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FOUNDATIONS

Some generalizations of p-semisimple BCI algebras and groups

Andrzej Walendziak[1]

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Abstract

We introduce and investigate the strong p-semisimple property for some generalizations of BCI algebras. For BCI algebras, the strong p-semisimple property is equivalent to the p-semisimple property. We describe the connections of strongly p-semisimple algebras and various generalizations of groups (such as, for example, involutive moons and goops). Moreover, we present some examples of proper strongly p-semisimple algebras.

Keywords RM, aRM, aRM**, BZ, BCI algebra · Moon · Goop · p-semisimplicity

1 Introduction

Iséki (1966) introduced BCI algebras as algebraic models of BCI-logic. Hu and Li (1983) defined BCH algebras, which are a generalization of BCI algebras. Later on, Ye (1991) introduced the notion of BZ algebras. It is known that BCI algebras are contained in the class of BZ algebras. Next, the new class of algebras called BH algebras was introduced by Jun et al. (1998). These algebras are a common generalization of BCH and BZ algebras (hence also a generalization of BCI algebras). Recently, Iorgulescu (2016) introduced new generalizations of BCI algebras such as aRM**, *aRM**, BCH** algebras, and many others. All of the algebras mentioned above are contained in the class of RM algebras (a RM algebra is an algebra $(A; \to, 1)$ of type $(2, 0)$ satisfying the equations: $x \to x = 1$ and $1 \to x = x$). The implicative and commutative properties for some subclasses of the class of RM algebras were studied by Walendziak (2018, 2019). Lei and Xi (1985) defined p-semisimple BCI algebras and proved that p-semisimple BCI algebras are equivalent with abelian groups. The p-semisimple BCI algebras have been extensively investigated in many papers (for example Meng (1987), Hoo (1990), Aslam and Thaheem (1991), Zhang (1991), Kim and Park (2005)). Zhang and Ye (1995) showed that p-semisimple BZ algebras are equivalent with groups.

In this paper, we introduce the notion of strongly p-semisimple RM algebra. For BZ and BCI algebras, the strong p-semisimple property is equivalent to the p-semisimple property. We describe the connections of strongly p-semisimple algebras and various generalizations of groups (such as, for example, involutive moons and goops, which were introduced by Iorgulescu (2018)). In particular, we prove that strongly p-semisimple RM algebras are equivalent with involutive moons. Moreover, we give some examples of proper strongly p-semisimple algebras.

2 Preliminaries

2.1 Generalizations of BCI algebras

Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type $(2, 0)$. We consider the following list of properties (cf. Iorgulescu 2016) that can be satisfied by $\mathcal{A}$:

(An) $x \to y = 1 = y \to x \Longrightarrow x = y$,
(B) $(y \to z) \to [(x \to y) \to (x \to z)] = 1$,
(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(D) $x \to ((x \to y) \to y) = 1$,
(D1) $x \to ((x \to 1) \to 1) = 1$,
(Ex) $x \to (y \to z) = y \to (x \to z)$,
(M) $1 \to x = x$,
(Re) $x \to x = 1$,
(*) $y \to z = 1 \Longrightarrow (x \to y) \to (x \to z) = 1$,

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✉ Andrzej Walendziak
walent@interia.pl

[1] Faculty of Exact and Natural Sciences, Institute of Mathematics, Siedlce University of Natural Sciences and Humanities, 3 Maja 54, 08110 Siedlce, Poland

(**) $y \to z = 1 \Longrightarrow (z \to x) \to (y \to x) = 1$,
(Tr) $x \to y = 1 = y \to z \Longrightarrow x \to z = 1$.

Lemma 2.1 (Iorgulescu 2016) *Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) (Re) + (Ex) + (An) imply (M),
(ii) (M) + (B) imply (*) and (**),
(iii) (M) + (*) imply (Tr),
(iv) (M) + (**) imply (Tr),
(v) (M) + (BB) imply (B),
(vi) (Re) + (Ex) imply (D),
(vii) (Re) + (Ex) + (*) imply (BB),
(viii) (M) + (BB) + (An) imply (Ex).
(ix) (Ex) + (B) imply (BB).

An algebra $\mathcal{A} = (A; \to, 1)$ is a *BCH algebra* if (Re), (Ex) and (An) are fulfilled, instead a BCH algebra $\mathcal{A}$ is a *BCI algebra* if (B) holds. Additionally, an algebra $\mathcal{A}$ is a *RM algebra* if (Re) and (M) hold. By Lemma 2.1(i), BCH and BCI algebras are particular cases of RM algebras.

