ik} \neq 0$, then $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} \geq m_{jl}^{ik}$;
4. If linguistic assessment of alternative a_j is discrete fuzzy number on linguistic term set S and $m_{jl}^{ik} \neq 0$, then $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} < m_{jl}^{ik}$.

Proof 1) According to Property 1 (1), if linguistic assessment of alternative a_j is distinct or single-linguistic term, then $\sum_{l=0}^g m_{jl}^{ik} = 1$. Based on Eq. (12), $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} = m_{jl}^{ik}$ for any $l = 0, \dots, g$.

- 2) According to Property 1 (3), if linguistic assessment of alternative a_j is hesitant linguistic term set, then $\sum_{l=0}^g m_{jl}^{ik} = |D_{ik}^j| > 1$ and $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{|D_{ik}^j|}$, it is obvious that $u_{jl}^{ik} = m_{jl}^{ik} = 0$ when $m_{jl}^{ik} = 0$ and $u_{jl}^{ik} < m_{jl}^{ik}$ when $m_{jl}^{ik} = 1$.
- 3) According to Property 2 (1), if linguistic assessment of alternative a_j is probabilistic linguistic term set, then $\sum_{l=0}^g m_{jl}^{ik} \leq 1$, when $m_{jl}^{ik} \neq 0$, $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} \geq m_{jl}^{ik}$, especially, when $\sum_{l=0}^g m_{jl}^{ik} = 1$, then $u_{jl}^{ik} = m_{jl}^{ik}$.
- 4) According to Property 2 (2), if linguistic assessment of alternative a_j is discrete fuzzy number on linguistic term set S , then $\sum_{l=0}^g m_{jl}^{ik} \in (1, g)$ and $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}} < m_{jl}^{ik}$ if $m_{jl}^{ik} \neq 0$. □

Example 2 Continues Example 1. Based on Eq. (13), multi-linguistic decision matrix D_{ik} is converted into trust degree linguistic decision matrix D'_{ik} , i.e.,

$$D_{ik} = \begin{matrix} & s_0 s_1 s_2 s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ \begin{matrix} J \\ T \\ S \\ TS \\ U \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.7 & 1 & 0.8 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

$$D'_{ik} = \begin{matrix} & s_0 s_1 s_2 s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ \begin{matrix} J \\ T \\ S \\ TS \\ U \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{36} & \frac{7}{36} & \frac{5}{18} & \frac{2}{9} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}.$$

4.3 Aggregation and exploitation of LS-MLDM

According to Fig. 2 and trust degree linguistic decision matrix D'_{ik} Eq. (13), linguistic aggregation operators can be selected to aggregate linguistic assessments with trust degrees, and then exploitation of LS-MLDM can be carried out.

4.3.1 Aggregation of multi-linguistic assessments

Theoretically, Definition 7 and Property 3 show that trust degree u_{jl}^{ik} of linguistic term s_l ($l = 0, \dots, g$) can be employed as weights of linguistic terms to aggregate multi-linguistic assessments of alternatives, i.e., for decision maker d_i and criterion c_k , multi-linguistic assessments of alternatives in D'_{ik} (Eq. (13)) can be aggregated by

$$E_j^{ik} = \Delta \left(\sum_{l=0}^g (u_{jl}^{ik} \times l) \right) = \Delta \left(\frac{\sum_{l=0}^g (m_{jl}^{ik} \times l)}{\sum_{l=0}^g m_{jl}^{ik}} \right) = (s_{jl'}^{ik}, \alpha_j^{ik}), \quad (14)$$

where Δ is transformation from numerical values $[0, g]$ to 2-TLMs on S defined by Eq. (1), $l' \in \{0, 1, \dots, g\}$ and $(s_{jl'}^{ik}, \alpha_j^{ik})$ is a 2-tuple linguistic term on $S = \{s_0, \dots, s_g\}$. Based on Eq. (14), multi-linguistic assessments of alternatives in D_{ik}' are uniformly represented by 2-TLM $E_j^{ik} = (s_{jl'}^{ik}, \alpha_j^{ik})$, such as in Example 2, multi-linguistic assessments of five clothes provided by customer d_i with respect to quality are uniformly represented by 2-TLMs $(s_6, 0)$ of J , $(s_5, -0.5)$ of T , $(s_6, -0.4)$ of S , $(s_6, 0.08)$ of TS and $(s_0, 0)$ of U , respectively. Accordingly, multi-linguistic assessments provided by decision maker d_i with respect to all criteria are represented by 2-tuple fuzzy linguistic decision matrix, *i.e.*,

$$D_i = (E_j^{ik})_{n \times r} = \begin{matrix} & c_1 & \cdots & c_r \\ \begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} & \begin{pmatrix} E_1^{i1} & \cdots & E_1^{ir} \\ \vdots & \vdots & \vdots \\ E_n^{i1} & \cdots & E_n^{ir} \end{pmatrix} \end{matrix}, \quad (15)$$

Based on the 2-tuple fuzzy linguistic decision matrix D_i , if weights $(w_1, \dots, w_m)$ of decision makers are considered, then 2-tuple linguistic assessment of a_j provided by all decision makers with respect to criterion c_k is

$$E_j^k = \Delta \left(\sum_{i=1}^m (w_i \Delta^{-1}(E_j^{ik})) \right) = (s_{jl'}^k, \alpha_j^k), \quad (16)$$

in which $s_{jl'}^k \in S = \{s_0, s_1, \dots, s_g\}$ and $\alpha_j^k \in [-0.5, 0.5]$. If weights $(\omega_1, \dots, \omega_r)$ of criteria are considered, then 2-tuple linguistic assessment of a_j provided by decision maker d_i with respect to criteria is

$$E_j^i = \Delta \left(\sum_{k=1}^r (\omega_k \Delta^{-1}(E_j^k)) \right) = (s_{jl'}^i, \alpha_j^i), \quad (17)$$

in which $s_{jl'}^i \in S = \{s_0, s_1, \dots, s_g\}$ and $\alpha_j^i \in [-0.5, 0.5]$. Accordingly, the final aggregation of a LS-MLDM problem can be divided into two categories, *i.e.*,

1. First aggregating linguistic assessments of decision makers then criteria:

$$R(a_j) = \Delta \left(\sum_{k=1}^r (\omega_k \Delta^{-1}(E_j^k)) \right) = (s_{jl'}^i, \alpha_j^i), \quad (18)$$

in which, E_j^k is decided by Eq. (16).

