Markov frameworks and stock market decision making

Kavitha Koppula¹ · Babushri Srinivas Kedukodi¹ · Syam Prasad Kuncham¹

Published online: 18 May 2020
© The Author(s) 2020

Abstract

In this paper, we present applications of Markov rough approximation framework (MRAF). The concept of MRAF is defined based on rough sets and Markov chains. MRAF is used to obtain the probability distribution function of various reference points in a rough approximation framework. We consider a set to be approximated together with its dynamacy and the effect of dynamacy on rough approximations is stated with the help of Markov chains. An extension to Pawlak's decision algorithm is presented, and it is used for predictions in a stock market environment. In addition, suitability of the algorithm is illustrated in a multi-criteria medical diagnosis problem. Finally, the definition of fuzzy tolerance relation is extended to higher dimensions using reference points and basic results are established.

Keywords Rough set · Markov chain · Rough approximation framework

1 Introduction

Pawlak (1982) introduced the notion of rough sets by defining the lower approximation of a set X as the collection of all the elements of the universe whose equivalence classes are contained in X , and the upper approximation of X as the set of all the elements of the universe whose equivalence classes have a non-empty intersection with X . We often consider the universe to be an algebraic structure and study the corresponding algebraic properties of rough approximations. The concept of rough approximation framework was defined by Ciucci (2008), basically as a collection of rough approximations of the set. A rough approximation framework is said to be regular if all the approximations of the set are inscribed in one another. An illustration of rough approximation framework was given by Kedukodi et al. (2010) using reference points.

In Markov chains, the probabilities which finally decide the stability of a system are represented in terms of a matrix known as transition probability matrix. Markov chains have been used in several applications to predict the future possibilities in dynamic and uncertain systems. One such area that has been the focus of intense research is the prediction of the performance of stock markets. In a typical stock market environment, customer's either SELL, BUY or HOLD a particular stock by assessing and predicting its performance utilizing previous and current performance of the stock, inputs from rating agencies, etc (Tavana et al. 2017). Such an assessment of past performance of the stock with the available empirical data of a stock and predicting the future performance or value of the stock is a challenging task due to its very dynamic nature and multiple variables that affect its performance.

Despite this, a variety of mathematical models have been proposed in the literature (Aleksandar et al. 2018; Xiongwen et al. 2020; Emrah and Taylan 2017; Sudan et al. 2019; Gour et al. 2018; Prasenjit and Ranadive 2019, etc.) Such methods often apply a number of ideas from Markov chains, fuzzy sets (Chan 2015; Choji et al. 2013; Rezaie et al. 2013), rough sets, artificial neural network (Suk et al. 2010) or other interesting methods (Chen et al. 2007; Gong et al. 2019; Markovic et al. 2017).

Recently, Koppula et al. (2018) introduced the concept of Markov rough approximation framework (MRAF) by using Markov chains and rough sets. MRAF helps to assign the

Communicated by V. Loia.

✉ Babushri Srinivas Kedukodi
babushrisrinivas.k@manipal.edu
Kavitha Koppula
kavitha.koppula@manipal.edu
Syam Prasad Kuncham
syamprasad.k@manipal.edu

¹ Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education (MAHE), Manipal, Karnataka 576104, India

probabilities for various reference points in the rough approximation framework. In the present work, we present explicit examples of MRAF. The first example focuses on explaining the mathematical ideas involved, and the second example demonstrates how to apply the concept in a practical situation. In the second example, we use MRAF along with Pawlak decision algorithm (Pawlak 2002) to analyze data from the different rating agencies (reference points) in the stock market environment. We arrive at a recommendation on the future prospects of a set of stocks along with the probabilities for the suggested recommendation to be correct. Usually, the rating agencies (experts) do not get evaluated for the quality of their recommendations. However, the algorithm proposed in this paper takes care of this aspect through the idea of Markov chain transition probability matrix which yields a telescopic way of assignment of weights. This method naturally evaluates the quality of recommendations made by the rating agencies (experts). Further, the suitability of the proposed algorithm is also validated by evaluating its efficacy in arriving at a decision in a multi-criteria medical problem. We extend the fuzzy similarity-based rough set approach used in multi-criteria decision problems to identify key or crucial attribute among multiple attributes.

2 Basic definitions and preliminaries

In this section, we provide basic definitions and results. We refer Pawlak (1982), Ciucci (2008), Kedukodi et al. (2010), Davaaz (2004, 2006) for basic definitions of rough set and rough approximation framework.

Definition 2.1 (Medhi 2009) The stochastic process $\{X_n, n = 0, 1, 2, \dots\}$ is called a Markov chain; if, for $j, k, j_1, \dots, j_{n-1} \in N$, $Pr\{X_n = k | X_{n-1} = j, X_{n-2} = j_1, \dots, X_0 = j_{n-1}\} = Pr\{X_n = k | X_{n-1} = j\} = P_{jk}$ (say) whenever the first member is defined. The transition probabilities P_{jk} satisfy $P_{jk} \geq 0$, $\sum_k P_{jk} = 1$ for all j .

Definition 2.2 (Ching and Ng 2006) A vector $\pi = (\pi_0, \pi_1, \dots, \pi_{k-1})^t$ is said to be a stationary distribution of a finite Markov chain if it satisfies:

- (i) $\pi_i \geq 0$ and $\sum_{i=0}^{k-1} \pi_i = 1$.
- (ii) $\sum_{j=0}^{k-1} P_{ij} \pi_j = \pi_i$.

The results on fuzzy ideals of semirings, near-rings and the ideals of seminear-rings can be found in Bhavanari et al. (2010), Jagadeesha et al. (2016a,b), Kedukodi et al. (2007, 2009), Kedukodi et al. (2017), Koppula et al. (2019), Kuncham et al. (2016, 2017), Nayak et al. (2018) and Akram and Dudek (2008).

Definition 2.3 (Koppula et al. 2018) Let $S = \{B_1, B_2, \dots, B_m\}$ be a collection of reference points in the universe U . Consider a sequence of trails $Y_n, n \geq 1$ of selection of reference points from the set S . Let the trails dependence be connected by a Markov chain with a transition probability matrix

$$C = \begin{matrix} & & Y_n \\ & & \begin{matrix} B_1 & B_2 & \dots & B_m \end{matrix} \\ \begin{matrix} B_1 \\ B_2 \\ \dots \\ B_m \end{matrix} & Y_{n-1} & \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1m} \\ q_{21} & q_{22} & \dots & q_{2m} \\ \dots & \dots & \dots & \dots \\ q_{m1} & q_{m2} & \dots & q_{mm} \end{pmatrix} \end{matrix}$$

Let E be the rough approximation framework formed by S . Then (E, C) is called a Markov rough approximation framework (MRAF).

The size of MRAF is the size of the rough approximation framework E .

Definition 2.4 (Huang 1992) quadruple $IS = (U, AT, V, h)$ is called an information system, where $U = \{u_1, u_2, \dots, u_n\}$ is a non-empty finite set of objects, called the universe of discourse, $AT = \{a_1, a_2, \dots, a_n\}$ is a non-empty finite set of attributes. $V = \cup_{a \in AT} V_a$ where V_a is the set of attribute values associated with each attribute $a \in AT$ and $h : U \times AT \rightarrow V$ is an information function that assigns particular values to the objects against attribute set such that $\forall a \in AT, \forall u \in U, h(u, a) \in V_a$.

Definition 2.5 (Huang 1992) In an information system, if each attribute has a single entity as attribute value, then it is called single-valued information system, otherwise it is known as set-valued information system. Set-valued information system is a generalization of the single-valued information system, in which an object can have more than one attribute values.

Definition 2.6 (Orlowska 1985) For a set-valued information system $IS = (U, AT, V, h)$, $\forall b \in AT$ and $u_i, u_j \in U$, tolerance relation is defined as

$$T_b = \{(u_i, u_j) \mid b(u_i) \cap b(u_j) \neq \emptyset\}$$

For $B \subseteq AT$, a tolerance relation is defined as

$$T_B = \{(u_i, u_j) \mid b(u_i) \cap b(u_j) \neq \emptyset, \forall b \in B\} = \bigcap_{b \in B} T_b,$$

where $(u_i, u_j) \in T_B$ implies that u_i and u_j are indiscernible (tolerant) with respect to a set of attributes B .