Following Iorgulescu (2016), we say that a RM algebra $\mathcal{A}$ is an *aRM algebra* if it satisfies (An). We note that aRM algebras are also called BH algebras [see, for example, Yu et al. (1999), Zhang et al. (2001), Jun et al. (2004)].

Now, we will define the following algebras (Iorgulescu 2016):

- An *aRM** algebra* is an aRM algebra satisfying (**).
- A *BCH** algebra is* a BCH algebra satisfying (**).
- An **aRM** algebra* is an aRM** algebra satisfying (*).
- An *aRM**(D1) algebra* (resp. **aRM**(D1) algebra*) is an aRM** algebra (resp. *aRM** algebra) satisfying (D1).
- A *BZ algebra* is an aRM algebra satisfying (B).

Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\le$ by: for all $x, y \in A$,

$$x \le y \Longleftrightarrow x \to y = 1.$$

In RM algebras, $\le$ is a reflexive relation; in aRM algebras, it is reflexive and antisymmetric. If $\mathcal{A}$ is an aRM** algebra, then $\le$ is also transitive by Lemma 2.1(iv). Therefore, $\le$ is an order relation in aRM** algebras (hence also in aRM** (D1), *aRM**, *aRM**(D1), BZ, BCH**, and BCI algebras).

Lemma 2.2 *Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

(i) (Re) + (Ex) $\Longrightarrow$ (D1),
(ii) (Re) + (M) + (B) $\Longrightarrow$ (D1).

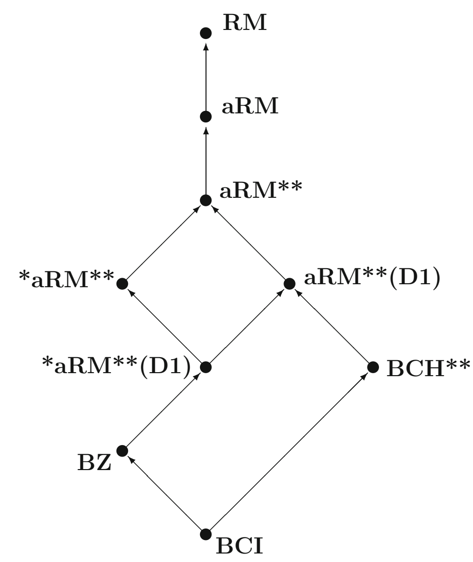

Fig. 1 The hierarchy between **BCI** and **RM**

Proof

(i) This follows easily from Lemma 2.1(vi).
(ii) Let $x \in A$. By (M), (B) and (Re),

$$x = 1 \to x \le (x \to 1) \to (x \to x) = (x \to 1) \to 1,$$

that is, (D1) is satisfied. □

Denote by **RM**, **aRM**, **aRM****, **aRM**(D1)**, ***aRM****, ***aRM**(D1)**, **BZ**, **BCH**,** and **BCI** the classes of RM, aRM, aRM**, aRM**(D1), *aRM**, *aRM**(D1), BZ, BCH**, and BCI algebras, respectively. From the definitions and Lemmas 2.1(ii) and 2.2(ii) it follows that

$$\mathbf{BCI} \subset \mathbf{BZ} \subset \mathbf{{*}aRM{*}{*}(D1)} \subset \mathbf{{*}aRM{*}{*}} \subset \mathbf{aRM{*}{*}} \subset \mathbf{aRM} \subset \mathbf{RM}.$$

By Lemma 2.2(i), **BCH**** $\subset$ **aRM**(D1)**. The interrelationships between the classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if $\mathbf{X}$ and $\mathbf{Y}$ are classes of algebras, then $\mathbf{X} \longrightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

2.2 Generalizations of groups

Iorgulescu (2018) introduced and studied new generalizations of groups such as moons, goops, and many others.

Definition 2.3 A *moon* is an algebra $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ of type $(2, 1, 0)$ satisfying

(U) $\quad x \cdot 1 = x = 1 \cdot x$,

(Iv) $x \cdot x^{-1} = 1 = x^{-1} \cdot x$.

A moon is *involutive* if it satisfies

(DN) $(x^{-1})^{-1} = x$.