2. First aggregating linguistic assessments of criteria then decision makers:

$$R(a_j) = \Delta \left(\sum_{i=1}^m (w_i \Delta^{-1}(E_j^i)) \right) = (s_{jl'}^i, \alpha_j^i), \quad (19)$$

in which, E_j^i is decided by Eq. (17).

Where $R(a_j) = (s_{jl'}^i, \alpha_j^i)$ on $S = \{s_0, \dots, s_g\}$.

4.3.2 Exploitation of alternatives

Exploitation phase of LS-MLDM problems is depended on linguistic aggregation result $(s_{jl'}^i, \alpha_j^i)$ calculated by Eqs. (18) or (19), formally, for any two alternatives a_j and $a_{j'}$, $a_j \preceq a_{j'}$ ($a_{j'}$ prefer to a_j) if and only if $R(a_j) = (s_{jl'}^i, \alpha_j^i) \preceq R(a_{j'}) = (s_{jl''}^i, \alpha_{j'}^i)$ if and only if $\Delta^{-1}(s_{jl'}^i, \alpha_j^i) \leq \Delta^{-1}(s_{jl''}^i, \alpha_{j'}^i)$ (Herrera and Martínez 2000), and the best alternatives is

$$R_{max} = \{a_j | R(a_j) = \max\{R(a_1), \dots, R(a_n)\}\}. \quad (20)$$

According to Fig. 2 and Eqs. (6)-(20), Algorithm 1 can be designed to carry out LS-MLDM problems.

Algorithm 1 Carrying out LS-MLDM problems

Input n, r, m, g and D_{ik} .

Output Solution set of the best alternatives

- 1) **for** $j := 1 : n$ **do**
- 2) **for** $k := 1 : r$ **do**
- 3) **for** $i := 1 : m$ **do**
- 4) **for** $l := 0 : g$ **do**
- 5) $u_{jl}^{ik} = \frac{m_{jl}^{ik}}{\sum_{l=0}^g m_{jl}^{ik}}$ (Eq. (12)), $E_j^{ik} = \Delta(\sum_{l=0}^g (u_{jl}^{ik} \times l))$ (Eq. (14))
- 6) **end for**
- 7) $E_j^k = \Delta(\sum_{i=1}^m (w_i \Delta^{-1}(E_j^{ik})))$ (Eq. (16))
- 8) **end for**
- 9) $E_j^i = \Delta(\sum_{k=1}^r (\omega_k \Delta^{-1}(E_j^k)))$ (Eq. (17))
- 10) **end for**
- 11) $R(a_j) = \Delta(\sum_{k=1}^r (\omega_k \Delta^{-1}(E_j^k)))$ (Eq. (18)) or $R(a_j) = \Delta(\sum_{i=1}^m (w_i \Delta^{-1}(E_j^i)))$ (Eq. (19))
- 12) **end for**
- 13) **sort** $\{R(a_1), \dots, R(a_n)\}$ with the descending order of $R(a_j)$
- 14) return the best alternatives R_{max} (Eq. (20)).

5 Case study

Shared economy is significant in Chinese market, which has continuously permeated and changed ourselves traditional lifestyle over the past several years. As a most representative form of shared economy, shared bikes are improving the efficiency of transport and solving the problem of scarcity of

transportation resources (Tian et al. 2018). In the section, user experience of shared bikes is utilized to show the proposed method and compare with decision making methods with 2-TLMs (Herrera and Martínez 2000), HFLTSs (Rodríguez et al. 2012), PLTSs (Pang et al. 2016) and FLM-DFNs (Masanet et al. 2014), respectively.

5.1 Multi-linguistic assessments of user experience of shared bikes

Shared bikes are useful and effective schemes to solve short trips, many shared bikes are rapidly appeared over the past several years, such as “Meituanbike” (a_1), “hellobike” (a_2) and “green orange” (a_3) in Chengdu city. Because users of shared bikes may be anyone of individuals with different knowledge level, user experiences of shared bikes become a LS-MLDM problem. In the example, “Fund Security” (c_1), “Riding Safety” (c_2), “Riding Convenience” (c_3), “Riding Comfort” (c_4), “Perfect Emergency Response Mechanism” (c_5) and “Sustainability” (c_6) (Tian et al. 2018) are criteria to finish user experiences of $\{a_1, a_2, a_3, a_4\}$, linguistic terms $S_6 = \{s_0$ (very poor), s_1 (poor), s_2 (slightly poor), s_3 (fair), s_4 (slightly good), s_5 (good), s_6 (very good)) are utilized by users to evaluate user experiences of shared bikes $\{a_1, a_2, a_3, a_4\}$ with respect to criteria $\{c_1, c_2, c_3, c_4, c_5, c_6\}$. According to the evaluation-based voting, users are partitioned by users’ knowledge level (social context or experience), users in the same partition have the similar knowledge level (social context or experience) and utilize the same fuzzy linguistic representation model to provide evaluation information about user experiences of shared bikes. In the example, three partitions (students, office workers and citizens) are employed, and every partition consists of more than 20 users to provide evaluation information. For the sake of simplicity, one student d_1 , office worker d_2 and citizen d_3 are selected from partitions, respectively, and their evaluation information about user experiences of shared bikes is shown follows:

1) User experiences of shared bikes provided by student d_1 are

$$D_{11} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 1 & 0.6 \\ 0 & 0 & 0 & 0 & 0.3 & 1 & 0.7 \end{pmatrix} \end{matrix}$$

$$D_{12} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0.4 & 1 & 0.6 \\ 0 & 0 & 0 & 0 & 0.2 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 1 & 0.7 \end{pmatrix} \end{matrix}$$

$$D_{13} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

$$D_{14} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

$$D_{15} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 1 & 0.7 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0.6 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0.7 \end{pmatrix} \end{matrix}$$

$$D_{16} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

where D_{1i} ($i = 1, 2, 3, 4, 5, 6$) provided by student d_1 is linguistic assessments of user experiences of shared bikes with respect to criterion c_i .

