Roman domination problem with uncertain positioning and deployment costs

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Abstract

In a connected simple graph, the weighted Roman domination problem is considered at which the cost of positioning at each vertex is imposed in addition to the costs of potential deployments from a vertex to some of its neighboring vertices. Proper decision in practice is prone to a high degree of indeterminacy, mostly raised by unpredictable events that do not obey the rules and prerequisites of the probability theory. In this study, we model this problem with such assumptions in the context of the uncertainty theory initiated by Liu (Uncertainty theory. Studies in fuzziness and soft computing, Springer, Berlin, 2007). Two different optimization models are presented, and a concrete example is provided for illustrative purposes. Weaknesses of the probability theory and fuzzy theory in dealing with this problem are also mentioned in detail.

Keywords Roman-dominating set problem · Positioning and deployment costs · Integer linear programming · Uncertainty theory · Uncertain variable

1 Introduction

Roman domination problem has historical significance and dates back to the fourth century when the emperor of Rome, Constantine the Great, decreed that two types of legions should be positioned in Roman provinces. The first type of legion was particularly skilled agile combatants who could be promptly deployed to an adjacent province for defending against any potential attack. The latter would behave as a local force permanently located in the given province. In addition, no legion could ever depart a province in order to defend another one, if such action leaves the base province unprotected.

In the graph theory language, the problem has been originally introduced by Ian Stewart as the “Roman domination problem” Stewart (1999), each province was denoted by a vertex and linkages between provinces were depicted as edges. The set of vertices in an instance positioned with one or two legions is referred to as a “Roman

dominating set,” and minimum cardinality of such a set is known as “Roman domination number.”

In addition to army placement, the same sort of mathematics is also useful when a planner wants to know the best place in a town to construct a new public service facility such as hospital, fire station and emergency forces’ bases. Such optimization problems can be modeled by Roman domination or its variants. For instance in a fire station location problem, if necessity is declared as “The region, at which just one fire station is established, is allowed to serve its region only. While the region with no fire station must be adjacent to another region with two fire stations, and in the case of accidents in the region, one of the stations in the latter must cover its neighboring region with not fire station. This problem is an example of non-military application of the Roman domination problem.

Another instance is the ad hoc wireless networks as a special type of wireless mobile networks in which a collection of mobile hosts with wireless network interfaces form a temporary network, without the aid of any established infrastructure or centralized administration. In these networks, wireless hubs are more expensive but can serve neighboring locations. The applications of ad hoc wireless networks range from civilian use (distributed computing, sensor networks) to disaster recovery (search-and-rescue) and military use (battlefield) Wu and Li (2000). This

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problem also can be readily modeled as a Roman domination problem and its variants.

Most of studies on this problem and its different versions have been focused on finding tighter bounds for the associated domination numbers [See, e.g., Burger et al. (2013), Chambers et al. (2009), Chun-Hung and Chang (2012)]. Meanwhile, exact solutions are available for some special graphs such as paths, cycles and their Cartesian products Pavlič and Žerovnik (2012). Optimization models are not only capable of identifying the Roman domination number itself in a general graph, but also provide an instance of the associated set. However, almost all models are in the form of linear binary program [See, e.g., Ivanović (2016); ReVelle and Rosing (2000)] which suffers from NP-hardness. Relaxations [See, e.g., Ghaffari-Hadigheh and Djahangiri (2015)] and evolutionary methods Hedar and Ismail (2010) enable reasonable approximations to the solution.

The Roman domination problem is away from practice to a great extent if the cost of positioning of legions and the expenditure imposed by the possible deployment of some legions to their neighboring provinces are overlooked. Moreover, exogenous and endogenous sources of uncertainty exist in the nature of the problem. In civilian applications, consider the problem of identifying fire stations places for a new establishing city for an instance. Construction costs of stations might have almost exact values, while deployment cost of a fire engine from an station to an accident is not known before establishment of the city and inhabitants settlement. One may argue this claim and say that there exist some data, collected in similar situations, which could be a basis for approximating a probability distribution of this unknown cost, and therefore stochastic approach is justifiable. However, a question here still is unanswered “Does this distribution good fit to the uncertain parameters of an unsettled city?”.