Definition 2.7 (Shivani et al. 2020) Let $SVIS = (U, AT, V, h)$, $\forall b \in AT$ be a set-valued information system, then a fuzzy

relation R_b is defined as

$$\mu_{R_b}(u_i, u_j) = 2 \frac{|b(u_i) \cap b(u_j)|}{|b(u_i)| + |b(u_j)|}.$$

For a set of attributes $B \subseteq A$, a fuzzy relation R_B can be defined as $\mu_{R_B}(u_i, u_j) = \inf_{b \in B} \mu_{R_b}(u_i, u_j)$

Definition 2.8 (Shivani et al. 2020) Binary relation using a threshold value α is defined as $T_b'' = \{(u_i, u_j) \mid \mu_{R_b}(u_i, u_j) \geq \alpha\}$, where $\alpha \in (0, 1)$ is a similarity threshold, which gives a level of similarity for insertion of objects within tolerance classes.

For a set of attributes $B \subseteq AT$, the binary relation is defined as $T_B'' = \{(u_i, u_j) \mid \mu_{R_B}(u_i, u_j) \geq \alpha\}$.

3 Examples of Markov frameworks

We begin with a mathematical example.

Proposition 3.1 Let $U = \mathbb{R} \times \mathbb{R}$ and $\bar{a}, \bar{b} \in U$. Define a relation R on U by $\bar{a} R \bar{b}$ if there exists a square S centered at the origin with radius $r > 0$ (the radius of a square defined to be the distance from the center $(0, 0)$ of the square to one of the vertices) such that $\bar{a}, \bar{b}$ lie on the square. Then R is an equivalence relation on U .

Proof (i) Let $\bar{a} = (x, y) \in U$ and δ_x, δ_y are perpendicular distances from $\bar{a}$ to X -axis and Y -axis, respectively. Take $\delta_{\bar{a}} = \max\{\delta_x, \delta_y\}$. Form a square S with centre $(0, 0)$ and radius $\sqrt{2}\delta_{\bar{a}}$. We claim that $\bar{a}$ lies on S .

Let $\delta_y \geq \delta_x$. Then $\delta_{\bar{a}} = \delta_y$. This implies $|y| = \delta_{\bar{a}}$, $-\delta_{\bar{a}} \leq x \leq \delta_{\bar{a}}$. Hence $\bar{a}$ lies on one of its sides (x, δ_y) or $(x, -\delta_y)$.

Similarly, when $\delta_x \geq \delta_y$ we can prove that $\bar{a}$ lies on one of its sides (δ_x, y) or $(-\delta_x, y)$ of the square S . Therefore $\bar{a} R \bar{a}$. Hence R is reflexive.

(ii) Let $\bar{a} R \bar{b}$. Then there exists a square centered at the origin with radius r such that $\bar{a}, \bar{b}$ lie on the square. This implies $\bar{b} R \bar{a}$. Hence R is symmetric.

(iii) Let $\bar{a} R \bar{b}$, $\bar{b} R \bar{c}$. Then there exist square S_1 centered at the origin with radius k_1 such that $\bar{a}, \bar{b}$ lie on S_1 and square S_2 centered at the origin with radius k_2 such that $\bar{b}, \bar{c}$ lie on S_2 . This implies $k_1 = k_2$ (because $\bar{b}$ lies on S_1 and S_2). Hence $S_1 = S_2$. Therefore $\bar{a} R \bar{c}$. Hence R is transitive. Therefore R is an equivalence relation on U . $\square$

Notation 3.2 Let $\bar{a} \in U$. Denote the rotations (in anticlockwise sense) of a square centered at the origin with radius $\sqrt{2}\delta_{\bar{a}}$ by an angle θ (reference points) as $S_{(\bar{a}, \theta)}$, $0 \leq \theta \leq 2\pi$.

In particular, the square defined by Proposition 3.1 is denoted by $S_{(\bar{a}, 0)}$ (as shown in Fig. 1).

Proposition 3.3 Let $U = \mathbb{R} \times \mathbb{R}$ and $\bar{x}, \bar{y} \in U$. Define a relation R_θ on U by $\bar{x} R_\theta \bar{y}$ if there exists θ such that $S_{(\bar{x}, \theta)} =$

Fig. 1 Square with rotation zero degrees

$S_{(\bar{y}, \theta)}$. Then R_θ defines a MRAF of size greater than or equal to 1 on U corresponding to the set $X_k = \{(x, y) \mid x^2 + y^2 \leq k\}$.

Proof First we prove that R_θ is an equivalence relation on U .

(i) Let $\bar{x} \in U$. Take $\theta = 0$. By Proposition 3.1, $\bar{x}$ lies on the square $S_{(\bar{x}, 0)}$. This implies $\bar{x} R_\theta \bar{x}$. Hence R_θ is reflexive.

(ii) Let $\bar{x} R_\theta \bar{y}$. Then there exists θ such that $S_{(\bar{x}, \theta)} = S_{(\bar{y}, \theta)}$. This implies $\bar{y} R_\theta \bar{x}$. Hence R_θ is symmetric.

(iii) Let $\bar{x} R_\theta \bar{y}$ and $\bar{y} R_\theta \bar{z}$. Then there exists θ such that $S_{(\bar{x}, \theta)} = S_{(\bar{y}, \theta)}$ and $S_{(\bar{y}, \theta)} = S_{(\bar{z}, \theta)}$. This implies $S_{(\bar{x}, \theta)} = S_{(\bar{z}, \theta)}$. Hence R_θ is transitive. Therefore R_θ is an equivalence relation on U .

The equivalence classes corresponding to the above equivalence relation are given by $[\bar{x}]_\theta = \{\bar{y} \in U \mid \bar{x} R_\theta \bar{y}\} = \{S_{(\bar{x}, \theta)} \mid 0 \leq \theta \leq 2\pi\}$.

The sets $L(X_k) = \cup\{\bar{x} \in U \mid [\bar{x}]_\theta \subseteq X_k\} = \cup\{\bar{x} \in U \mid S_{(\bar{x}, \theta)} \subseteq X_k\} = \{(x, y) \mid |x| = \sqrt{\frac{k}{2}}, |y| \leq \sqrt{\frac{k}{2}} \text{ or } |y| = \sqrt{\frac{k}{2}}, |x| \leq \sqrt{\frac{k}{2}}\}$ and $U(X_k) = \cup\{\bar{x} \in U \mid [\bar{x}]_\theta \cap X_k \neq \emptyset\} = \cup\{\bar{x} \in U \mid S_{(\bar{x}, \theta)} \cap X_k \neq \emptyset\} = \{(x, y) \mid |x| = k, |y| \leq k \text{ or } |y| = k, |x| \leq k\}$ are, respectively, the lower approximation and upper approximations of the set X_k . Note that whenever the number of rotations of a square is zero, we get MRAF of size 1. If the number of rotations of a square is more than one, then we get a MRAF of size greater than one. $\square$

Suppose we take four rotations of the square given by $\theta = \{0, 22.5, 45, 67.5\}$ in Proposition 3.3 with the transition probability matrix given by

$$T = \begin{matrix} & \begin{matrix} 0 & 22.5 & 45 & 67.5 \end{matrix} \\ \begin{matrix} 0 \\ 22.5 \\ 45 \\ 67.5 \end{matrix} & \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \end{matrix}$$

Fig. 2 MRAF of size 4

with $a_i + b_i + c_i + d_i = 1$, $i = 1, 2, 3, 4$. Then (S_θ, T) is a MRAF of size 4. A visual representation is shown in Fig. 2, wherein a circle approximated by MRAF of size 4.

Proposition 3.4 Let $U = \mathbb{R} \times \mathbb{R}$ and $\bar{a}, \bar{b} \in U$. Define a relation R_0 on U by $\bar{a} R_0 \bar{b}$ if there exists a circle centered at the origin with radius $r(> 0)$ such that $\bar{a}, \bar{b}$ lie on the circle. Then R_0 defines MRAF of size 1 on U corresponding to the set $X_k = \{(x, y) \mid |x| = k, |y| \leq k \text{ or } |y| = k, |x| \leq k\}$.

Proof We first prove that R_0 is an equivalence relation on U .

(i) Let $\bar{a} = (x, y) \in U$. Then $\bar{a}$ lies on the circle centered at the origin with radius $\sqrt{(x-0)^2 + (y-0)^2}$. Therefore $\bar{a} R_0 \bar{a}$. Hence R_0 is reflexive.

(ii) Let $\bar{a} R_0 \bar{b}$. Then there exists a circle centered at the origin with radius r such that $\bar{a}, \bar{b}$ lie on the circle. This implies $\bar{b} R_0 \bar{a}$. Hence R_0 is symmetric.