A moon is *associative* if it satisfies

(Ass) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

A moon is *commutative* if it satisfies

(Com) $x \cdot y = y \cdot x$.

Note that the associative moon is just the group.

Definition 2.4 (Iorgulescu 2018) A *goop* is an algebra $(G; \cdot, ^{-1}, 1)$ of type (2, 1, 0) satisfying (U) and the following conditions:

$$\text{(GP1)} \quad y \cdot x^{-1} = 1 \Longleftrightarrow x^{-1} \cdot y = 1,$$
$$\text{(GP2)} \quad y \cdot x^{-1} = 1 \Longleftrightarrow x = y.$$

Proposition 2.5 *An algebra $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ of type* (2, 1, 0) *is a goop if and only if it is an involutive moon satisfying*

$$\textit{(GP)} \quad y \cdot x^{-1} = 1 \Longrightarrow x = y.$$

Proof Let $\mathcal{G}$ be a goop. Observe that it satisfies (Iv). Indeed, let $x \in G$. We have

$$x = x \overset{\text{(GP2)}}{\Longleftrightarrow} x \cdot x^{-1} = 1 \overset{\text{(GP1)}}{\Longleftrightarrow} x^{-1} \cdot x = 1.$$

Therefore, (Iv) holds. By (Iv), $(x^{-1})^{-1} \cdot x^{-1} = 1$. From (GP2) we conclude that $(x^{-1})^{-1} = x$. Thus $\mathcal{G}$ is an involutive moon. By (GP2), it satisfies (GP).

Conversely, let $\mathcal{G}$ satisfy (U), (Iv), (DN) and (GP). Let $x, y \in G$. To prove (GP1), we first assume that $y \cdot x^{-1} = 1$. By (GP), $x = y$. Then $x^{-1} \cdot y = x^{-1} \cdot x \overset{\text{(Iv)}}{=} 1$. Now suppose that $x^{-1} \cdot y = 1$. Therefore, $1 = x^{-1} \cdot y \overset{\text{(DN)}}{=} x^{-1} \cdot (y^{-1})^{-1}$. Applying (GP), we see that $x^{-1} = y^{-1}$. Hence $y \cdot x^{-1} = y \cdot y^{-1} = 1$. Consequently, (GP1) holds. Using (GP) and (Iv), we have (GP2). Thus, $\mathcal{G}$ is a goop. □

Definition 2.6 We say that an algebra $(G; \cdot, ^{-1}, 1)$ of type (2, 1, 0) is a *weakly goop* if it is an involutive moon satisfying

$$\text{(wGP)} \quad y \cdot x^{-1} = 1 = x \cdot y^{-1} \Longrightarrow x = y.$$

Example 2.7 The algebra $\mathcal{G} = (\{a, b, 1\}; \cdot, ^{-1}, 1)$, with

$\cdot$	a	b	1
a	1	a	a
b	1	1	b
1	a	b	1

and $x^{-1} = x$ for $x \in \{a, b, 1\}$, is a weakly goop. Since $b \cdot a^{-1} = b \cdot a = 1$ and $a^{-1} \cdot b = a \cdot b = a \neq 1$, $\mathcal{G}$ does not satisfy (GP1). Therefore, it is not a goop.

Let **involutive moon**, **weakly goop**, **goop**, **group**, and **abelian group** denote the class of all involutive moons, weakly goops, goops, groups, and abelian groups, respectively. From the definitions we obtain

involutive moon $\subset$ **weakly goop** $\subset$ **goop** $\subset$ **group** $\subset$ **abelian group**.

3 The (strong) p-semisimple property

3.1 p-semisimple and strongly p-semisimple algebras

Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type (2, 0). Consider the following properties that can be satisfied by $\mathcal{A}$:

$$\text{(p-s)} \quad x \leq y \Longrightarrow x = y,$$
$$\text{(p-s1)} \quad x \leq 1 \Longrightarrow x = 1,$$
$$\text{(D1=)} \quad x = (x \to 1) \to 1,$$
$$\text{(PS)} \quad (x \to 1) \to y = (y \to 1) \to x.$$

Note that, in Iorgulescu (2018), the concept of negation, $^{-1}$, is defined by $x^{-1} = x \to 1$, and hence

$$\text{(D1 =)} \Longleftrightarrow (x^{-1})^{-1} = x \text{ and}$$
$$\text{(PS)} \Longleftrightarrow x^{-1} \to y = y^{-1} \to x.$$

Thus (D1=) is in fact the double negation property (DN) and (PS) is the property (pDNeg2), in the commutative case, from the book Iorgulescu (2018).