In some cases, this sort of data could be even misleading and result in disastrous consequences. Consider augmenting of temporary hospitals to the existing healthcare system as one solution for dealing with a surge of patients related to war, pandemic disease outbreaks or natural disasters. These situations are almost always unprecedented and unrepeatability characteristics of the problem in question, and lack of historical data does not permit the use of probability theory in the model.

In such cases, a reasonable approach is to refer to an expert and model the human reasoning in a fashionable way. One may pose the fuzzy theory first. However, this theory has self-contradictory in using some fuzzy numbers which make the fuzzy theory far from applicability in dealing with indeterminacy of parameter [see Sect. 7 for detailed discussion]. In addition, one of the important questions (and still an open unsolved problem) in the fuzzy

theory is the evaluation of an appropriate membership value. Moreover, it was shown that fuzzy theory may not be an appropriate tool to deal with large-scale decision-making problems Biswas (2016).

Uncertainty theory, introduced by Liu in (2007) and then completed in 2010 Liu (2010), is a mathematical framework to model human reasoning. Question and answers would be provided to express the uncertainty on parameters. For example, how much could be your belief that the cost of positioning a legion at province i is not less than a certain value, say a_i ? What is your belief degree that this cost would not be more than an amount, say b_i ? For a value $x \in (a_i, b_i)$, what could be your belief rate that the cost is not more than x ? Provided answers for such questions can be used to construct an uncertainty distribution to an uncertain parameter of the problem Liu (2007). Uncertainty theory has proved its ability in modeling such problems in many fields. Applications in decision making abound and include DEA [See for recent developments Lio and Liu (2017), Kang et al. (2014), Nejad and Ghaffari-Hadigheh (2018)] and weighted domination problem Djahangiri and Ghaffari-Hadigheh (2018), to name only a few.

In this paper, we consider the problem in the context of the uncertainty theory, at which costs of positioning and deployment are uncertain variables. Binary uncertain linear optimization models are introduced for solving the raised problem, and a simple illustrative example is are provided.

This paper is organized as follows. Definition of the weighted Roman domination problem with positioning and deployment costs is given in Sect. 2. A binary linear programming formulation is provided in this section as well. Some basic notions of the uncertainty theory are reviewed in Sect. 3. Uncertain version of the problem is studied in Sect. 4, and an associated optimization model is followed in Sect. 5. A simple concrete example is presented in Sect. 6. Drawbacks of the probability theory and the fuzzy theory in dealing with the problem under study are mentioned in detail in Sect. 7. Concluding remarks and outlook on the further works direction are given in the final section.

2 Problem definition

Let $G = (V, E)$ represent an undirected simple graph with a vertex set V and edge set E . Here, each vertex $v \in V$ represents a province and each edge $e \in E$ represents an existing linkage between two adjacent provinces. The open neighborhood set N_v is the set at which each vertex $u \in N_v$ is adjacent to vertex v . Additionally, the function $f : V \rightarrow \{0, 1, 2\}$ corresponding to the number of legions assigned to a province (represented by vertex v). This function has to satisfy the condition that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N_v$ with $f(u) = 2$. This

means that an undefended province v must be adjacent to at least one province with two stationed legions, one of them is skilled and agile.

Let w_i denote the cost of positioning a legion at vertex i when $f(i) = 1$ and w'_i denote this cost when $f(i) = 2$. Naturally, $w'_i \geq w_i$, while there is no proportional relation between them nor such restriction reduces the generality of the problem. Further, let there be an unavoidable cost of deployment from vertex i to vertex j , as c_{ij} only if $f(i) = 2$ and $f(j) = 0$.