(iii) Let $\bar{a} R_0 \bar{b}$, $\bar{b} R_0 \bar{c}$. Then there exist circle C_1 centered at the origin with radius k_1 such that $\bar{a}, \bar{b}$ lie on C_1 and circle C_2 centered at the origin with radius k_2 such that $\bar{b}, \bar{c}$ lie on C_2 . This implies $k_1 = k_2$ (because $\bar{b}$ lies on C_1 and C_2). Hence $C_1 = C_2$. Therefore $\bar{a} R_0 \bar{c}$. Hence R_0 is transitive. Therefore R_0 is an equivalence relation on U .

Let $\bar{a} \in U$. Denote $C_{\bar{a}}$ as the circle centered at the origin and radius is the distance between $(0, 0)$ and $\bar{a}$. Then the equivalence classes corresponding to the above equivalence relation are given by

$$[\bar{a}] = \{\bar{b} \in U \mid \bar{a} R_0 \bar{b}\} = C_{\bar{a}}.$$

The sets

$$\begin{aligned} L(X_k) &= \cup\{\bar{a} \in U \mid [\bar{a}] \subseteq X_k\} = \cup\{\bar{a} \in U \mid C_{\bar{a}} \subseteq X_k\} \\ &= \{(x, y) \mid x^2 + y^2 \leq k\} \end{aligned}$$

and

$$\begin{aligned} U(X_k) &= \cup\{\bar{a} \in U \mid [\bar{a}] \cap X_k \neq \emptyset\} = \cup\{\bar{a} \in U \mid C_{\bar{a}} \cap X_k \neq \emptyset\} \\ &= \{(x, y) \mid x^2 + y^2 \leq \sqrt{2}k\} \end{aligned}$$

Fig. 3 MRAF of size 1

Fig. 4 Telescoping MRAF (Recursive implementation of Figs. 2b and 3b)

are, respectively, called lower approximation and upper approximation of the set X_k . We get MRAF of size 1 because all the rotations of circle yield the same circle. Thus we get a trivial transition probability matrix having just an entry of 1 in it. $\square$

A visual representation of MRAF of size 1 is shown in Fig. 3, wherein a collection of squares are approximated by MRAF of size 1.

Figure 4 represents telescoping MRAF, wherein circles and squares are alternatively rough approximations of each other.

Example 3.5 Now we illustrate an application of MRAF in the stock market environment. In the process, we give an extension to Pawlak's decision algorithm and address the problem of decision making in a situation involving data from several sources (rating agencies/experts) with varying

criteria (attributes) for a particular stock (company/service provider). Usually, listed companies in a share market release their quarterly performance reports in terms of EBITDA (Earnings Before Interest, Tax, Depreciation and Amortization), EBITDA Margin, EBIT (Earnings Before Interest and Tax), EBIT Margin, PAT (Profit After Tax), Adjusted PAT, Revenue, Sales, etc (attributes). Among these, various rating agencies/experts select few parameters as performance indicators of a given stock, predict its future performance for the next quarter and give suggestions to the existing or prospective customers either to BUY, SELL or HOLD the shares of a given company. From the customer point of view, making a decision either to SELL, HOLD or BUY the shares of a particular company is a challenging task due to the disparity in performance indicators selected by various rating agencies/experts (reference points) and also because of their contrary views sometimes on the same stock. MRAF together with Pawlak's decision algorithm is very useful to tackle such problems.

Algorithm 3.6 Extension of Pawlak's Decision Algorithm:

Step 1: Gather information from multiple sources (reference points) $R_1, R_2, \dots, R_p$ for $U = \{u_1, u_2, \dots, u_m\}$ with respect to the attributes $A = \{a_1^k, a_2^k, \dots, a_n^k\}$ and tabulate them as a decision table for the reference point R_k .

Step 2: Form decision rules associated with the decision table for each reference point R_k satisfying the conditions given by Pawlak (2002).

Step 3: $Support_s(\Phi, \Psi) = Supp_s(\Phi, \Psi) \times P(R_k) \neq 0$, $Supp_s(\Phi, \Psi)$ is given by Pawlak (2002), $P(R_k)$ is the probability of the reference point R_k obtained from MRAF.

Step 4: Combine the decision rules from the various decision algorithms with respect to various reference points and form decision table.

Step 5: Verify whether the decision rules satisfy the Pawlak's (2002) decision algorithm.

Step 6: Calculate strength, certainty and coverage factors for the decision algorithm given in Step 5 by using the following formulae.

$$Cer_s(\Phi, \Psi) = \frac{|\llbracket \Phi \wedge \Psi \rrbracket_s|}{|\llbracket \Phi \rrbracket_s|} = \frac{support_s(\Phi, \Psi)}{|\llbracket \Phi \rrbracket_s|}$$

$$Cov_s(\Phi, \Psi) = \frac{|\llbracket \Phi \wedge \Psi \rrbracket_s|}{|\llbracket \Psi \rrbracket_s|} = \frac{support_s(\Phi, \Psi)}{|\llbracket \Psi \rrbracket_s|},$$

where $\llbracket \Phi \rrbracket_s \neq \phi$, $\llbracket \Psi \rrbracket_s \neq \phi$.

Step 7: Arrive at decision using the values of certainty and coverage factors.

Now to demonstrate the utility of the above algorithm with a stock market case scenario as described above, we have collected relevant data of 15 companies [$C_1, C_2, C_3, \dots, C_{15}$] from four different rating agencies (R_1, R_2, R_3 and R_4) with

EBITDA	PAT	Sales	EBIT	Adjusted PAT	EBITDA Margin	EBIT Margin	Revenue	CMP	TP	Rec.	Reference Points (Rating Agencies)
miss	inline	miss	-	-	-	-	-	1673	1833	Hold	R_1
-	-	-	inline	beat	-	-	miss	1673	1875	Buy	R_2
inline	beat	-	-	-	-	-	inline	1673	1835	Buy	R_3
-	-	-	-	-	inline	inline	miss	1673	1650	Hold	R_4

Fig. 5 Quarterly performance of a company according to rating agencies (CMP current market price, TP target price, Rec. recommendation)

Condition (raw interval)	Likely scenario
$CMP - E(TP) \leq -15\% E(TP)$	CMP is lower than E(TP)
$-15\% E(TP) < CMP - E(TP) \leq -5\% E(TP)$	CMP is lower than E(TP)
$-5\% E(TP) < CMP - E(TP) \leq -0\% E(TP)$	CMP is closer to E(TP)
$0\% E(TP) < CMP - E(TP) \leq 5\% E(TP)$	CMP is closer to E(TP)
$5\% E(TP) < CMP - E(TP) \leq 15\% E(TP)$	CMP is higher than E(TP)
$CMP - E(TP) > 15\% E(TP)$	CMP is higher than E(TP)

Fig. 6 Criteria for company performance based on E(TP) and CMP

respect to various attributes. Representative data obtained for a single company (C_4) are presented in Fig. 5.

In Fig. 5, as the target price projections by various rating agencies are different, an expected target price is computed using MRAF. For the sake of simplicity, we consider MRAF given in the example of Koppula et al. (2018) and the corresponding stationary probability distribution [1/6, 2/6, 1/6, 2/6] for the rating agencies R_1, R_2, R_3 and R_4 , respectively. Both current market price and the calculated expected target price are compared as per conditions in Fig. 6.

Subsequently, all the companies are grouped based on the similarity among the various attributes along with the likely scenario (Fig. 6) of the stock and is presented in Fig. 7.

From Fig. 7, a decision algorithm is formed as shown in Fig. 8.

The certainty and coverage factors for the above decision algorithm are calculated using Algorithm 3.6 and are shown in Fig. 9.

Information in Fig. 9 can be used to arrive at best possible decision as per the decision algorithm presented in Fig. 8.

For example, if (EBITDA, inline), (PAT, beat) and (CMP-TP, closer to E(TP)) then the Recommendation is BUY with the probability 0.909.

If (EBITDA, miss), (PAT, inline) and (CMP-TP, lower to E(TP)) then the Recommendation is HOLD with probability 0.667.

If Recommendation is SELL, then (EBITDA margin, miss), (EBIT margin, miss) and (CMP-TP, higher to E(TP)) with probability 0.5.