Remark that RM algebras satisfying (PS) were studied in Walendziak (2020).

First we present connections between the conditions in the above list.

Lemma 3.1 *If an algebra $\mathcal{A}$ verifies* (Re) *or* (M)*, then* (D1=) *implies* (p-s1).

Proof The proof is immediate. □

Proposition 3.2 *If $\mathcal{A}$ is an aRM** algebra, then* (p-s1) $\Longleftrightarrow$ (p-s).

Proof Let $\mathcal{A}$ be an aRM** algebra. Suppose that (p-s1) holds in $\mathcal{A}$. Let $x, y \in A$ and $x \leq y$. By (**), $y \to x \leq x \to x = 1$. Hence, using (p-s1), we have $y \to x = 1$, that is, $y \leq x$. From (An) we conclude that $x = y$. Thus (p-s) is satisfied. The converse is obvious. □

Proposition 3.3 *Let $\mathcal{A}$ be an aRM**(D1) algebra. Then*

(p-s1) $\Longleftrightarrow$ (p-s) $\Longleftrightarrow$ (D1=).

Proof From Proposition 3.2 we see that (p-s1) $\Longleftrightarrow$ (p-s). Let (p-s) hold and $x \in A$. By (D1), $x \leq (x \to 1) \to 1$. Applying (p-s), we obtain (D1=). Suppose now that (D1=) holds. Let $x \leq 1$. Then $x \to 1 = 1$, and hence $x = (x \to 1) \to 1 = 1 \to 1 = 1$. We thus get (p-s1). □

Proposition 3.4 *If $\mathcal{A}$ is a BCH** algebra, then*

(p-s1) $\Longleftrightarrow$ (p-s) $\Longleftrightarrow$ (D1=) $\Longleftrightarrow$ (PS).

Proof By definition and Lemma 2.1(vi), $\mathcal{A}$ satisfies (An), (Re), (M), (Ex), (**), (D). Applying Proposition 3.3, we conclude that (p-s1) $\Longleftrightarrow$ (p-s) $\Longleftrightarrow$ (D1=).

(D1=) $\Longrightarrow$ (PS): We have $(y \to 1) \to x \overset{(D1=)}{=} (y \to 1) \to ((x \to 1) \to 1) \overset{(Ex)}{=} (x \to 1) \to ((y \to 1) \to 1) \overset{(D1=)}{=} (x \to 1) \to y$.

(PS) $\Longrightarrow$ (D1=): Putting $y = 1$ in (PS), and using (M), we obtain (D1=). □

Definition 3.5 A RM algebra is called *p-semisimple* (*strongly p-semisimple*) if it satisfies (p-s1) (resp. (D1=)).

Note that from Lemma 3.1 it follows that every strongly p-semisimple RM algebra is p-semisimple.

Example 3.6 (Iorgulescu 2016) Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

$\to$	a	b	c	1
a	1	a	a	a
b	a	1	a	a
c	a	a	1	a
1	a	b	c	1

Properties (Re), (M), (An), (*), (**) (hence (Tr)), and (p-s) are satisfied. The algebra $\mathcal{A} = (A; \to, 1)$ does not satisfy (D1=). Therefore, $\mathcal{A}$ is a p-semisimple *aRM** algebra that is not strongly p-semisimple.

From Proposition 3.3 we obtain

Corollary 3.7 *For aRM**(D1) algebras (hence also for *aRM**(D1), BZ, BCH**, and BCI algebras), the strong p-semisimple property is equivalent to the p-semisimple property.*

Proposition 3.8 *Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type* (2, 0). *The following are equivalent:*

*(i) $\mathcal{A}$ is a strongly p-semisimple aRM** algebra,*
*(ii) $\mathcal{A}$ is a p-semisimple aRM**(D1) algebra,*
*(iii) $\mathcal{A}$ is a strongly p-semisimple *aRM** algebra,*
*(iv) $\mathcal{A}$ is a p-semisimple *aRM**(D1) algebra.*

Proof (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) are obvious.

(ii) $\Longrightarrow$ (iii): From Proposition 3.3 we conclude that $\mathcal{A}$ verifies (p-s) and (D1=). Hence $\mathcal{A}$ also verifies (*), and consequently it is a strongly p-semisimple *aRM** algebra.