2.1 Binary linear programming formulation

Let us first introduce a crisp optimization model for solving the proposed problem. For such a function f , let

$$x_i = \begin{cases} 1 & f(i) = 1 \\ 0 & o.w. \end{cases}, y_i = \begin{cases} 1 & f(i) = 2 \\ 0 & o.w. \end{cases},$$

$$t_{ij} = \begin{cases} 1 & f(i) = 2 \text{ and } f(j) = 0 \\ 0 & o.w. \end{cases}.$$

Then,

$$\min \sum_{i \in V} w_i x_i + \sum_{i \in V} w'_i y_i + \sum_{(i,j) \in E} c_{ij} t_{ij} \quad (1)$$

$$\text{s.t. } x_i + y_i + \sum_{j \in N_i} y_j \geq 1 \forall i \in V \quad (2)$$

$$x_i + y_i \leq 1 \forall i \in V \quad (3)$$

$$y_j - x_i \leq t_{ij}, i \in V, j \in N_i \quad (4)$$

$$x_i, y_i, t_{ij} \in \{0, 1\}, \quad (5)$$

can solve the problem in question. Objective function (1) not only considers the total cost of positioning the legions, but also respects the cost of potential deployment of legions throughout the network. Constraints (2) guarantee that an undefended province (the one with $f(i) = 0$) has to be in the neighborhood of at least one province with two assigned legions (with the one with $f(j) = 2$). Furthermore, Conditions (3) say that there is no need to station a province by one legion when it is positioned by two, and vice versa. In respect of constraint (4), accompanying the positivity of c_{ij} guarantees that $t_{ij} = 1$ only when $f(i) = 2$ and $f(j) = 0$.

3 Some basic notions from uncertainty theory

Consider a nonempty set Γ as a universal set and a σ -algebra $\mathcal{L}$ over Γ consisting of its subsets. The pair $(\Gamma, \mathcal{L})$ is called a measurable space, and each element of $\mathcal{L}$ is called an event. A measurable function f is a function from

the measurable space $(\Gamma, \mathcal{L})$ to $\mathbb{R}$ if $f^{-1}(B) = \{v \in \Gamma \mid f(v) \in B\} \in \mathcal{L}$ for any Borel set B of real numbers.

An uncertain measure $\mathcal{M}$ is defined as a function from the σ -algebra $\mathcal{L}$ to $[0, 1]$ satisfying the following axioms. Here, $\mathcal{M}\{A\}$ represents the belief degree that the event A will happen.

Axiom 1 (Normality) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ .

Axiom 2 (Duality) $\mathcal{M}\{A\} + \mathcal{M}\{A^c\} = 1$ for any event A .

Axiom 3 (Subadditivity) $\mathcal{M}\left\{\bigcup_{k=1}^{\infty} A_k\right\} \leq \sum_{k=1}^{\infty} \mathcal{M}\{A_k\}$ for every countable sequence of events $A_k, k \geq 1$.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is referred to as an uncertainty space. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots$. Set $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots$, a measurable rectangle in Γ is defined as $A = A_1 \times A_2 \times \dots$, where $A_k \in \mathcal{L}_k$ for $k = 1, 2, \dots$. The product σ -algebra $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \dots$ is the smallest σ -algebra containing all measurable rectangles of Γ . The product uncertain measure $\mathcal{M}$ on the product σ -algebra $\mathcal{L}$ is defined by Liu (2010) as follows.

Axiom 4 (Product) For uncertainty spaces $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k), k = 1, 2, \dots$, the product uncertain measure $\mathcal{M}$ is the one satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} A_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{A_k\},$$

where A_k is an event in $\mathcal{L}_k, k = 1, 2, \dots$.

It is important to notice that this axiom differentiates the probability measure from the uncertainty measure Liu (2015).