2) User experiences of shared bikes provided by office worker d_2 are

$$D_{21} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

$$D_{22} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

$$D_{23} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

$$D_{24} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

$$D_{25} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

$$D_{26} = \begin{matrix} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Table 1 User experiences of shared bikes with respect to c_1, c_2, c_3, c_4, c_5 and c_6

	E_j^1	E_j^2	E_j^3	E_j^4	E_j^5	E_j^6	$R(a_j)$
a_1	$(s_5, 0.10)$	$(s_6, -0.30)$	$(s_5, 0.17)$	$(s_6, -0.17)$	$(s_6, -0.25)$	$(s_5, 0.30)$	$(s_5, 0.48)$
a_2	$(s_5, 0.03)$	$(s_5, 0.40)$	$(s_5, 0.33)$	$(s_5, 0.17)$	$(s_5, 0.01)$	$(s_3, 0.33)$	$(s_5, -0.12)$
a_3	$(s_5, 0.23)$	$(s_5, 0.40)$	$(s_5, 0.00)$	$(s_6, -0.50)$	$(s_6, -0.31)$	$(s_6, 0.00)$	$(s_5, 0.47)$

3) User experiences of shared bikes provided by citizen d_3 are

$$D_{31} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$D_{32} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$D_{33} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

$$D_{34} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$D_{35} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

$$D_{36} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Accordingly, d_1, d_2 and d_3 provide multi-linguistic decision matrices to express their user experiences of shared bikes. Using Algorithm 1, user experience of each shared bike can be obtained as follows:

- (1) Normalization of user experiences of shared bikes. Using Eq.(12), each D_{ik} is converted into trust degree linguistic decision matrix D'_{ik} ($i = 1, 2, 3, k = 1, 2, 3, 4, 5, 6$), for example,

$$D'_{11} = \begin{matrix} s_0 s_1 s_2 s_3 s_4 s_5 s_6 \\ a_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0.8 & 0 \end{pmatrix}, \\ a_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0.5 & 0.3 \end{pmatrix}, \\ a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0.15 & 0.5 & 0.35 \end{pmatrix} \end{matrix}$$

Table 2 User experiences of shared bikes provided by d_1, d_2 and d_3

	E_j^1	E_j^2	E_j^3	$R(a_j)$
a_1	$(s_5, 0.17)$	$(s_6, -0.10)$	$(s_5, 0.37)$	$(s_5, 0.48)$
a_2	$(s_5, 0.22)$	$(s_5, 0.08)$	$(s_4, 0.33)$	$(s_5, -0.12)$
a_3	$(s_5, 0.41)$	$(s_6, -0.17)$	$(s_5, 0.17)$	$(s_5, 0.47)$

- (2) Aggregation of user experiences of shared bikes in each D'_{ik} ($i = 1, 2, 3, k = 1, 2, 3, 4, 5, 6$). Using Eq.(14), 2-tuple fuzzy linguistic decision matrices D_1, D_2 and D_3 of d_1, d_2 and d_3 can be obtained, respectively.

$$D_1 = \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ a_1 \begin{pmatrix} (s_5, -0.2) & (s_5, 0.1) & s_5 & (s_6, -0.5) & (s_5, 0.26) & s_5 \end{pmatrix}, \\ a_2 \begin{pmatrix} (s_5, 0.1) & (s_5, 0.2) & s_3 & (s_6, -0.5) & (s_5, 0.04) & s_5 \end{pmatrix}, \\ a_3 \begin{pmatrix} (s_5, 0.2) & (s_5, 0.2) & (s_6, -0.5) & (s_6, -0.5) & (s_5, 0.08) & s_6 \end{pmatrix} \end{matrix}$$

$$D_2 = \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ a_1 \begin{pmatrix} (s_6, -0.5) & s_6 & (s_6, -0.5) & s_6 & s_6 & s_6 \end{pmatrix}, \\ a_2 \begin{pmatrix} s_5 & s_5 & (s_6, -0.5) & s_5 & s_5 & s_5 \end{pmatrix}, \\ a_3 \begin{pmatrix} (s_6, -0.5) & s_6 & (s_6, -0.5) & s_6 & s_6 & s_6 \end{pmatrix} \end{matrix}$$

$$D_3 = \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ a_1 \begin{pmatrix} s_5 & s_6 & s_5 & s_5 & s_6 & s_5 \end{pmatrix}, \\ a_2 \begin{pmatrix} s_5 & s_6 & s_5 & s_5 & s_5 & s_0 \end{pmatrix}, \\ a_3 \begin{pmatrix} s_5 & s_5 & s_4 & s_5 & s_6 & s_6 \end{pmatrix} \end{matrix}$$

Suppose users d_1, d_2 and d_3 (or criteria c_1, c_2, c_3, c_4, c_5 and c_6) with the same weight information, according to Eqs. (16)–(19), user experience of each shared bike can be obtained, which are shown in Tables 1 (considering weights of d_1, d_2 and d_3) and 2 (considering weights of c_1, c_2, c_3, c_4, c_5 and c_6).

- (3) Exploitation of user experiences of shared bikes provided by student d_1 , office worker d_2 and citizen d_3 with respect to the six criteria. According to Table 1 or 2, $R(a_1) = \max\{R(a_1), R(a_2), R(a_3)\} = (s_5, 0.48)$ due to $(s_5, 0.48) > (s_5, 0.47) > (s_5, -0.12)$, i.e., user experience of “meituanbike” is the best one in $\{a_1, a_2, a_3\}$.