(iv) $\Longrightarrow$ (i): This follows from Proposition 3.3. □

Denote by **strongly p-s-aRM**** the class of all strongly p-semisimple aRM** algebras (= the class of all p-semisimple aRM**(D1) algebras = the class of all strongly p-semisimple *aRM** algebras = class of all p-semisimple *aRM**(D1) algebras).

Proposition 3.9 *Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type* (2, 0). *The following are equivalent:*

*(i) $\mathcal{A}$ is a p-semisimple BCH** algebra,*
(ii) $\mathcal{A}$ is a p-semisimple BCI algebra.

Proof Let $\mathcal{A}$ be a p-semisimple BCH** algebra. By Proposition 3.2, $\mathcal{A}$ satisfies (p-s). From (p-s) we deduce that $\mathcal{A}$ also satisfies (*). Applying Lemma 2.1(vii) and (v) we see that (B) holds in $\mathcal{A}$. Consequently, $\mathcal{A}$ is a BCI algebra. The converse is obvious. □

Denote by **p-s-BCI** the class of all p-semisimple BCI algebras (= the class of all p-semisimple BCH** algebras). Let **p-s-BZ** (resp. **strongly p-s-RM**, **strongly p-s-aRM**) denote the class of all p-semisimple BZ algebras (resp. strongly p-semisimple RM algebras, strongly p-semisimple aRM algebras).

3.2 Connections between RM algebras, moons and goops

In this subsection, we establish the connections between:

strongly p-semisimple RM algebras and involutive moons,
strongly p-semisimple aRM algebras and weakly goops,
strongly p-semisimple aRM** algebras and goops.

Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type (2, 0). Define $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ by: for all $x, y \in A$, $x \cdot y = (y \to 1) \to x$ and $x^{-1} = x \to 1$.

Let $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ be an algebra of type (2, 1, 0). Define $\Psi(\mathcal{G}) = (G; \to, 1)$ by: for all $x, y \in G$, $x \to y = y \cdot x^{-1}$.

Theorem 3.10

(i) Let $\mathcal{A} = (A; \to, 1)$ textitbe a strongly p-semisimple RM algebra. Then $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ is an involutive moon.
(ii) Let $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ be an involutive moon. Then $\Psi(\mathcal{G}) = (G; \to, 1)$ is a strongly p-semisimple RM algebra.

(iii) Given $\mathcal{A}$ and $\mathcal{G}$ as above we have $\Psi\Phi(\mathcal{A}) = \mathcal{A}$ and $\Phi\Psi(\mathcal{G}) = \mathcal{G}$.

Proof

(i) Let $x \in A$. By (M), $x \cdot 1 = (1 \to 1) \to x = x$. Applying (D1=), we get $1 \cdot x = (x \to 1) \to 1 = x$. Therefore, $\Phi(\mathcal{A})$ satisfies (U). From (D1=) and (Re) we obtain

$$x \cdot x^{-1} = ((x \to 1) \to 1) \to x = x \to x = 1 \quad \text{and}$$
$$x^{-1} \cdot x = (x \to 1) \to (x \to 1) = 1.$$

Then (Iv) holds in $\Phi(\mathcal{A})$. Again using (D1=), we have (DN). Thus $\Phi(\mathcal{A})$ is an involutive moon.

(ii) Let $x \in G$. By (Iv), $x \to x = x \cdot x^{-1} = 1$, that is, (Re) holds in $\Psi(\mathcal{G})$. Applying (U) and (Iv), we get

$$1 \to x = x \cdot 1^{-1} = x \cdot (1 \cdot 1^{-1}) = x \cdot 1 = x,$$

i.e., $\Psi(\mathcal{G})$ satisfies (M). Thus $\Psi(\mathcal{G})$ is a RM algebra. Since, by (U) and (DN), $(x \to 1) \to 1 = 1 \cdot (1 \cdot x^{-1})^{-1} = x$, it is strongly p-semisimple.

(iii) Suppose $\mathcal{A} = (A; \to, 1)$ is a strongly p-semisimple RM algebra and $x \in A$. Then with $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ we have $x^{-1} = x \to 1$ and $y \cdot x^{-1} = y \cdot (x \to 1) = ((x \to 1) \to 1) \to y = x \to y$. Thus $\Psi\Phi(\mathcal{A}) = \mathcal{A}$.