EBITDA	PAT	Sales	CMP-E(TP)	Recommendation	Rating Agency (Reference Point)	Companies (number)
Miss	Miss	Miss	Closer	Hold	R_1	C_9 (1)
Miss	Miss	Inline	Closer	Sell	R_1	C_6, C_{14} (2)
Miss	Inline	Miss	Lower	Hold	R_1	$C_2, C_7, C_9, C_{10}, C_{11}, C_{12}, C_{13}$ (8)
Inline	Beat	Inline	Lower	Buy	R_1	C_1, C_3, C_{15} (3)
Inline	Beat	Miss	Lower	sell	R_1	C_6 (1)
EBITDA	PAT	Revenue	CMP-E(TP)	Recommendation	Rating Agency	Companies
Miss	Inline	Beat	Lower	Buy	R_3	C_9, C_{12} (2)
Inline	Beat	Inline	Lower	Buy	R_3	$C_4, C_9, C_7, C_9, C_{10}, C_{14}$ (6)
Inline	Beat	Beat	Closer	Buy	R_3	C_1, C_{13}, C_{15} (3)
Inline	Beat	Inline	Closer	Hold	R_3	C_2, C_3, C_6, C_{11} (4)
EBIT	Adjusted PAT	Revenue	CMP-E(TP)	Recommendation	Rating Agency	Companies
Inline	Miss	Inline	Closer	Hold	R_2	C_2, C_9, C_9 (3)
Inline	Beat	Miss	Lower	Buy	R_2	$C_{11}, C_{13}, C_9, C_9, C_{11}, C_{12}, C_{13}, C_{15}$ (8)
Inline	Beat	Beat	Lower	Hold	R_2	C_5, C_{14} (2)
Inline	Miss	Beat	Closer	Buy	R_2	C_7, C_{10} (2)
EBITDA Margin	EBIT Margin	Revenue	CMP-E(TP)	Recommendation	Rating Agency	Companies
Miss	Miss	Inline	Higher	Hold	R_4	C_5 (1)
Miss	Miss	Beat	Closer	Buy	R_4	C_{10}, C_{12} (2)
Inline	Inline	Inline	Closer	Buy	R_4	C_{15} (1)
Inline	Inline	Miss	Closer	Hold	R_4	C_4, C_7, C_{13}, C_{14} (4)
Miss	Miss	Miss	Higher	Sell	R_4	C_2, C_9, C_9, C_{11} (5)
Miss	Miss	Inline	Closer	Sell	R_4	C_1, C_3 (2)

Fig. 7 Grouping of companies based on similarity criteria (numerical in the bracket indicates number of companies satisfying the corresponding decision criteria)

1	If (EBITDA, miss), (PAT, miss) and (CMP-TP, closer to E(TP)) then the (Recommendation, Hold)
2	If (EBITDA, miss), (PAT, miss) and (CMP-TP, closer to E(TP)) then the (Recommendation, Sell)
3	If (EBITDA, miss), (PAT, inline) and (CMP-TP, lower to E(TP)) then the (Recommendation, Hold)
4	If (EBITDA, inline), (PAT, beat) and (CMP-TP, lower to E(TP)) then the (Recommendation, Buy)
5	If (EBITDA, miss), (PAT, inline) and (CMP-TP, lower to E(TP)) then the (Recommendation, Buy)
6	If (EBITDA, inline), (PAT, beat) and (CMP-TP, closer to E(TP)) then the (Recommendation, Buy)
7	If (EBITDA, inline), (PAT, beat) and (CMP-TP, closer to E(TP)) then the (Recommendation, Hold)
8	If (EBITDA, inline), (PAT, beat) and (CMP-TP, lower to E(TP)) then the (Recommendation, Sell)
9	If (EBIT margin, inline), (Adjusted PAT, miss) and (CMP-TP, closer to E(TP)) then the (Recommendation, Hold)
10	If (EBIT margin, inline), (Adjusted PAT, beat) and (CMP-TP, lower to E(TP)) then the (Recommendation, Buy)
11	If (EBIT margin, inline), (Adjusted PAT, beat) and (CMP-TP, lower to E(TP)) then the (Recommendation, Hold)
12	If (EBIT margin, inline), (Adjusted PAT, miss) and (CMP-TP, closer to E(TP)) then the (Recommendation, Buy)
13	If (EBITDA margin, miss), (EBIT margin, miss) and (CMP-TP, higher to E(TP)) then the (Recommendation, Hold)
14	If (EBITDA margin, miss), (EBIT margin, miss) and (CMP-TP, closer to E(TP)) then the (Recommendation, Buy)
15	If (EBITDA margin, inline), (EBIT margin, inline) and (CMP-TP, closer to E(TP)) then the (Recommendation, Buy)
16	If (EBITDA margin, inline), (EBIT margin, inline) and (CMP-TP, closer to E(TP)) then the (Recommendation, Hold)
17	If (EBITDA margin, miss), (EBIT margin, miss) and (CMP-TP, higher to E(TP)) then the (Recommendation, Sell)
18	If (EBITDA margin, miss), (EBIT margin, miss) and (CMP-TP, closer to E(TP)) then the (Recommendation, Sell)

Fig. 8 Decision algorithm associated with decision Fig. 7

To validate the results of Example 3.5, we have considered three companies (viz. ACC Limited, Wipro Limited and Maruti Suzuki India Limited designated as C_4 , C_9 , C_{15} , respectively) and the data from their quarterly reports. Also, recommendation reports from four different rating agencies (reference points) are noted. From these data, expected values for various parameters are computed and a recommendation is suggested based on MRAF. Subsequently, actual share prices of these companies are plotted and compared with the suggestion arrived from MRAF. The validation is shown in Fig. 10.

The graph indicates that the recommendation derived from MRAF and extended Pawlak's decision algorithm seems

Decision rule	Support	Strength	Certainty	Coverage
1	0.167	0.075	0.75	0.75
2	0.334	0.025	0.333	0.111
3	1.336	0.05	0.667	0.182
4	1.503	0.025	0.125	0.091
5	0.334	0.025	0.25	0.5
6	0.501	0.25	0.909	0.833
7	0.668	0.025	0.091	0.5
8	0.167	0.025	0.111	0.111
9	0.999	0.2	0.889	0.727
10	2.664	0.175	0.875	0.778
11	0.666	0.05	0.667	0.167
12	0.666	0.025	0.333	0.5
13	0.333	0.025	0.5	0.25
14	0.666	0.025	0.5	0.5
15	0.333	0.025	0.5	0.5
16	1.332	0.025	0.5	0.5
17	1.665	0.025	0.5	0.5
18	0.666	0.025	0.5	0.5

Fig. 9 Certainty and coverage factors for the decision algorithm (Fig. 8). Support is obtained by multiplying the number (Fig. 7) with $P(R_i)$, $i = 1, 2, 3, 4$ as defined in Algorithm 3.6

Fig. 10 Graph depicting the actual share prices of selected companies from July to November of 2016 along with benchmark index Nifty (Source: www.nseindia.com)

to be appropriate for the selected companies as indicated by share price variations of these companies. (An increase in share price of Maruti Suzuki Ltd was observed and an appropriate recommendation 'BUY' is obtained from the algorithm. Similarly, changes in share price of ACC cements are not significant and the recommendation obtained is 'HOLD'. Example 3.5 can be extended to a telescoping MRAF over a period of 4 quarters in a typical financial year.

	$E(y_1)$	$E(y_2)$	$E(y_3)$	$E(y_4)$	$E(y_5)$
x_1	0.15	0.1833	0.35	0.6167	0.2
x_2	0.2667	0.367	0.5	0.25	0.1667
x_3	0.4333	0.7	0.417	0.233	0.2667
x_4	0.3	0.4833	0.4	0.3167	0.3833
x_5	0.2167	0.31667	0.633	0.4167	0.3667

Fig. 11 The expected values of criteria

	$E(y_1)$	$E(y_2)$	$E(y_3)$	$E(y_4)$	$E(y_5)$	Rec.
x_1	Very Low	Very Low	Average	High	Very Low	No
x_2	Low	Average	High	Low	Very Low	No
x_3	Average	Very High	Average	Very Low	Low	Yes
x_4	Low	Average	Average	Low	Average	No
x_5	Very Low	Low	High	Average	Average	No

Fig. 12 The linguistic terms to the expected values

It is to be noted that the rating agencies (experts) do not get evaluated for their recommendations. However, in our method Markov chain transition probability matrix yields a telescopic way of assignment of weights which naturally evaluates the quality of recommendations made by the rating agencies (experts). The above example indicates the potential applications of MRAF in real world situations.

Example 3.7 In order to demonstrate that the proposed method can be useful in other similar uncertain and dynamic situations, we consider Example 4.3 from Bingzhen et al. (2019) that describes a multi-criteria decision-making problem and utilizes multi-granulation vague rough set method to arrive at a decision in medical diagnosis. With the help of MRAF, probability distribution function was assigned to experts R_1, R_2, R_3 as $[1/3 \ 1/6 \ 1/2]$. Later, the expected values of criteria are calculated using the probabilities of R_1, R_2, R_3 and are shown in Fig. 11.