3.1 Uncertain variable

Expression of quantities in an uncertain environment is fulfilled by the concept of uncertain variables. An uncertain variable ξ is a measurable function on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ Liu (2007). An uncertain variable ξ is called nonnegative if $\mathcal{M}\{\xi < 0\} = 0$, and positive if $\mathcal{M}\{\xi \leq 0\} = 0$. Incomplete information on uncertain variables is described by uncertainty distribution. For uncertain variable ξ , a real-valued function Φ defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \forall x \in \mathbb{R}, \quad (6)$$

is called uncertainty distribution of ξ Liu (2007). For a continuous uncertainty distribution Φ , $\mathcal{M}\{\xi < x\} = \Phi(x)$ also is satisfied. A continuous and strictly increasing uncertainty distribution with respect to x that $0 < \Phi(x) < 1, \lim_{x \rightarrow -\infty} \Phi(x) = 0$ and $\lim_{x \rightarrow +\infty} \Phi(x) = 1$ is said to be regular Liu (2010). A regular uncertainty distribution $\Phi(x)$ for which $\Phi(x) \in (0, 1)$ has a well-defined

inverse function $\Phi^{-1}(\alpha)$ on $(0, 1)$ and is called inverse uncertainty distribution of ξ . Some uncertain variables which can be separately defined on different uncertainty spaces are independent. Formally, independency is defined as follows Liu (2009). Uncertain variables $\xi_1, \dots, \xi_n$ satisfying

$$\mathcal{M}\left\{\bigcap_{k=1}^n (\xi_k \in B_k)\right\} = \bigwedge_{k=1}^n \mathcal{M}\{\xi_k \in B_k\}, \quad (7)$$

for any Borel sets $B_1, \dots, B_n$ are said to be independent.

The following theorem plays a key role in reducing our uncertainty model to a deterministic optimization problem.

Theorem 1 Liu (2015) Let $\xi_1, \dots, \xi_n$ be independent uncertain variables with inverse uncertainty distributions $\Phi_1^{-1}, \dots, \Phi_n^{-1}$, respectively. Let the function $f(\xi_1, \dots, \xi_n)$ be strictly increasing with respect to $\xi_1, \dots, \xi_m$, $m \leq n$, and strictly decreasing with respect to $\xi_{m+1}, \dots, \xi_n$. Then, $\mathcal{M}\{f(\xi_1, \dots, \xi_n) \leq 0\} \geq \alpha$ if and only if $f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)) \leq 0$.

The expected value of an uncertain variable ξ is defined by Liu (2007)

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi > r\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\} dr,$$

provided that at least one of the two integrals is finite. For more details on uncertainty theory, the interested reader is referred to Liu (2015).

There are different uncertain variables in the literature. Without loss of generality, we only consider the linear uncertain variable, defined as

$$\Phi(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x \geq b, \end{cases}$$

and denoted by $\mathcal{L}(a, b)$, provided that $a < b$. Its inverse is $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$, and its expected value is $\frac{b+a}{2}$.

4 Uncertain weighted Roman domination models

As mentioned beforehand, parameters in the weighted Roman domination problem are highly affected by different sorts of uncertainties. For instance, the cost of situating a legion at a province over a specific time span, i.e., w_i , is a parameter which is not known in advance, and in a critical situation, it is not even predictable by statistical methods.

Consequently, applying the probability theory is misleading and the only trustful option would be inquiring some experts for inferring a reasonable decision. The expert may contribute that this cost would never be less than an amount, say a_i and neither more than some higher value, say b_i . For a value x in (a_i, b_i) , he may consent to deliver his opinion as a linear uncertain variable. In this case, the uncertainty distribution of w_i is

$$\varphi_i(x) = \begin{cases} 0 & x \leq a_i \\ \frac{x-a_i}{b_i-a_i} & a_i \leq x \leq b_i \\ 1 & x \geq b_i. \end{cases} \quad (8)$$