5.2 Comparative analyses

In the subsection, the proposed LS-MLDM method for user experiences of shared bikes is compared with LDM methods with 2-TLM, HFLTS, PLTS and FLM-DFN assessments of alternatives. Because user experiences of shared bikes pro-

Table 3 The LDM methods to evaluate user experiences provided by d_1 , d_2 and d_3

Methods	D_{11}, D_{12}, D_{15}	$D_{13}, D_{14}, D_{21}, D_{23}$	$D_{16}, D_{22}, D_{24} \sim D_{26}, D_{31} \sim D_{36}$
2-TLM			✓
HFLTS		✓	
PLTS	✓		
FLM-DFN	✓		

Table 4 Decision-making table provided by d_1 , d_2 and d_3 with respect to criterion c_6

	linguistic term of d_1	linguistic term of d_2	linguistic term of d_3
a_1	s_5	s_6	s_5
a_2	s_5	s_5	$\emptyset$
a_3	s_6	s_6	s_6

vided by d_1 , d_2 and d_3 are multi-linguistic decision matrices, LDM methods with 2-TLM, HFLTS, PLTS and FLM-DFN assessments of alternatives can partly carry out user experiences of shared bikes. Table 3 points out that the LDM methods can be utilized to evaluate user experiences of shared bikes with respect to c_1 , c_2 , c_3 , c_4 , c_5 and c_6 .

Based on Table 3, user experiences of shared bikes are carried out as follows:

(1) For criterion c_6 (Sustainability), d_1 , d_2 and d_3 provide single-linguistic terms to assess user experiences, hence 2-tuple fuzzy linguistic aggregation operator with weights of d_1 , d_2 and d_3 can be utilized to obtain user experience of each shared bike with respect to c_6 , where decision-making table provided by d_1 , d_2 and d_3 is shown in Table 4. By using LDM method in Herrera and Martínez (2000), user experiences of shared bikes provided by d_1 , d_2 and d_3 with respect to c_6 are shown as follows:

$$\begin{aligned}
 & E^6(D_{16}, D_{26}, D_{36}) \\
 &= \Delta \left(\frac{\Delta^{-1}(D_{16}) + \Delta^{-1}(D_{26}) + \Delta^{-1}(D_{36})}{3} \right) \\
 &= \Delta \left(\frac{1}{3} \left(\begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} \right) \right) \\
 &\doteq \Delta \left(\begin{pmatrix} 5.33 \\ 3.33 \\ 6 \end{pmatrix} \right) = \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} \begin{pmatrix} (s_5, 0.33) \\ (s_3, 0.33) \\ (s_6, 0) \end{pmatrix}.
 \end{aligned}$$

(2) For criteria c_3 (Riding Convenience) and c_4 (Riding Comfort), d_1 and d_2 provides HFLTSs to evaluate user experiences, d_3 provides single-linguistic terms to evaluate user experience, here the LDM method with HFLTSs (Rodríguez et al. 2012) can be utilized to obtain user experiences of shared bikes with respect to c_3 and c_4 , i.e., according to hesitant linguistic decision-making tables provided by d_1 , d_2 and d_3 shown in Tables 5 and 6, for each shared bike, $H(a_j) = [H_{\min}(a_j), H_{\max}(a_j)]$ ($j = 1, 2, 3, 4$) is employed

to aggregate user experiences provided by d_1 , d_2 and d_3 with respect to c_3 and c_4 , in which

$$\begin{aligned}
 H_{\min}(a_j) &= \max\{H_{S^-}^{d_1}(a_j), H_{S^-}^{d_2}(a_j), H_{S^-}^{d_3}(a_j)\}, \\
 H_{\max}(a_j) &= \min\{H_{S^+}^{d_1}(a_j), H_{S^+}^{d_2}(a_j), H_{S^+}^{d_3}(a_j)\}.
 \end{aligned}$$

For example, for shared bike a_1 with respect to c_3 , $H_{\min}(a_1) = \max\{s_5, s_5, s_5\} = s_5$ and $H_{\max}(a_1) = \min\{s_5, s_6, s_5\} = s_5$, hence $H(a_1) = [s_5, s_5]$. Others are shown in Tables 5 and 6.

(3) For criteria c_2 (Riding Safety) and c_5 (Perfect Emergency Response Mechanism), d_1 provides FLM-DFNs to evaluate user experiences of $\{a_1, a_3, a_4\}$ and PLTSs to evaluate user experience of a_2 . However, d_2 and d_3 provide single-linguistic terms to evaluate user experiences of shared bikes, and linguistic decision-making tables provided by d_1 , d_2 and d_3 are shown in Tables 7 and 8. Considering the single-linguistic term is FLM-DFN with membership degree 1 or PLTS with probability 1, here user experiences of shared bikes with respect to c_2 and c_5 can be carried out by using LDM methods with PLTSs or FLM-DFNs, respectively, i.e.,

a) Using the LDM method with FLM-DFNs and aggregation function on the finite chain $S = \{s_0, s_1, \dots, s_6\}$ (Masanet et al. 2014), i.e., $F(x_1, x_2, x_3) = \max\{\min\{x_1, x_2, x_3\}, \max\{x_1, x_2, x_3\} - 3\}$, user experiences of a_1 and a_3 with respect to c_2 are

$$\begin{aligned}
 H_{c_2}(a_1) &= F(\{0.4/s_4, 1/s_5, 0.6/s_6\}, \{1/s_6\}, \\
 &\quad \{1/s_6\}) = \{0.4/s_4, 1/s_5, 0.6/s_6\}, \\
 H_{c_2}(a_3) &= F(\{0.3/s_4, 1/s_5, 0.7/s_6\}, \{1/s_6\}, \\
 &\quad \{1/s_5\}) = \{0.3/s_4, 1/s_5\}.
 \end{aligned}$$

User experiences of a_1 , a_2 and a_3 with respect to c_5 are

$$\begin{aligned}
 H_{c_5}(a_1) &= F(\{0.2/s_4, 1/s_5, 0.7/s_6\}, \{1/s_6\}, \\
 &\quad \{1/s_6\}) = \{0.2/s_4, 1/s_5, 0.7/s_6\}, \\
 H_{c_5}(a_2) &= F(\{0.5/s_4, 1/s_5, 0.6/s_6\}, \{1/s_5\},
 \end{aligned}$$