The expected values of various criteria are denoted as Very Low, Low, Average, High and Very High if $0.1 \leq$ expected value < 0.25 , $0.25 \leq$ expected value < 0.34 , $0.34 \leq$ expected value < 0.49 , $0.49 \leq$ expected value < 0.64 and expected value ≥ 0.64 , respectively. The values of Recommendation are taken as the sum of all the expected values corresponding to each alternative and denote them as ‘Yes’ if Recommendation value ≥ 2 and ‘No’ if Recommendation value < 2 as shown in Fig. 12.

The decision rules are formed for the criteria given in decision Fig. 12 and are presented in Fig. 13.

Coverage and certainty factors for the above decision algorithm are calculated using the proposed algorithm and are shown in Fig. 14.

From the above values, we can observe that if $(E(y_1), \text{Average}), (E(y_2), \text{Very High})$ and $(E(y_4), \text{Very Low})$, then the Recommendation is Yes to say that the patient has gas-

1	If $(E(y_1), \text{Very Low}), (E(y_2), \text{Very Low})$ and $(E(y_4), \text{High})$ then Recommendation is No.
2	If $(E(y_1), \text{Low}), (E(y_2), \text{Average})$ and $(E(y_4), \text{Low})$ then Recommendation is No.
3	If $(E(y_1), \text{Average}), (E(y_2), \text{Very High})$ and $(E(y_4), \text{Very Low})$ then Recommendation is Yes.
4	If $(E(y_1), \text{Low}), (E(y_2), \text{Average})$ and $(E(y_4), \text{Low})$ then Recommendation is No.
5	If $(E(y_1), \text{Very Low}), (E(y_2), \text{Low})$ and $(E(y_4), \text{Average})$ then Recommendation is No.

Fig. 13 The decision algorithm associated with Fig. 12

Decision rule	Support	Certainty	Coverage
1	1	1	0.25
2	1	0.5	0.25
3	1	1	1
4	1	0.5	0.25
5	1	1	0.25

Fig. 14 Certainty and coverage factors for the decision algorithm in Fig. 13

troenteritis with probability 1. This is in agreement with the decision obtained using multi-granulation vague rough set theory, thus confirming the usefulness of MRAF and Pawlak’s decision algorithm.

Fuzzy tolerance relation Shivani et al. (2020) introduced a novel approach for attribute selection in set-valued information system based on tolerance rough set theory and defined fuzzy relation between two objects using a similarity threshold. In this paper, we generalized this relation for alternative selection in set-valued information system by using the concepts Markov rough approximation frame work (MRAF) and reference points.

Definition 3.8 Let $(U, AL, V, h), \forall a \in AL$ be a set-valued information system, where $U = \{u_1, u_2, \dots, u_n\}$ is a non-empty finite set of objects (Criteria).

$AL = \{a_1, a_2, \dots, a_n\}$ is a non-empty set of alternatives, $V = \bigcup_{a \in AL} V_a$ where V_a is the set of criteria values associated with each alternative $a \in AL$, $h : U \times AL \rightarrow V$ such that $\forall a \in AL, \forall u \in U, h(u, a) = V_a$ and $X = \{R_1, R_2, \dots, R_n\}$ is the collection of reference points from the universe U , then the fuzzy relation is defined as

$$\sum_{k=1}^n \mu_a(u_i, u_j)_{R_k} P(R_k) = \sum_{k=1}^n 2 \frac{|a(u_i)_{R_k} \cap a(u_j)_{R_k}|}{|a(u_i)_{R_k}| + |a(u_j)_{R_k}|} P(R_k),$$

where $P(R_k), k = 1, 2, \dots, n$ is the probability distribution for the reference points $R_k, k = 1, 2, \dots, n$ obtained from MRAF.

The binary relation using a threshold value $\alpha \in (0, 1)$ is defined as

$$T_{(\alpha, \alpha)} = \left\{ (u_i, u_j) \mid \sum_{k=1}^n \mu_a(u_i, u_j)_{R_k} P(R_k) \geq \alpha \right\}.$$

Lemma 3.9 $T_{(a,\alpha)}$ is a tolerance relation.

Proof (1) Reflexive: Consider

$$\begin{aligned} & \sum_{k=1}^n \mu_a(u_i, u_i)_{R_k} P(R_k) \\ &= \sum_{k=1}^n 2 \frac{|a(u_i)_{R_k} \cap a(u_i)_{R_k}|}{|a(u_i)_{R_k}| + |a(u_i)_{R_k}|} P(R_k) \\ &= 2 \frac{|a(u_i)_{R_1} \cap a(u_i)_{R_1}|}{|a(u_i)_{R_1}| + |a(u_i)_{R_1}|} P(R_1) \\ &\quad + 2 \frac{|a(u_i)_{R_2} \cap a(u_i)_{R_2}|}{|a(u_i)_{R_2}| + |a(u_i)_{R_2}|} P(R_2) + \dots \\ &\quad + 2 \frac{|a(u_i)_{R_n} \cap a(u_i)_{R_n}|}{|a(u_i)_{R_n}| + |a(u_i)_{R_n}|} P(R_n). \\ &= \frac{2|a(u_i)_{R_1}|}{2|a(u_i)_{R_1}|} P(R_1) + \frac{2|a(u_i)_{R_2}|}{2|a(u_i)_{R_2}|} P(R_2) + \dots \\ &\quad + \frac{2|a(u_i)_{R_n}|}{2|a(u_i)_{R_n}|} P(R_n) \\ &= P(R_1) + P(R_2) + \dots + P(R_n) \\ &= 1 \geq \alpha. \end{aligned}$$

This implies $(u_i, u_i) \in T_{(a,\alpha)}$. Hence $T_{(a,\alpha)}$ is reflexive.

(2) Symmetry: Let $(u_i, u_j) \in T_{(a,\alpha)}$.
Then

$$\begin{aligned} & \sum_{k=1}^n \mu_a(u_i, u_j)_{R_k} P(R_k) \geq \alpha \\ &\Rightarrow [2 \frac{|a(u_i)_{R_1} \cap a(u_j)_{R_1}|}{|a(u_i)_{R_1}| + |a(u_j)_{R_1}|} P(R_1) \\ &\quad + 2 \frac{|a(u_i)_{R_2} \cap a(u_j)_{R_2}|}{|a(u_i)_{R_2}| + |a(u_j)_{R_2}|} P(R_2) + \dots \\ &\quad + 2 \frac{|a(u_i)_{R_n} \cap a(u_j)_{R_n}|}{|a(u_i)_{R_n}| + |a(u_j)_{R_n}|} P(R_n)] \geq \alpha \\ &\Rightarrow [2 \frac{|a(u_j)_{R_1} \cap a(u_i)_{R_1}|}{|a(u_j)_{R_1}| + |a(u_i)_{R_1}|} P(R_1) \\ &\quad + 2 \frac{|a(u_j)_{R_2} \cap a(u_i)_{R_2}|}{|a(u_j)_{R_2}| + |a(u_i)_{R_2}|} P(R_2) + \dots \\ &\quad + 2 \frac{|a(u_j)_{R_n} \cap a(u_i)_{R_n}|}{|a(u_j)_{R_n}| + |a(u_i)_{R_n}|} P(R_n)] \geq \alpha \\ &\Rightarrow \sum_{k=1}^n \mu_a(u_j, u_i)_{R_k} P(R_k) \geq \alpha. \end{aligned}$$

This implies $(u_j, u_i) \in T_{(a,\alpha)}$. Hence $T_{(a,\alpha)}$ is symmetric.
Thus $T_{(a,\alpha)}$ is a tolerance relation. $\square$

The tolerance class for an attribute u_i with respect to $T_{(a,\alpha)}$ is $[T_{(a,\alpha)}](u_i) = \{u_j \in U \mid u_i T_{(a,\alpha)} u_j\}$.

The lower and upper approximations of a non-empty subset $B \subseteq U$ as:

$$\begin{aligned} \underline{T}_{(a,\alpha)}(B) &= \{u_i \in U \mid [T_{(a,\alpha)}](u_i) \subseteq B\}. \\ \overline{T}_{(a,\alpha)}(B) &= \{u_i \in U \mid [T_{(a,\alpha)}](u_i) \cap B \neq \emptyset\}. \end{aligned}$$

Properties of lower and upper approximations with respect to fuzzy tolerance relation In the following, we assume that (U, AL, V, h) be a set-valued information system, $a \in AL$ and $B \subseteq U$, $\alpha \in (0, 1)$.

Theorem 3.10 $\underline{T}_{(a,\alpha)}(B) \subseteq B \subseteq \overline{T}_{(a,\alpha)}(B)$.