This means that the belief degree of the expert that this cost is less than x for $x \in (a, b)$ is

$$\mathcal{M}\{w_i \leq x\} = \varphi_i(x) = \frac{x-a_i}{b_i-a_i}.$$

Similarly, positioning cost of two legions at province i , w'_i , could be considered as a linear uncertain variable with the uncertainty distribution

$$\psi_i(x) = \begin{cases} 0 & x \leq a'_i \\ \frac{x-a'_i}{b'_i-a'_i} & a'_i \leq x \leq b'_i \\ 1 & x \geq b'_i. \end{cases} \quad (9)$$

Analogous to the positioning costs, deployment cost of a legion from a province to one of its neighbors c_{ij} has similar feature. This variable could be also expressed as a linear uncertain variable with uncertainty distribution as

$$\Psi_{ij}(x) = \begin{cases} 0 & x \leq a_{ij} \\ \frac{x-a_{ij}}{b_{ij}-a_{ij}} & a_{ij} \leq x \leq b_{ij} \\ 1 & x \geq b_{ij}. \end{cases} \quad (10)$$

Recall that we only consider all uncertain variables as linear. When additional information is accessible, these variables might be expressed as zigzag, empirical or even exponential uncertain variables [See Liu (2015) for more on other uncertain variables]. We assert that our methodology is independent of the type of uncertain variables.

5 Uncertain optimization models

Considering the uncertainties in the parameters of the problem, an uncertain version of the weighted Roman domination problem may have the following objective function while constraints (2)–(5) must be satisfied:

$$E \left\{ \sum_{i \in V} w_i x_i + \sum_{i \in V} w'_i y_i + \sum_{(i,j) \in E} c_{ij} t_{ij} \right\}. \quad (11)$$

When parameters w_i , w'_i and c_{ij} are considered as independent uncertain variables, (11) is equivalent to

$$\sum_{i \in V} E\{w_i\}x_i + \sum_{i \in V} E\{w'_i\}y_i + \sum_{(i,j) \in E} E\{c_{ij}\}t_{ij}. \quad (12)$$

Another model could be formulated by adding the following constraint to the problem and substituting the objective function with θ

$$\mathcal{M} \left\{ \sum_{i \in V} w_i x_i + \sum_{i \in V} w'_i y_i + \sum_{(i,j) \in E} c_{ij} t_{ij} \leq \theta \right\} \geq \alpha, \quad (13)$$

where α is a predetermined belief rate. With the assumption that uncertainty distributions defined as (8)–(10) are independent and respecting the nonnegativity of x_i , y_i and t_{ij} , constraint (13) is identical with

$$\sum_{i \in V} \varphi_i^{-1}(\alpha)x_i + \sum_{i \in V} \psi_i^{-1}(\alpha)y_i + \sum_{(i,j) \in E} \Psi_{ij}^{-1}(\alpha)t_{ij} \leq \theta. \quad (14)$$

6 Illustrative example

We implement the proposed models on an instance of the Roman Empire problem (See Tables 1–2 and Fig. 1). We assume that all uncertain variables are independent, and the objective function is in the form of (12). Each vertex

Table 1 Provinces and expected values of positioning costs of legions

Vertex (i)	Province	$E[w_i]$	$E[w'_i]$
1	Asia Minor	3	5
2	Britain	2	2
3	Egypt	3	3
4	Gaul	2	4
5	Iberia	1	3
6	Italy	4	4
7	North Africa	2	2
8	Thracia	1	3

Table 2 Expected values of deployment costs between provinces

Link	(1,3)	(1,8)	(2,4)	(2,5)	(3,6)	(3,7)	(3,8)
$E[c_{ij}]$	2	1	4	4	2	1	1
Link	(4,5)	(4,6)	(5,6)	(5,7)	(6,7)	(6,8)	
$E[c_{ij}]$	3	2	2	1	4	1	

(province) is labeled by two numbers, respectively, expected values of positioning's uncertain costs of one and two legions at a province. The label at each link refers to the expected value of the deployment's uncertain cost between vertices. Without loss of generality, we consider the graph to be undirected and the deployment cost is identical at any direction in a link. Nominal expected values of parameters are presented in Tables 1 and 2.