Table 5 Decision-making table provided by d_1 , d_2 and d_3 with respect to criterion c_3

	Linguistic terms of d_1	Linguistic terms of d_2	Linguistic terms of d_3	$H(a_j)$
a_1	$\{s_5\}$	$\{s_5, s_6\}$	$\{s_5\}$	$[s_5, s_5]$
a_2	$\{s_5, s_6\}$	$\{s_5, s_6\}$	$\{s_5\}$	$[s_5, s_5]$
a_3	$\{s_5, s_6\}$	$\{s_5, s_6\}$	$\{s_4\}$	$[s_4, s_4]$

Table 6 Decision-making table provided by d_1 , d_2 and d_3 with respect to criterion c_4

	Linguistic terms of d_1	Linguistic terms of d_2	Linguistic terms of d_3	$H(a_j)$
a_1	$\{s_5, s_6\}$	$\{s_6\}$	$\{s_5\}$	$[s_5, s_6]$
a_2	$\{s_5, s_6\}$	$\{s_5\}$	$\{s_5\}$	$[s_5, s_5]$
a_3	$\{s_5, s_6\}$	$\{s_6\}$	$\{s_5\}$	$[s_5, s_6]$

Table 7 Decision-making table provided by d_1 , d_2 and d_3 with respect to criterion c_2

	Linguistic terms of d_1	Linguistic terms of d_2	Linguistic terms of d_3
a_1	$\{0.4/s_4, 1/s_5, 0.6/s_6\}$	$\{1/s_6\}$	$\{1/s_6\}$
a_2	$\{s_4(0.2), s_5(0.4), s_6(0.4)\}$	$\{s_5(1)\}$	$\{s_6(1)\}$
a_3	$\{0.3/s_4, 1/s_5, 0.7/s_6\}$	$\{1/s_6\}$	$\{1/s_5\}$

$$\begin{aligned} \{1/s_5\} &= \{0.5/s_4, 1/s_5\}, \\ H_{c_5}(a_3) &= F(\{0.5/s_4, 1/s_5, 0.7/s_6\}, \{1/s_6\}, \\ \{1/s_6\}) &= \{0.5/s_4, 1/s_5, 0.7/s_6\}. \end{aligned}$$

b) Using the LDM method with PLTSs and the PLWA operator (Pang et al. 2016)

$$\begin{aligned} Z_j(w) &= w_1 L_{j1}(p) \oplus \dots \oplus w_n L_{jn}(p) \\ &= \cup_{L_{j1}^k \in L_{j1}(p)} \{w_1 p_{j1}^k L_{j1}^k\} \oplus \dots \oplus \cup_{L_{jn}^k \in L_{jn}(p)} \{w_n p_{jn}^k L_{jn}^k\}, \end{aligned}$$

user experiences of a_2 with respect to c_2 are

$$\begin{aligned} Z_2^{c_2} \left(\left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\} \right) &= \frac{1}{3} \{s_6(0.4), s_5(0.4), s_4(0.2)\} \oplus \frac{1}{3} \{s_5(1), \\ s_5(0), s_5(0)\} &\oplus \frac{1}{3} \{s_6(1), s_6(0), \\ s_6(0)\} &= \{s_{4.47}, s_{0.67}, s_{0.27}\}. \end{aligned}$$

(4) For criterion c_1 (Fund Security), d_1 , d_2 and d_3 provide single-linguistic term, HFLTS, PLTS or FLM-DFN to evaluate user experiences of shared bikes, respectively, and linguistic decision-making table provided by d_1 , d_2 and d_3 is shown in Table 9. Similarly, user experiences of shared bikes can be obtained as follows: User experience of a_1 is

$$\begin{aligned} Z_1^{c_1} \left(\left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\} \right) &= \frac{1}{3} \{s_5(0.8), s_4(0.2)\} \oplus \frac{1}{3} \{s_6(0.5), \\ s_5(0.5)\} &\oplus \frac{1}{3} \{s_5(1), s_5(0)\} = \{s_4, s_{1.1}\}, \end{aligned}$$

here HFLTS $\{s_5, s_6\}$ provided by d_2 is equal to PLTS $\{s_6(0.5), s_5(0.5)\}$. User experiences of a_2 and a_3 are

$$\begin{aligned} H_{c_1}(a_2) &= F(\{0.4/s_4, 1/s_5, 0.6/s_6\}, \{1/s_5\}, \\ \{1/s_5\}) &= \{0.4/s_4, 1/s_5\}, \\ H_{c_1}(a_3) &= F(\{0.3/s_4, 1/s_5, 0.7/s_6\}, \\ \{1/s_5, 1/s_6\}, \{1/s_5\}) &= \{0.3/s_4, 1/s_5\}, \end{aligned}$$

here HFLTS $\{s_5, s_6\}$ and $\{s_5\}$ provided by d_2 and d_3 are equal to FLM-DFNs $\{1/s_5, 1/s_6\}$ and $\{1/s_5\}$, respectively.