Next suppose $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ is an involutive moon. Then with $\Psi(\mathcal{G}) = (G; \to, 1)$ and $x, y \in G$, $x \to 1 = 1 \cdot x^{-1} = x^{-1}$ and $(y \to 1) \to x = y^{-1} \to x = x \cdot (y^{-1})^{-1} = x \cdot y$. Thus $\Phi\Psi(\mathcal{G}) = \mathcal{G}$. □

Hence, by above Theorem 3.10, we have the equivalence

strongly p-s-RM ≡ involutive moon,

that is, the strongly p-semisimple RM algebras are term equivalent to the involutive moons.

Theorem 3.11

*(i) Let $\mathcal{A} = (A; \to, 1)$ be a strongly p-semisimple aRM algebra (aRM** algebra). Then $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ is a weakly goop (goop, respectively).*

*(ii) Let $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ be a weakly goop (goop). Then $\Psi(\mathcal{G}) = (G; \to, 1)$ is a strongly p-semisimple aRM algebra (aRM** algebra, respectively).*

(iii) Given $\mathcal{A}$ and $\mathcal{G}$ as above we have $\Psi\Phi(\mathcal{A}) = \mathcal{A}$ and $\Phi\Psi(\mathcal{G}) = \mathcal{G}$.

Proof

(i) Let $\mathcal{A}$ be a strongly p-semisimple aRM algebra. By Theorem 3.10, $\Phi(\mathcal{A})$ is an involutive moon. Let $x, y \in A$ and suppose that $y \cdot x^{-1} = 1 = x \cdot y^{-1}$. Hence $x \leq y$ and $y \leq x$. Therefore $x = y$ by (An). Then (wGP) holds in $\Phi(\mathcal{A})$, that is, $\Phi(\mathcal{A})$ is a weakly goop.
Let now $\mathcal{A}$ be a strongly p-semisimple aRM** algebra. Then $\mathcal{A}$ satisfies (p-s1), and also (p-s) by Proposition 3.2. Let $y \cdot x^{-1} = 1$, and hence $x \leq y$. From (p-s) it follows that $x = y$. Consequently, (GP) holds in $\Phi(\mathcal{A})$, that is, $\Phi(\mathcal{A})$ is a goop by Proposition 2.5.

(ii) Let $\mathcal{G}$ be a weakly goop. By Theorem 3.10, $\Psi(\mathcal{G})$ is a strongly p-semisimple RM algebra. From (wGP) we deduce that (An) holds in $\Psi(\mathcal{G})$. Thus $\Psi(\mathcal{G})$ is a strongly p-semisimple aRM algebra.
Let now $\mathcal{G}$ be a goop. From (GP) it follows that (p-s) holds in $\Psi(\mathcal{G})$. Hence, obviously, $\Psi(\mathcal{G})$ satisfies (**). Thus $\Psi(\mathcal{G})$ is a strongly p-semisimple aRM** algebra.

(iii) See the proof of Theorem 3.10(iii).

□

Hence, by above Theorem 3.11, we have the equivalences:

strongly p-s-aRM ≡ weakly goop,
strongly p-s-aRM ≡ goop**,

that is, the strongly p-semisimple aRM algebras are term equivalent to the weakly goops and the strongly p-semisimple aRM** algebras are term equivalent to the goops.

Theorem 3.12 (Walendziak 2020) *If $\mathcal{A} = (A; \to, 1)$ is a RM algebra satisfying* (PS)*, then $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ is a commutative involutive moon. Conversely, if $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ is a commutative involutive moon, then $\Psi(\mathcal{G}) = (G; \to, 1)$ is a RM algebra with* (PS).

Theorem 3.13 (Walendziak 2020) *If $\mathcal{A} = (A; \to, 1)$ is an aRM algebra (*aRM** algebra) satisfying* (PS)*, then $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ is a commutative weakly goop (commutative goop, respectively). If $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ is a commutative weakly goop (commutative goop), then $\Psi(\mathcal{G}) = (G; \to, 1)$ is an aRM algebra (*aRM** algebra, respectively) with* (PS).

From Theorems 3.12 and 3.13 it follows that the RM algebras with (PS) are term equivalent to the commutative involutive moons, the aRM algebras with (PS) are term equivalent to the commutative weakly goops, and the *aRM** algebras with (PS) are term equivalent to the commutative goops.