Notation 3.11 Let $\{R_1, R_2, \dots, R_n\}$ be the reference points in the universe U . Then

$$\begin{aligned} \mu_a(u_i, u_j)_{\min} &= \text{Minimum of} \\ &\quad \{\mu_a(u_i, u_j)_{R_1}, \mu_a(u_i, u_j)_{R_2}, \dots, \mu_a(u_i, u_j)_{R_n}\} \\ \mu_a(u_i, u_j)_{\max} &= \text{Maximum of} \\ &\quad \{\mu_a(u_i, u_j)_{R_1}, \mu_a(u_i, u_j)_{R_2}, \dots, \mu_a(u_i, u_j)_{R_n}\} \end{aligned}$$

The binary relations using a threshold value $\alpha \in (0, 1)$ are defined as

$$\begin{aligned} T_{(a,\alpha)}^{\max} &= \{(u_i, u_j) \mid \mu_a(u_i, u_j)_{\max} \geq \alpha\}. \\ T_{(a,\alpha)}^{\min} &= \{(u_i, u_j) \mid \mu_a(u_i, u_j)_{\min} \geq \alpha\}. \end{aligned}$$

The tolerance classes for an attribute u_i with respect to $T_{(a,\alpha)}$ are

$$\begin{aligned} [T_{(a,\alpha)}^{\max}](u_i) &= \{u_j \in U \mid u_i T_{(a,\alpha)}^{\max} u_j\}. \\ [T_{(a,\alpha)}^{\min}](u_i) &= \{u_j \in U \mid u_i T_{(a,\alpha)}^{\min} u_j\}. \end{aligned}$$

The lower and upper approximations of a non-empty subset $B \subseteq U$ are

$$\begin{aligned} \underline{T}_{(a,\alpha)}^{\max}(B) &= \{u_i \in U \mid [T_{(a,\alpha)}^{\max}](u_i) \subseteq B\}. \\ \overline{T}_{(a,\alpha)}^{\max}(B) &= \{u_i \in U \mid [T_{(a,\alpha)}^{\max}](u_i) \cap B \neq \emptyset\}. \\ \underline{T}_{(a,\alpha)}^{\min}(B) &= \{u_i \in U \mid [T_{(a,\alpha)}^{\min}](u_i) \subseteq B\}. \\ \overline{T}_{(a,\alpha)}^{\min}(B) &= \{u_i \in U \mid [T_{(a,\alpha)}^{\min}](u_i) \cap B \neq \emptyset\} \end{aligned}$$

respectively.

Theorem 3.12 $\underline{T}_{(a,\alpha)}^{\max}(B) \subseteq \underline{T}_{(a,\alpha)}(B) \subseteq \underline{T}_{(a,\alpha)}^{\min}(B) \subseteq B \subseteq \overline{T}_{(a,\alpha)}^{\min}(B) \subseteq \overline{T}_{(a,\alpha)}(B) \subseteq \overline{T}_{(a,\alpha)}^{\max}(B)$.

Proof By the fundamental theorem of linear programming, the expression $\sum_{k=1}^n \mu_a(u_i, u_j)_{R_k} P(R_k)$ attains its extreme values at the vertices (corners) of the convex polytope (polygon) formed by it. Clearly, the minimum value is attained at $\mu_a(u_i, u_j)_{\min}$ and maximum value is attained at $\mu_a(u_i, u_j)_{\max}$.

Now, we prove that $\underline{T}_{(a,\alpha)\max}(B) \subseteq \underline{T}_{(a,\alpha)}(B)$. Let $z \in \underline{T}_{(a,\alpha)\max}(B) \Rightarrow [T_{(a,\alpha)\max}](z) \subseteq B$. Now, we prove that $[T_{(a,\alpha)}](z) \subseteq B$. Let $y \in [T_{(a,\alpha)}](z)$. Then

$$\sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \alpha,$$

As

$$\mu_a(y, z)_{\max} \geq \sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \alpha,$$

we get $\mu_a(y, z)_{\max} \geq \alpha$. This implies

$$y \in [T_{(a,\alpha)\max}](z).$$

This gives

$$[T_{(a,\alpha)}](z) \subseteq [T_{(a,\alpha)\max}](z).$$

As $[T_{(a,\alpha)\max}](z) \subseteq B$, we get $[T_{(a,\alpha)}](z) \subseteq B$. This implies $z \in \underline{T}_{(a,\alpha)}(B)$. Now, we prove that $\underline{T}_{(a,\alpha)}(B) \subseteq \underline{T}_{(a,\alpha)\min}(B)$. Let $z \in \underline{T}_{(a,\alpha)}(B) \Rightarrow [T_{(a,\alpha)}](z) \subseteq B$. Now, we prove that $[T_{(a,\alpha)\min}](z) \subseteq B$. Let $y \in [T_{(a,\alpha)\min}](z)$. Then $\mu_a(y, z)_{\min} \geq \alpha$. As

$$\sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \mu_a(y, z)_{\min} \geq \alpha,$$

we get

$$\sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \alpha.$$

This implies

$$y \in [T_{(a,\alpha)}](z).$$

This gives

$$[T_{(a,\alpha)\min}](z) \subseteq [T_{(a,\alpha)}](z).$$

As $[T_{(a,\alpha)}](z) \subseteq B$, we get $[T_{(a,\alpha)\min}](z) \subseteq B$. This implies

$$z \in \underline{T}_{(a,\alpha)\min}(B).$$

Clearly, from Theorem 3.10, we have $\underline{T}_{(a,\alpha)\min}(B) \subseteq B \subseteq \overline{T}_{(a,\alpha)\min}(B)$. Now, we prove that $\overline{T}_{(a,\alpha)\min}(B) \subseteq \overline{T}_{(a,\alpha)}(B)$.

Let $z \in \overline{T}_{(a,\alpha)\min}(B)$. Then $[T_{(a,\alpha)\min}](z) \cap B \neq \phi$. Now, take $y \in [T_{(a,\alpha)\min}](z)$. Then $\mu_a(y, z)_{\min} \geq \alpha$. As $\sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \mu_a(y, z)_{\min} \geq \alpha$, we get

R_1	a_1	a_2	a_3	a_4
u_1	$\{1, 2, 3\}$	$\{1, 2\}$	$\{3, 4\}$	1
u_2	$\{1, 2\}$	$\{1, 2, 3, 4\}$	$\{1, 2\}$	2
u_3	1	1	$\{3, 4\}$	$\{1, 2, 3\}$
u_4	2	$\{2, 3\}$	$\{1, 2, 3, 4\}$	3
u_5	$\{1, 2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3, 4\}$

R_2	a_1	a_2	a_3	a_4
u_1	$\{1, 2\}$	$\{1, 2, 3\}$	$\{1, 2, 3, 4\}$	$\{3, 4\}$
u_2	$\{2, 3\}$	2	1	$\{1, 2\}$
u_3	1	$\{3, 4\}$	$\{2, 3, 4\}$	$\{2, 3\}$
u_4	$\{3, 4\}$	$\{1, 2\}$	3	1
u_5	3	$\{2, 3\}$	$\{1, 2, 3\}$	4

Fig. 15 Set-valued information system with respect to the reference points R_1 and R_2

$y \in [T_{(a,\alpha)}](z)$. This implies $[T_{(a,\alpha)\min}](z) \subseteq [T_{(a,\alpha)}](z)$. As $[T_{(a,\alpha)\min}](z) \cap B \neq \phi$, we get $[T_{(a,\alpha)}](z) \cap B \neq \phi$. Hence $z \in \overline{T}_{(a,\alpha)}(B)$. Now, we prove that $\overline{T}_{(a,\alpha)}(B) \subseteq \overline{T}_{(a,\alpha)\max}(B)$. Let $z \in \overline{T}_{(a,\alpha)}(B)$. Then $[T_{(a,\alpha)}](z) \cap B \neq \phi$. Now, take $y \in [T_{(a,\alpha)}](z)$. Then $\sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \alpha$. As $\mu_a(y, z)_{\max} \geq \sum_{k=1}^n \mu_a(y, z)_{R_k} P(R_k) \geq \alpha$, we get $y \in [T_{(a,\alpha)\max}](z)$. This implies $[T_{(a,\alpha)}](z) \subseteq [T_{(a,\alpha)\max}](z)$. As $[T_{(a,\alpha)}](z) \cap B \neq \phi$, we get $[T_{(a,\alpha)\max}](z) \cap B \neq \phi$. Hence $z \in \overline{T}_{(a,\alpha)\max}(B)$. $\square$

Here we provide an example to select the best alternative in multi-criteria decision-making problem by using the tolerance relation and the proposed algorithm.