Based on the above-mentioned steps (1)-(4), assessment results of user experiences of shared bikes with respect to criteria c_1, c_2, c_3, c_4, c_5 and c_6 are shown in Table 10. Accordingly, it seems that we can do nothing, because linguistic assessment results of user experiences in Table 10 are very different; for example, user experience of a_1 with respect to c_1 is PLTS $\{s_4, s_{1.1}\}$, user experiences of a_1 with respect to c_2 and c_5 are FLM-DFNs $\{0.4/s_4, 1/s_5, 0.6/s_6\}$ and $\{0.2/s_4, 1/s_5, 0.7/s_6\}$, respectively, user experiences of a_1 with respect to c_3 and c_4 are HFLTSs $\{s_5, s_5\}$ and $\{s_5, s_6\}$,

Table 8 Decision-making table provided by d_1 , d_2 and d_3 with respect to criterion c_5

	Linguistic terms of d_1	Linguistic terms of d_2	Linguistic terms of d_3
a_1	$\{0.2/s_4, 1/s_5, 0.7/s_6\}$	$\{1/s_6\}$	$\{1/s_6\}$
a_2	$\{0.5/s_4, 1/s_5, 0.6/s_6\}$	$\{1/s_5\}$	$\{1/s_5\}$
a_3	$\{0.5/s_4, 1/s_5, 0.7/s_6\}$	$\{1/s_6\}$	$\{1/s_6\}$

Table 9 Decision-making table provided by d_1 , d_2 and d_3 with respect to criterion c_1

	Linguistic terms of d_1	Linguistic terms of d_2	Linguistic terms of d_3
a_1	$\{s_4(0.2), s_5(0.8)\}$	$\{s_5, s_6\}$	$\{s_5\}$
a_2	$\{0.4/s_4, 1/s_5, 0.6/s_6\}$	$\{s_5\}$	$\{s_5\}$
a_3	$\{0.3/s_4, 1/s_5, 0.7/s_6\}$	$\{s_5, s_6\}$	$\{s_5\}$

Table 10 Assessment results of user experiences of shared bikes with respect to criteria

	a_1	a_2	a_3
c_1	$\{s_4, s_{1.1}\}$	$\{0.4/s_4, 1/s_5\}$	$\{0.3/s_4, 1/s_5\}$
c_2	$\{0.4/s_4, 1/s_5, 0.6/s_6\}$	$\{s_{4.47}, s_{0.67}, s_{0.27}\}$	$\{0.3/s_4, 1/s_5\}$
c_3	$[s_5, s_5]$	$[s_5, s_5]$	$[s_4, s_4]$
c_4	$[s_5, s_6]$	$[s_5, s_5]$	$[s_5, s_6]$
c_5	$\{0.2/s_4, 1/s_5, 0.7/s_6\}$	$\{0.5/s_4, 1/s_5\}$	$\{0.5/s_4, 1/s_5, 0.7/s_6\}$
c_6	$(s_5, 0.33)$	$(s_3, 0.33)$	$(s_6, 0)$

Table 11 Comparing analysis of LDM methods in carrying out LDM problems

LDM problems	M_1	M_2	M_3	M_4	M_5
With single-linguistic terms	✓	✓	✓	✓	✓
With HFLTSS		✓	Partly	✓	✓
With PLTSS			✓		✓
With FLM-DFNs				✓	✓
With multi-linguistic terms	Partly	Partly	Partly	Partly	✓

respectively, user experience of a_1 with respect to c_6 is 2-TLM $(s_5, 0.33)$. Up to now, there is no any linguistic aggregation operator can be utilized to simultaneously aggregate them.

Summary, Table 11 shows differences among LDM methods, in which M_1 , M_2 , M_3 and M_4 are LDM methods with 2-TLMs, HFLTSS, PLTSS and FLM-DFNs, M_5 is the proposed LS-MLDM method with multi-linguistic assessments. It is obvious that M_1 , M_2 , M_3 and M_4 are useful and effective tools for dealing with LDM problems with 2-TLMs, HFLTSS, PLTSS and FLM-DFNs, respectively. However, they are invalid when linguistic assessments of alternatives are multi-linguistic terms. In LS-MLDM environment, multi-linguistic assessments widely exist because a huge amounts of decision makers employ several fuzzy linguistic representation models to assess alternatives. Unlike M_1 , M_2 , M_3 and M_4 , the proposed LS-MLDM method (M_5) is based on multi-linguistic decision matrix and trust degrees of linguistic terms, which can be utilized to not only handle LDM problems with 2-TLMs, HFLTSS, PLTSS or FLM-DFNs, but also carry out LS-MLDM problems.

6 Conclusions and future works

LS-MLDM problems are widely existed in large-scale linguistic decision-making environments because decision

makers maybe utilize several fuzzy linguistic representation models to evaluate alternatives with respect to criteria. To deal with LS-MLDM problems, evaluation-based voting is defined and multi-linguistic decision matrix as well as trust degrees of linguistic terms are developed in the paper, and the new decision-making framework of LS-MLDM problems is designed. Theoretically, multi-linguistic decision matrix and trust degrees of linguistic terms aim to uniformly represent multi-linguistic assessments of alternatives and normalize multi-linguistic decision matrix; based on trust degrees of linguistic terms, an uniform fusion method of multi-linguistic assessments is proposed to aggregate multi-linguistic assessments of alternatives with respect to criterion. Comparing analysis in user experience of shared bikes shows that the proposed method is an useful and effective tool to carry out not only classical LDM problems but also LS-MLDM problems.

In reality, huge amounts of decision makers maybe produce many problems besides multi-linguistic assessments of alternatives, such as conflict assessments of alternatives. To deal with the problem, the study of consensus-based decisions in the past 10 years has become a consensus issue of LDM, which is to guide decision makers to reach a consensus before making a decision and obtain acceptable linguistic decision results of alternatives (Labella et al. 2018; Rodriguez et al. 2018; Li and Dong 2019). In multi-linguistic decision matrices, multi-linguistic assessments of alterna-

tives provided by decision makers may be conflict, “how to analyze consensus in LS-MLDM problems” will be our future works. In addition, decision makers maybe utilize multi-granular linguistic terms or numerical and linguistic information to assess alternatives with respect to criteria, “how to manage, transform and deal with numerical, linguistic or multi-granular linguistic assessments of alternatives” may be interesting and important issues in LS-MLDM problems.