Theorem 3.14 (Zhang and Ye 1995) *If $\mathcal{A} = (A; \to, 1)$ is a p-semisimple BZ algebra, then $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ is* *a group. Conversely, if $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ is a group, then $\Psi(\mathcal{G}) = (G; \to, 1)$ is a p-semisimple BZ algebra.*

Theorem 3.15 (Lei and Xi 1985) *If $\mathcal{A} = (A; \to, 1)$ is a p-semisimple BCI algebra (or, equivalently, p-semisimple BCH** algebra), then $\Phi(\mathcal{A}) = \left(A; \cdot, ^{-1}, 1\right)$ is an abelian group. Conversely, if $\mathcal{G} = \left(G; \cdot, ^{-1}, 1\right)$ is an abelian group, then $\Psi(\mathcal{G}) = (G; \to, 1)$ is a p-semisimple BCI algebra.*

From Theorems 3.14 and 3.15 we see that

$$\textbf{p-s-BZ} \equiv \textbf{group} \quad \text{and} \quad \textbf{p-s-BCI} \equiv \textbf{abelian group},$$

that is, the p-semisimple BZ algebras are term equivalent to the groups and the p-semisimple BCI algebras are term equivalent to the abelian groups.

4 Examples of proper strongly p-semisimple algebras

Definition 4.1

P1. A *proper strongly p-semisimple RM algebra* is a strongly p-semisimple RM algebra (i.e., verifying (Re), (M), (D1=)) not verifying (An), (Ex), (Tr) (hence not (B), (BB), (*), (**), by Lemma 2.1(ii)–(v)).

P2. A *proper strongly p-semisimple aRM algebra* is a strongly p-semisimple aRM algebra (i.e., verifying (An), (Re), (M), (D1=)) not verifying (Ex), (Tr) (hence not (B), (BB), (*), (**)).

P3. A *proper strongly p-semisimple aRM** algebra* is a strongly p-semisimple aRM** algebra (i.e., verifying (An), (Re), (M), (**), (*), (Tr), (D1=)) not verifying (B), (Ex) (hence not (BB), by Lemma 2.1 (viii)).

P4. A *proper p-semisimple BZ algebra* is a p-semisimple BZ algebra (i.e., verifying (An), (Re), (M), (B), (*), (**), (Tr), (D1=)) not verifying (BB) (hence not (Ex), by Lemma 2.1(ix)).

Example 4.2 Consider the set $A = \{a, b, c, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	1
a	1	1	b	a
b	1	1	c	b
c	1	b	1	c
1	a	b	c	1

We can observe that the properties (Re), (M), and (D1=) (hence (p-s1)) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$; (Ex) and (Tr) are not satisfied for $(x, y, z) = (c, a, b)$. Hence, $(A; \to, 1)$ is a proper strongly p-semisimple RM algebra.

Example 4.3 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

$\to$	a	b	c	1
a	1	1	b	a
b	c	1	c	b
c	1	b	1	c
1	a	b	c	1

The algebra $\mathcal{A} = (A; \to, 1)$ satisfies properties (An), (Re), (M), and (D1=) (hence (p-s1)). It does not satisfy (Ex) and (Tr) for $(x, y, z) = (c, a, b)$. Then $\mathcal{A}$ is a proper strongly p-semisimple aRM algebra.

Example 4.4 Let $A = \{a, b, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	1
a	1	a	a
b	b	1	b
1	a	b	1

It is easy to see that the properties (An), (Re), (M), (*), (**), (Tr), and (D1=) (hence (p-s1) and (p-s)) are satisfied; (B) and (Ex) are not satisfied for $(x, y, z) = (a, b, 1)$. Hence, $(A; \to, 1)$ is a proper strongly p-semisimple aRM** algebra.

Example 4.5 Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	e	1
a	1	d	e	b	c	a
b	e	1	d	c	a	b
c	d	e	1	a	b	c
d	b	c	a	1	d	e
e	c	a	b	e	1	d
1	a	b	c	d	e	1

Then the properties (An), (Re), (M), (B) (hence (*), (**), (Tr)), (p-s1) (hence (p-s), (D1=)) are satisfied. (BB) is not satisfied for $(x, y, z) = (a, d, c)$. Therefore, $(A; \to, 1)$ is a proper p-semisimple BZ algebra.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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