Example 3.13 The set-valued information system with alternatives a_1, a_2, a_3, a_4 and their corresponding attributes u_1, u_2, u_3, u_4, u_5 with respect to reference points R_1 and R_2 is given in Fig. 15.

By using MRAF, the probability assigned to the reference points $[R_1, R_2]$ as $[\frac{1}{4}, \frac{3}{4}]$. Then the values obtained by tolerance relation (Definition 3.8) are presented in Fig. 16.

In order to apply the decision algorithm, the sum of each column in Fig. 16 is compiled and the compiled values (u_i, a_j) are denoted as Low, Average, Good and Very Good if $(u_i, a_j) \leq 1.75$, $1.75 \leq (u_i, a_j) < 2.5$, $2.5 \leq (u_i, a_j) < 3.25$, $3.25 \leq (u_i, a_j) < 4$, respectively. The values of Recommendation are taken as the sum of all values corresponding to each alternative and denoted them as 'Yes' if Recommendation value ≥ 14 and 'No' if Recommendation value < 14 and are presented in Fig. 17.

Then by using the proposed algorithm, the alternatives a_2 and a_3 can be selected with the probability 1.

The above example clearly shows that fuzzy tolerance relation can be used to arrive at a best alternative in multi-criteria decision problems.

$\sum_{k=1}^n \mu_{a_1(u_i, u_j)_{R_k P(R_k)}}$					$\sum_{k=1}^n \mu_{a_2(u_i, u_j)_{R_k P(R_k)}}$					$\sum_{k=1}^n \mu_{a_3(u_i, u_j)_{R_k P(R_k)}}$					$\sum_{k=1}^n \mu_{a_4(u_i, u_j)_{R_k P(R_k)}}$				
1	0.575	0.6275	0.125	0.25	1	0.5425	0.4675	0.725	0.725	1	0.3	0.8928	0.4675	0.7678	1	0	0.5	0	0.5025
0.575	1	0.1675	0.5425	0.7025	0.5425	1	0.1	0.67	0.67	0.3	1	0	0.1675	0.5	0	1	0.5	0.5025	0.125
0.6275	0.1675	1	0	0.125	0.4675	0.1	1	0	0.375	0.8928	0	1	0.5425	0.6275	0.5	0.5	1	0.125	0.1675
0.125	0.5425	0	1	0.6275	0.725	0.67	0	1	0.625	0.4675	0.1675	0.5425	1	0.5425	0	0.5025	0.125	1	0.125
0.25	0.7025	0.125	0.6275	1	0.725	0.67	0.375	0.625	1	0.7678	0.5	0.6275	0.5425	1	0.5025	0.125	0.1675	0.125	1

Fig. 16 Tolerance relation values

	a_1	a_2	a_3	a_4
u_1	2.5775(Average)	3.46(Very Good)	3.4281(Very Good)	2.0025(Average)
u_2	2.9875(Good)	2.9825(Good)	1.9675(Average)	2.1275(Average)
u_3	1.92(Average)	1.9425(Average)	3.0628(Good)	2.2925(Good)
u_4	2.295(Good)	3.02(Good)	2.72(Good)	1.7525(Average)
u_5	2.705(Average)	3.395(Very Good)	3.4378(Very Good)	1.92(Average)
Rec.	12.4850(No)	14.8000(Yes)	14.6162(Yes)	10.0950(No)

Fig. 17 Compilation values of (u_i, a_j)

	b_1	b_2	b_3	b_4
u_1	1	2	{2, 3, 4}	{1, 2, 3}
u_2	{1, 2}	{1, 2}	{3, 4}	1
u_3	{2, 3, 4}	{1, 3}	{1, 2, 3}	{2, 3, 4}
u_4	{2, 3}	{1, 2, 3, 4}	1	{2, 3}
u_5	2	1	{2, 3}	{3, 4}

	b_1	b_2	b_3	b_4
u_1	{2, 3}	{2, 3, 4}	{1, 2}	4
u_2	{1, 2}	{1, 2, 3}	1	{1, 2}
u_3	{2, 3}	{1, 2}	4	{2, 3}
u_4	{2, 3, 4}	{2, 3}	{2, 3}	{3, 4}
u_5	{1, 2, 3}	{2, 3, 4}	{1, 2}	{1, 2, 3}

Fig. 18 Set-valued information system with respect to the reference points R_1 and R_2

Now, we generalize the fuzzy tolerance relation given by Shivani et al. (2020) for selecting the attributes in set-valued information system by using MRAF.

Definition 3.14 Let $(U, AT, V, h), \forall b \in AT$ be a set-valued information system, where $U = \{u_1, u_2, \dots, u_n\}$ is a non-empty finite set of objects, $AT = \{b_1, b_2, \dots, b_n\}$ is a non-empty set of attributes, $V = \bigcup_{b \in AT} V_b$ where V_b is the set of alternative values associated with each attribute $b \in AT$, $h : U \times AT \rightarrow V$ such that $\forall b \in AT, \forall u \in U, h(u, b) = V_b$ and $X = \{R_1, R_2, \dots, R_n\}$ is the collection of reference points from the universe U , then the fuzzy relation is defined as $\sum_{k=1}^n \mu_b(u_i, u_j)_{R_k} P(R_k) = \sum_{k=1}^n 2 \frac{|b(u_i)_{R_k} \cap b(u_j)_{R_k}|}{|b(u_i)_{R_k}| + |b(u_j)_{R_k}|} P(R_k)$, where $P(R_k)$, $k = 1, 2, \dots, n$ is the probability distribution for the reference points R_k , $k = 1, 2, \dots, n$ obtained from MRAF.

For a set of alternatives $B \subseteq AT$, a fuzzy relation is defined as $\mu_B(u_i, u_j) = \inf_{b \in B} (\sum_{k=1}^n \mu_b(u_i, u_j)_{R_k} P(R_k))$.

The binary relation using a threshold value $\alpha \in (0, 1)$ is defined as

$$T_{(b, \alpha)} = \left\{ (u_i, u_j) \mid \sum_{k=1}^n \mu_b(u_i, u_j)_{R_k} P(R_k) \geq \alpha \right\}.$$

$\sum_{k=1}^n \mu_{b_1(u_i, u_j)_{R_k P(R_k)}}$					$\sum_{k=1}^n \mu_{b_2(u_i, u_j)_{R_k P(R_k)}}$					$\sum_{k=1}^n \mu_{b_3(u_i, u_j)_{R_k P(R_k)}}$					$\sum_{k=1}^n \mu_{b_4(u_i, u_j)_{R_k P(R_k)}}$				
1	0.5567	0.67	0.125	0.533	1	0.67	0.2667	0.67	0.67	1	0.6133	0.2233	0.3333	0.9333	1	0.1667	0.2233	0.6133	0.133
0.5567	1	0.4667	0.4333	0.7567	0.67	1	0.7	0.7567	0.67	0.6133	1	0.1333	0	0.6133	0.1667	1	0.333	0	0.533
0.67	0.4667	1	0.8	0.7	0.2667	0.7	1	0.5567	0.7567	0.2233	0.1333	1	0.1667	0.2667	0.2233	0.333	1	0.933	0.8
0.5333	0.4333	0.8	1	0.67	0.67	0.7567	0.5567	1	0.67	0.3333	0	0.1667	1	0.3333	0.6133	0	0.933	1	0.433
0.5333	0.7567	0.7	0.67	1	0.67	0.67	0.7567	0.67	1	0.9333	0.6133	0.2667	0.3333	1	0.133	0.533	0.8	0.433	1

Fig. 19 Tolerance relation values

	b_1	b_2	b_3	b_4
u_1	3.2933(Good)	3.2767(Good)	3.1032(Good)	2.1363(Average)
u_1	3.2134(Good)	3.7967(Good)	2.3599(Average)	2.0327(Average)
u_2	3.6367(Good)	3.2801(Good)	1.79(Low)	3.2893(Good)
u_3	3.0283(Good)	3.6534(Good)	1.8333(Low)	2.9793(Average)
u_4	3.6597(Good)	3.7667(Good)	3.1466(Good)	2.899(Average)
Rec.	16.8314 (Yes)	17.7736 (Yes)	12.233(No)	13.3366(No)

Fig. 20 Compilation values of (u_i, b_j)

For a set of attributes $B \subseteq AT$, we define a relation as

$$T_{(B, \alpha)} = \{(u_i, u_j) \mid \mu_B(u_i, u_j) \geq \alpha\}.$$

Theorem 3.15 $T_{(b, \alpha)}$ is a tolerance relation.