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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References

- Bai C, Zhang R, Qian L et al (2017) Comparisons of probabilistic linguistic term sets for multi-criteria decision making. *Knowl-Based Syst* 119:284–291
- Chao XR, Kou G, Peng Y, Herrera-Viedma E (2021) Large-scale group decision-making with non-cooperative behaviors and heterogeneous preferences: an application in financial inclusion. *Eur J Oper Res* 288:271–293
- Chen ZS, Chin KS et al (2017) Generating HFLTS possibility distribution with an embedded assessing attitude. *Information Sci* 394–395:141–166
- Chen ZS, Chin KS, Li YL, Yang Y (2016) Proportional hesitant fuzzy linguistic term set for multiple criteria group decision making. *Information Sci* 357:61–87
- Chen ZS, Martínez L, Chin KS, Tusi KL (2018) Two-stage aggregation paradigm for HFLTS possibility distributions: a hierarchical clustering perspective. *Expert Syst Appl* 104:43–66
- Chen ZS, Chin KS, Martínez L, Tusi KL (2018) Customizing semantics for individuals with attitudinal HFLTS possibility distributions. *IEEE Transactions Fuzzy Syst* 26(6):3452–3466
- Chen ZS, Chin KS, Tusi KL (2019) Constructing the geometric Bonferroni mean from the generalized Bonferroni mean with several extensions to linguistic 2-tuples for decision-making. *Appl Soft Comput* 78:595–613
- Ding RX, Palomares I et al (2020) Large-Scale decision-making: characterization, taxonomy, challenges and future directions from an artificial intelligence and applications perspective. *Information Fus* 59:84–102
- Dong YC, Xu YF, Yu S (2009) Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model. *IEEE Transactions Fuzzy Syst* 17(6):1366–1378
- Dong YC, Zhang GQ, Hong WC, Xu YF (2013) Linguistic computing model based on 2-tuples and intervals. *IEEE Transactions Fuzzy Syst* 21:1006–1018
- Dong YC, Li CC, Herrera F (2016) Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. *Information Sci* 367–368:259–278
- Feng XQ, Zhang Q, Jin LS (2020) Aggregation of pragmatic operators to support probabilistic linguistic multi-criteria group decision-making problems. *Soft Comput* 24:7735–7755
- Filos-Ratsikas A, Micha E, Voudouris AA (2020) The distortion of distributed voting. *Artif Intell* 286:103343
- Gou XJ, Xu ZS, Liao HC (2017) The hesitant fuzzy linguistic possibility degree-based linear assignment method for multiple criteria decision making. *Int J Information Technol Decis Mak* 16:1–29
- Gou XJ, Liao HC, Xu ZS, Herrera F (2017) Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: a case of study to evaluate the implementation status of haze controlling measures. *Information Fus* 38:22–34
- Herrera F, Martínez L (2000) A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions Fuzzy Syst* 8(6):746–752
- Herrera F, Herrera-Viedma E, Martínez L (2008) A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions Fuzzy Syst* 16(2):354–370
- Jiang LS, Liao HC (2020) Mixed fuzzy least absolute regression analysis with quantitative and probabilistic linguistic information. *Fuzzy Sets Syst* 387:35–48
- Krishankumar R, Ravichandran KS, Kar S, Gupta P, Mehla MK (2021) Double-hierarchy hesitant fuzzy linguistic term set-based decision framework for multi-attribute group decision-making. *Soft Comput* 25:2665–2685
- Labella A, Liu Y, Rodriguez RM, Martinez L (2018) Analyzing the performance of classical consensus models in large scale group decision making: a comparative study. *Appl Soft Comput* 67:677–690
- Lawry J (1998) A voting mechanism for fuzzy logic. *Int J Approx Reason* 19:315–333
- Li CC, Dong YC et al (2019) An overview on managing additive consistency of reciprocal preference relations for consistency-driven decision making and fusion: Taxonomy and future directions. *Information Fus* 52:143–156
- Li CC, Dong YC, Herrera F, Herrera-Viedma E, Martínez L (2017) Personalized individual semantics in computing with words for supporting linguistic group decision making. *Information Fus* 33:29–40
- Li CC, Rodríguez RM, Martínez L, Dong Y, Herrera F (2018) Personalized individual semantics based on consistency in hesitant linguistic group decision making with comparative linguistic expressions. *Knowl-Based Syst* 145:156–165
- Li CC, Dong YC, Herrera F (2019) A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups. *IEEE Transactions Fuzzy Syst* 27(2):221–233