The objective function value of this instance is 13, and vertices in the optimal solution are partitioned into three parts. Vertices 2, 4, 5 and 7 are positioned with one legion, Vertex 8 is with 2 legions and the others with no legions. The result is shown in Fig. 1. The first type of vertices is in green, the second is in red and the others are in gray. The potential edges for legions deployments are in blue. As it is seen from the result, six legions are required respecting the costs. However, disregarding the costs leads to identifying only vertices 3 and 5, or vertices 8 and 5; each is positioned with two legions. The number of required legions is four, therefore.

Observe that Tables 1 and 2 only reflect the expected values of uncertain variables without any information on their distribution. Now, suppose that these uncertain variables are linear as reported in Tables 3 and 4. Here, the following constraint is added to the problem at which α is a prespecified belief rate and θ is a free real variable,

Fig. 1 A prototype solution for the uncertain weighted Roman Empire Problem

Table 3 Uncertain positioning costs of legions at each province

Vertex (i)	Province	$w_i = \mathcal{L}(a_i, b_i)$	$w'_i = \mathcal{L}(a'_i, b'_i)$
1	Asia Minor	(2,4)	(3,7)
2	Britain	(1,3)	(0.5,3.5)
3	Egypt	(2,4)	(2.5,3.5)
4	Gaul	(1,3)	(2,6)
5	Iberia	(0.5,1.5)	(2,4)
6	Italy	(3,5)	(3,5)
7	North Africa	(1,3)	(1,3)
8	Thracia	(0.5,1.5)	(2,4)

Table 4 Uncertain deployment costs between provinces

Link	(1,3)	(1,8)	(2,4)	(2,5)	
$c_{ij} = \mathcal{L}(a_{ij}, b_{ij})$	(1,3)	(0.25,1.75)	(3,5)	(2,6)	
Link	(3, 6)	(3, 7)	(3, 8)	(4, 5)	(4, 6)
$c_{ij} = \mathcal{L}(a_{ij}, b_{ij})$	(1,3)	(0.5,1.5)	(0.5,1.5)	(1,5)	(0,4)
Link	(5,6)	(5,7)	(6,7)	(6,8)	
$c_{ij} = \mathcal{L}(a_{ij}, b_{ij})$	(1.5,2.5)	(0.5,1.5)	(1,7)	(0.75,1.25)	

$$\sum_{i \in V} ((1 - \alpha)a_i + \alpha b_i)x_i + \sum_{i \in V} ((1 - \alpha)a'_i + \alpha b'_i)y_i + \sum_{(i,j) \in E} ((1 - \alpha)a_{ij} + \alpha b_{ij})t_{ij} \leq \theta. \quad (15)$$

Fig. 2 Optimal value for different belief degrees α

The objective of this model is to minimize the value of θ . For computational purpose, value of α is fixed at 0.9. Therefore, (15) takes the following specific form:

$$\begin{aligned} & [3.8x_1 + 2.8x_2 + 3.2x_3 + 2.8x_4 + 1.4x_5 + 4.8x_6 + 2.8x_7 + 1.4x_8] \\ & + [6.6y_1 + 3.2y_2 + 3.4y_3 + 5.6y_4 + 3.8y_5 + 4.8y_6 + 2.8y_7 + 3.8y_8] \\ & + [2.8t_{13} + 1.6t_{18} + 4.8t_{24} + 5.6t_{25} + 2.8t_{36} + 1.4t_{37} + 1.4t_{38} + \\ & + 4.6t_{45} + 3.6t_{46} + 2.4t_{56} + 1.4t_{57} + 6.4t_{67} + 1.2t_{68}] \leq \theta. \end{aligned}$$