Example 3.16 The set-valued information system with criteria b_1, b_2, b_3, b_4 and their corresponding alternatives u_1, u_2, u_3, u_4, u_5 with respect to reference points R_1 and R_2 are given in Fig. 18.

By using MRAF, the probability assigned to the reference points $[R_1, R_2]$ as $[\frac{1}{3}, \frac{2}{3}]$. Then the values obtained by tolerance relation are presented in Fig. 19.

In order to apply the decision algorithm, the sum of each column in Fig. 19 is compiled and the compiled values (u_i, b_j) are denoted as Low, Average and Good if $1 \leq (u_i, b_j) < 2$, $2 \leq (u_i, b_j) < 3$ and ≥ 3 , respectively. The values of Recommendation are taken as the sum of all values corresponding to each alternative and denoted them as ‘Yes’ if Recommendation value ≥ 15 and ‘No’ if Recommendation (Rec.) value < 15 and are presented in Fig. 20.

Then by using the proposed algorithm, the attributes b_1 and b_2 can be selected with the probability 1.

4 Conclusions

We have provided applications of Markov frameworks in dealing with decision problems in dynamic and uncertain environments. The MRAF enables the possibility of assigning probabilities in a telescopic manner and Pawlak’s decision algorithm enables in arriving at decisions in multi-criteria decision problems. This method can also be used to supplement other multi-criteria decision-making methods for deeper analysis of the data especially when the values of the ranked alternatives are very close to each other. The proposed algorithm is explained in the stock market environment to give the recommendation SELL, BUY or HOLD for a particular stock. The proposed algorithm is validated in the medical diagnosis problem to give the recommendation. In addition, fuzzy tolerance relation is used along with MRAF as a tool to ascertain critical attributes as well as alternatives in multi-criteria decision problems. Methods presented

in this paper can be used in the development of other methods which deal with multi-criteria decision problems.

Acknowledgements The authors acknowledge anonymous reviewers and the editor for their valuable comments and suggestions. All authors acknowledge the support and encouragement of Manipal Institute of Technology, Manipal Academy of Higher Education (MAHE), India.

Funding Open access funding provided by Manipal Academy of Higher Education, Manipal.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical standard This article does not contain any studies with human participants or animals performed by any of the authors.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Akram M, Dudek WA (2008) Intuitionistic fuzzy left k-ideals of semirings. *Soft Comput* 12:881–890
- Aleksandar R, Vlado S, Bratislav P, Sanja M (2018) An automated system for stock market trading based on logical clustering. *Tech Gaz* 25(4):970–978
- Bingzhen S, Weimin M, Xiangtang C, Xiong Z (2019) Multigranulation vague rough set over two universes and its application to group decision making. *Soft comput* 23:8927–8956
- Bhavanari S, Kuncham SP, Kedukodi BS (2010) Graph of a nearring with respect to an ideal. *Commun Algebra* 38:1957–1962
- Chan KC (2015) Market share modelling and forecasting using Markov chains and alternative models. *Int J Innov Comput Inf Control* 11(4):1205–1218
- Chen MC, Lin CL, Chen AP (2007) Constructing a dynamic stock portfolio decision-making assistance model: using the Taiwan 50 Index constituents as an example. *Soft Comput* 11(12):1149–1156
- Choji DN, Eduno SN, Kassem GT (2013) Markov chain model application on share price movement in stock market. *Comput Eng Intell Syst* 4:84–95
- Ching W, Ng MK (2006) Markov chains: models, algorithms and applications. Springer, New York
- Ciucci D (2008) A unifying abstract approach for rough models. *Lect Notes Artif Int* 5009:371–378
- Davaaz B (2004) Roughness in rings. *Inf Sci* 164:147–163
- Davaaz B (2006) Roughness based on fuzzy ideals. *Inf Sci* 176:2417–2437

- Emrah O, Taylan AA (2017) Financial performance evaluation of Turkish construction companies in Istanbul stock exchange (BIST). *Int J Acad Res Account Finance Manag Sci* 7(3):108–113
- Gong XL, Liu XH, Xiong X, Zhuang XT (2019) Forecasting stock volatility process using improved least square support vector machine approach. *Soft Comput*. <https://doi.org/10.1007/s00500-018-03743-0>
- Gour SMT, Rupak B, Seema S (2018) Stock portfolio selection using Dempster–Shafer evidence theory. *J King Saud Univ Comput Inf Sci* 30:223–235
- Huang SY (ed) (1992) *Intelligent decision support: handbook of applications and advances of rough sets theory*. Springer, Berlin
- Jagadeesha B, Kedukodi BS, Kuncham SP (2016a) Implications on a lattice. *Fuzzy Inf Eng* 8(4):411–425
- Jagadeesha B, Kedukodi BS, Kuncham SP (2016b) Interval valued L-fuzzy ideals based on t-norms and t-conorms. *J Intell Fuzzy Syst* 28(6):2631–2641
- Kedukodi BS, Kuncham SP, Bhavanari S (2007) C-prime fuzzy ideals of nearrings. *Soochow J Math* 33(4):891–901
- Kedukodi BS, Kuncham SP, Bhavanari S (2009) Equiprime, 3-prime and C-prime fuzzy ideals of nearrings. *Soft Comput* 13(10):933–944
- Kedukodi BS, Kuncham SP, Bhavanari S (2010) Reference points and roughness. *Inf Sci* 180(17):3348–3361
- Kedukodi BS, Kuncham SP, Jagadeesha B (2017) Interval valued L-fuzzy prime ideals, triangular norms and partially ordered groups. *Soft comput*. <https://doi.org/10.1007/s00500-017-2798-x>
- Koppula K, Kedukodi BS, Kuncham SP (2018) Markov chains and rough sets. *Soft comput* 23(15):6441–6453
- Koppula K, Kedukodi BS, Kuncham SP (2019) On prime strong ideals of a seminearring. *Mat Vesnik*. http://www.vesnik.math.rs/inpress/mv2019_003.pdf
- Kuncham SP, Jagadeesha B, Kedukodi BS (2016) Interval valued L-fuzzy cosets of nearrings and isomorphism theorems. *Afrika Math* 27(3):393–408
- Kuncham SP, Kedukodi BS, Hari Krishnan P, Bhavanari S (2017) *Nearrings, nearfields and related topics*. World Scientific, Singapore. ISBN 978-981-3207-35-6
- Markovic I, Stojanovic M, Stankovic J, Stankovic M (2017) Stock market trend prediction using AHP and weighted kernel LS-SVM. *Soft Comput* 21(18):5387–5398
- Medhi J (2009) *Stochastic processes*. New Age International Publishers Limited, New Delhi
- Nayak H, Kuncham SP, Kedukodi BS (2018) $\theta\Gamma$ N-group. *Mat Vesnik* 70:64–78
- Orlowska E (1985) Logic of nondeterministic information. *Stud Log* 44(1):91–100
- Pawlak Z (1982) Rough sets. *Int J Comput Inf Sci* 11:341–356
- Pawlak Z (2002) Rough sets and intelligent data analysis. *Inf Sci* 147(1–4):1–12
- Prasenjit M, Ranadive AS (2019) Fuzzy multi-granulation decision-theoretic rough sets based on fuzzy preference relation. *Soft comput* 23:85–99. <https://doi.org/10.1007/s00500-018-3411-7>
- Rezaie K, Majazi DV, Hatami SL, Shirkouhi N (2013) Efficiency appraisal and ranking of decision-making units using data envelopment analysis in fuzzy environment: a case study of Tehran stock exchange. *Neural Comput Appl* 23(Suppl 1):1. <https://doi.org/10.1007/s00521-012-1209-6>
- Shivani S, Shivam S, Tanmoy S, Gaurav S (2020) A fuzzy similarity-based rough set approach for attribute selection in set-valued information systems. *Soft comput* 24:4675–4691
- Sudan J, Raghyendra K, Le HS, Jyotir MC, Manju K, Yadav Navneet, Florentin S (2019) Neutrosophic soft set decision making for stock trending analysis. *Evol Syst* 10:621–627
- Suk JL, Jae JA, Kyong JO, Tae YK (2010) Using rough set to support investment strategies of real-time trading in futures market. *Appl Intell* 32:364–377
- Tavana M, Khanjani SR, Di CD (2017) A chance-constrained portfolio selection model with random-rough variables. *Neural Comput Appl*. <https://doi.org/10.1007/s00521-017-3014-8>
- Xiongwen P, Yanqiang Z, Pan W, Weiwei L, Victor C (2020) An innovative neural network approach for stock market prediction. *J Supercomput*. <https://doi.org/10.1007/s11227-017-2228-y>

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.