- Li Y, Chen X, Dong YC, Herrera F (2020) Linguistic group decision making: Axiomatic distance and minimum cost consensus. *Information Sci* 541:242–258
- Liao HC, Xu ZS (2015) Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSS and their application in qualitative decision making. *Expert Syst Appl* 42:5328–5336
- Liao HC, Xu ZS, Zeng XJ (2015) Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making. *IEEE Transactions Fuzzy Syst* 23(5):1343–1355
- Liu PD, Teng F (2019) Probabilistic linguistic TODIM method for selecting products through online product reviews. *Information Sci* 485:441–455
- Liu YY, Rodríguez RM, Hagaras H et al (2019) Type-2 fuzzy envelope of hesitant fuzzy linguistic term set: a new representation model of comparative linguistic expression. *IEEE Transactions Fuzzy Syst* 27(12):2312–2326
- Ma XY, Zhao M, Zou X (2019) Measuring and reaching consensus in group decision making with the linguistic computing model based on discrete fuzzy numbers. *Appl Soft Comput* 77:135–154
- Malhotra T, Gupta A (2020) A systematic review of developments in the 2-tuple linguistic model and its applications in decision analysis. *Soft Comput* (<https://doi.org/10.1007/s00500-020-05031-2>)
- Mao XB, Wu M, Dong JY et al (2019) A new method for probabilistic linguistic multi-attribute group decision making: application to the selection of financial technologies. *Appl Soft Comput* 77:155–175
- Martínez L, Herrera F (2012) An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges. *Information Sci* 207(1):1–18
- Martínez L, Rodríguez RM, Herrera F (2015) The 2-tuple linguistic model-computing with words in decision making. Springer, Switzerland
- Massanet S, Riera JV, Torrens J, Herrera-Viedma E (2014) A new linguistic computational model based on discrete fuzzy numbers for computing with words. *Information Sci* 258:277–290
- Morente-Molinera JA, Pérez IJ, Ureña MR, Herrera-Viedma E (2015) On multi-granular fuzzy linguistic modeling in group decision making problems: A systematic review and future trends. *Knowl-Based Syst* 74:49–60
- Pang Q, Wang H, Xu ZH (2016) Probabilistic linguistic term sets in multi-attribute group decision making. *Information Sci* 369:128–143
- Pei Z, Ruan D, Liu J, Xu Y (2012) A linguistic aggregation operator with three kinds of weights for nuclear safeguards evaluation. *Knowl-Based Syst* 28:19–26
- Pei Z, Liu J, Hao F, Zhou B (2019) FLM-TOPSIS: the fuzzy linguistic multiset TOPSIS method and its application in linguistic decision making. *Information Fus* 45:266–281
- Riera JV, Torrens J (2014) Aggregation functions on the set of discrete fuzzy numbers defined from a pair of discrete aggregations. *Fuzzy Sets Syst* 241:76–93
- Riera JV, Torrens J (2015) Using discrete fuzzy numbers in the aggregation of incomplete qualitative information. *Fuzzy Sets Syst* 264:121–137
- Riera JV, Massanet S, Herrera-Viedma E, Torrens J (2015) Some interesting properties of the fuzzy linguistic model based on discrete fuzzy numbers to manage hesitant fuzzy linguistic information. *Appl Soft Comput* 36:383–391
- Rodríguez RM, Martínez L, Herrera F (2012) Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions Fuzzy Syst* 20(1):109–119
- Rodríguez RM, Martínez L, Herrera F (2013) A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Information Sci* 241(1):28–42
- Rodríguez RM, Martínez L, Torra V, Xu ZS, Herrera F (2014) Hesitant fuzzy sets: state of the art and future directions. *Int J Intell Syst* 29(6):495–524
- Rodríguez RM, Labella A, Martínez L (2016) An overview on fuzzy modelling of complex linguistic preferences in decision making. *Int J Comput Intell Syst* 9(1):81–94
- Rodríguez RM, Labella A, Tre GD, Martínez L (2018) A large scale consensus reaching process managing group hesitation. *Knowl-Based Syst* 159(1):86–97
- Roszkowska E, Kacprzak D (2016) The fuzzy saw and fuzzy TOPSIS procedures based on ordered fuzzy numbers. *Information Sci* 369:564–584
- Tian ZP, Wang JQ, Wang J, Zhang HY (2018) A multi-phase QFD-based hybrid fuzzy MCDM approach for performance evaluation: a case of smart bike-sharing programs in Changsha. *J Clean Prod* 171:1068–1083
- Voxman W (2001) Canonical representations of discrete fuzzy numbers. *Fuzzy Sets Syst* 118(3):457–466
- Wang JH, Hao JY (2006) A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions Fuzzy Syst* 143(3):435–445
- Wang H, Xu ZH, Zeng XJ (2018) Modeling complex linguistic expressions in qualitative decision making: an overview. *Knowl-Based Syst* 144:174–187
- Wang J, Wang HW, Zhao M, Cao JN, Li Z, Guo M (2020) Joint Topic-Semantic-aware social matrix Factorization for online voting recommendation. *Knowl-Based Syst* 210:106433
- Wei CP, Rodríguez RM, Martínez L (2018) Uncertainty measures of extended hesitant fuzzy linguistic term sets. *IEEE Transactions Fuzzy Syst* 26:1763–1768
- Wu ZB, Xu JP (2016) Possibility distribution-based approach for MAGDM with hesitant fuzzy linguistic information. *IEEE Transactions Cybernet* 46(3):694–705
- Wu ZB, Xu JP (2018) A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters. *Information Fus* 41:217–231
- Wu P, Zhou LG, Chen HY, Tao ZF (2019) Additive consistency of hesitant fuzzy linguistic preference relation with a new expansion principle for hesitant fuzzy linguistic term sets. *IEEE Transactions Fuzzy Syst* 27(4):716–730
- Wu ZB, Xu JP, Jiang XL, Zhong L (2019) Two MAGDM models based on hesitant fuzzy linguistic term sets with possibility distributions: VIKOR and TOPSIS. *Information Sci* 473:101–120
- Wu YZ, Dong YC, Qin JD, Pedrycz W (2020) Flexible linguistic expressions and consensus reaching with accurate constraints in group decision-making. *IEEE Transactions Cybernet* 50(6):2488–2501
- Wu YZ, Zhang Z, Kou G et al (2021) Distributed linguistic representations in decision making: taxonomy. *Key Elem Appl Chall Data Sci Explain Artif Intell Information Fus* 65:165–178
- Xu YJ, Cabrerizo FJ, Herrera-Viedma E (2017) A consensus model for hesitant fuzzy preference relations and its application in water allocation management. *Appl Soft Comput* 58:265–284
- Yan L, Pei Z, Ren FL (2019) Constructing and managing multi-granular linguistic values based on linguistic terms and their fuzzy sets. *IEEE Access* 7:152928–152943
- Yang XW, Liang C et al (2017) Collaborative filtering-based recommendation of online social voting. *IEEE Transactions Computational Soc Syst* 4(1):1–13
- Yang Q, Li YL, Chin KS (2019) Constructing novel operational laws and information measures for proportional hesitant fuzzy linguistic term sets with extension to phfl-vikor for group decision making. *Int J Comput Intell Syst* 12(2):998–1018
- Zadeh LA (1996) Fuzzy logic = computing with words. *IEEE Transactions Fuzzy Syst* 4(2):103–111

- Zhang Z, Guo CH, Martínez L (2017) Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making. *IEEE Transactions Syst* 47(11):3063–3077
- Zhao M, Liu MY, Su J, Liu T (2019) A shape similarity-based ranking method of hesitant fuzzy linguistic preference relations using discrete fuzzy number for group decision making. *Soft Comput* 23:13569–13589
- Zhong X, Xu XH (2020) Clustering-based method for large group decision making with hesitant fuzzy linguistic information: Integrating correlation and consensus. *Appl Soft Comput* 87:105973

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