The optimal solution is as the same as in the first model, while the objective value increases to 23.2. We also implemented this prototype example with different belief degrees $\alpha \in [0, 1]$ to observe the sensitivity of the model on different belief degrees (See Fig. 2). We realized that the optimal solution is invariant for all belief rates while this is not the case in all instances, and completely in accordance with sensible expectation, it was observed that as the belief degree rises, the cost of actions increases as well and vice versa. Our further computational experiences with different deployment costs revealed that when these costs are higher enough, regarding the belief rate, the optimal solution assigns all vertices to be stationed by only one legion. This reasonable result has been also observed when the cost of positioning two legions in a vertex is much higher than of one.

Note that similarity of solutions is not the case in general.

7 Challenge in other approaches

There are different approaches in dealing with indeterminacy. Here, we only consider the unbefitting of probability theory and fuzzy theory for the problem under consideration.

Let us consider for a moment that the costs in model (12) are stochastic random variables. Observe that the main difference between the uncertainty theory and the probability theory is in the way they define the independency of the events. Consider the stochastic event “The cost of deployment in edge (1,3) is grater than 4.” is denoted by ξ . For the sake of simplicity, let us assume that $[3, 6]$ is a possible cost interval with the uniform distribution. With such assumption, $Pr(\xi) = \frac{2}{3}$. Now consider ten experts are independently assessing the cost on this linkage with identical opinions. The probability that all experts believe to ξ is

$$Pr(\underbrace{\xi \wedge \dots \wedge \xi}_{10\text{times}}) = \left(\frac{2}{3}\right)^{10} \simeq 0.017,$$

which is clearly a contradiction with the reality. However, the belief on event ξ in the uncertainty theory is $\frac{2}{3}$ (Using Axiom 4), which is more compatible with the expectation.

Furthermore, complexity still emerges in applying the probability theory. For instance, constraint (13) is replaced with the following inequality in the probability theory:

$$Pr\left\{\sum_{i \in V} w_i x_i + \sum_{i \in V} w'_i y_i + \sum_{(i,j) \in E} c_{ij} t_{ij} \leq \theta\right\} \geq \alpha, \quad (16)$$

Even with the known probability densities of costs w_i , w'_i and t_{ij} , finding the aggregate probability density of $\sum_{i \in V} w_i x_i + \sum_{i \in V} w'_i y_i + \sum_{(i,j) \in E} c_{ij} t_{ij}$ is not an easy task in many cases. Computational burden of this density function could be another factor hindering the use of probability theory. For example, if all indeterminate variables are assumed to be independently normally distributed random variables, then (16) is identical with

$$\begin{aligned} & \sum_{i=1}^n E[w_i]x_i + \varphi^{-1}(\alpha) \sqrt{\sum_{i=1}^n \text{Var}(w_i)x_i^2} \\ & + \sum_{i=1}^n E[w'_i]y_i + \psi^{-1}(\alpha) \sqrt{\sum_{i=1}^n \text{Var}(w'_i)y_i^2} \\ & + \sum_{(i,j) \in E} E[c_{ij}]t_{ij} + \Psi^{-1}(\alpha) \sqrt{\sum_{i=1}^n \text{Var}(c_{ij})t_{ij}^2} \leq \theta, \end{aligned}$$

where φ , ψ and Ψ here stand for standard normal distribution functions [See Liu and Liu (2009), Theorem 4.9]. Since x_i , y_i and t_{ij} are binary variables, $x_i^2 = x_i$, $y_i^2 = y_i$ and

$t_{ij}^2 = t_{ij}$. However, the reduced constraint is still nonlinear in terms of x_i , y_i and t_{ij} 's. In spite, one of privileges of the uncertainty theory is that if the original uncertain program in linear, the reduced problem will be linear as well, disregarding the type of uncertain variable. In addition, as mentioned beforehand, probability theory is applicable to the cases when enough experimental data exist for approximating a proper probability distribution function. This is not the situation in the problem under study in this paper.

One may pose the fuzzy theory as another approach for treating the indeterminacy on the costs. Consider for the moment the example in Sec. 6, at which only the cost of deployment on the linkage (1, 3) is uncertain and treated as a fuzzy number. The quantitative cost could be from 3 to 6, without loss of generality. Now, assume that this output is a fuzzy variable, say ε , characterized by a trapezoidal membership function as

$$\mu(x) = \begin{cases} 0 & x \leq 3 \\ x - 3 & 3 \leq x \leq 4 \\ 1 & 4 \leq x \leq 5 \\ 6 - x & 5 \leq x \leq 6 \\ 0 & x \geq 6. \end{cases}$$

The possibility of “The cost is 4.” is $\text{pos}\{\varepsilon = 4\} = 1$, and the possibility of “The cost is not 4.” is $\text{pos}\{\varepsilon \neq 4\} = 1$. This means that these two contradicting terms have identical possibility measure 1. Obviously, applying possibility measure to model this sort of uncertain quantity leads to a paradox. In addition, there are different fuzzy numbers and finitely many ordering rules in the fuzzy theory, and considering each of them may lead to different and sometimes contradictory results.

8 How to obtain the uncertainty distributions?

A methodology for collecting and interpreting expert's experimental data by uncertainty theory has been started in (2010) by Liu. The author proposed a questionnaire survey for collecting expert's experimental data. Inviting some domain experts for completing a questionnaire about the notion of an uncertain variable is the starting point. For instance, let w_i be the uncertain cost of positioning one legion in the province i . Therefore, the question might be “How much could the cost of positioning one legion in the province i .”

We first ask the domain expert to choose a possible value x (say \$5000) that the uncertain variable w_i may take, and then question him “How likely is w_i less than or equal to x ?” Denoting the belief degree of the expert with α , the

pair of (x, α) is obtained. Recall that the expert's belief degree of w_i greater than x must be $1 - \alpha$ due to the self-duality of uncertain measure. Repeating this process, the following expert's experimental data are obtained by the questionnaire,

$$(x_1, \alpha_1), (x_2, \alpha_2), \dots, (x_n, \alpha_n). \quad (17)$$

Having these data, different methods of extracting an appropriate uncertainty distribution exist in the literature, such as producing an empirical uncertainty distribution Liu (2010). The principle of least squares that minimizes the sum of the squares of the distance of the expert's experimental data to the uncertainty distribution has been also proposed Liu (2010). Wang and Peng further suggested a method of moments to estimate the unknown parameters of an uncertainty distribution Wang and Peng (2014).

When there are more several experts of domain, one may use the convex combination of different uncertainty distributions to obtain a single one Liu (2010). Delphi method that was originally developed in the 1950s by the RAND Corporation may also be used to identify a suitable uncertainty distribution Wang et al. (2012). For more detail and illustrative examples, we refer the interested reader to Liu (2015).

9 Concluding remarks

In this paper, we considered the uncertain weighted Roman domination problem at which not only vertices have uncertain costs as weights, but also each link has a nominal uncertain deployment cost. The problem was modeled as an uncertain optimization problem, and the results were depicted by a simple example. This point of view can be applied in other similar situations, such as eternal domination problem, where referring to an expert is the only sensible option. An eternal dominating set Cockayne et al. (2004) is a set of locations on which mobile guards are initially located (at most one guard may be located on any vertex), and this set must be such that for any infinite sequence of attacks occurring sequentially at vertices, the set can be adjusted by moving a guard from an adjacent vertex to the attacked vertex, provided the attacked vertex has no guard on it at the time it is attacked. This problem has a very important application in protecting the Web sites and data centers against infinitely hacking attacks, while the parameters are facing much higher degree of uncertainty on their nature.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflicts of interest.

Informed consent Further, the research involves no human participants and animals and consequently there is no need for informed consent.

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