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Boundary controllability for a 1D degenerate parabolic equation with a Robin boundary condition

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Abstract
In this paper, we prove the null controllability of a one-dimensional degenerate parabolic equation with a weighted Robin boundary condition at the left endpoint, where the potential has a singularity. We use some results from the singular Sturm–Liouville theory to show the well-posedness of our system. We obtain a spectral decomposition of a degenerate parabolic operator with Robin conditions at the endpoints, we use Fourier–Dini expansions and the moment method introduced by Fattorini and Russell to prove the null controllability and to obtain an upper estimate of the cost of controllability. We also get a lower estimate of the cost of controllability by using a representation theorem for analytic functions of exponential type.

Keywords Degenerate parabolic equation · Robin boundary condition · Sturm–Liouville theory · Boundary controllability · Moment method

Mathematics Subject Classification 35K65 · 34B24 · 30E05 · 93B05 · 93B60

1 Introduction and main results

Let $T > 0$ and set $Q_T := (0, 1) \times (0, T)$. For $\alpha, \beta \in \mathbb{R}$ with $0 \le \alpha < 2$, consider the equation

$$u_t - (x^\alpha u_x)_x - \beta x^{\alpha-1} u_x - \frac{\mu}{x^{2-\alpha}} u = 0 \text{ in } Q_T, \tag{1}$$

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provided that $\mu \in \mathbb{R}$ satisfies

$$-\infty < \mu < \mu(\alpha+\beta), \quad \text{where} \quad \mu(\delta) := \frac{(1-\delta)^2}{4}, \quad \delta \in \mathbb{R}. \tag{2}$$

In this work, we consider a weighted Robin boundary condition at the left endpoint of the form

$$\lim_{x\to 0^+} \left(a x^{(\alpha+\beta-1)/2+\sqrt{\mu(\alpha+\beta)-\mu}} u(x,t) + x^{(\alpha+\beta+1)/2+\sqrt{\mu(\alpha+\beta)-\mu}} u_x(x,t) \right) = f(t),$$

and a usual Robin boundary condition at the right endpoint of the form

$$a u(1,t) + u_x(1,t) = g(t),$$

where

$$a := a(\alpha,\beta,\mu) = \frac{\alpha+\beta-1}{2} - \sqrt{\mu(\alpha+\beta)-\mu}. \tag{3}$$

The goal of this work is to prove the null controllability of the following system, with a control $f(t) \in L^2(0,T)$ acting at the left endpoint,

$$\begin{cases} u_t - (x^\alpha u_x)_x - \beta x^{\alpha-1} u_x - \dfrac{\mu}{x^{2-\alpha}} u = 0 & \text{in } Q_T, \\ \left[u(\cdot,t), x^{-a}\right](0) = f(t), \quad a u(1,t) + u_x(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{4}$$

where our Lagrange form $[\cdot,\cdot]$ is given by

$$[u,v](x) = (upv' - vpu')(x), \quad \text{with } p(x) = x^{\alpha+\beta}, \text{ and } ' = \frac{\mathrm{d}}{\mathrm{d}x}.$$

Consider the weighted Lebesgue space $L^2_\beta(0,1) := L^2((0,1); x^\beta dx)$, $\beta \in \mathbb{R}$, endowed with the inner product

$$\langle f, g\rangle_\beta := \int_0^1 f(x) g(x) x^\beta dx,$$

and its corresponding norm is denoted by $\|\cdot\|_\beta$.

Here, we use some results from the singular Sturm–Liouville theory to see the well-posedness of the system (4) with initial data in $L^2_\beta(0,1)$, although the solution $u(t)$ lives in an interpolation space $\mathcal{H}^{-s}$. We say the system (4) is null controllable in $L^2_\beta(0,1)$ at time $T > 0$ with controls in $L^2(0,T)$, if for any $u_0 \in L^2_\beta(0,1)$ there exists $f \in L^2(0,T)$ such that the corresponding solution satisfies $u(\cdot,T) \equiv 0$.

We are also interested in the behavior of the cost of the controllability. Consider the set of admissible controls given by

$$U(T,\alpha,\beta,\mu,u_0) := \{f \in L^2(0,T) : u \text{ is solution of the system (4) that satisfies } u(\cdot,T) \equiv 0\}.$$

If X is a subspace in $L^2_\beta(0,1)$, we define the cost of controllability for initial data in X as follows:

$$\mathcal{K}_X(T,\alpha,\beta,\mu) := \sup_{u_0\in X, \|u_0\|_\beta=1} \inf\{|f|_{L^2(0,T)} : f \in U(T,\alpha,\beta,\mu,u_0)\}.$$

The main result of this work is the following.

Theorem 1 *Let $T>0$, $0\le\alpha<2$, $\beta\in\mathbb{R}$, and μ satisfying (2). The next statements hold.*

1. *Existence of a control. For any $u_0 \in L^2_\beta(0,1)$ there exists a control $f\in L^2(0,T)$ such that the solution u to (4) satisfies $u(\cdot,T)\equiv 0$.*
2. *Upper bound of the cost. There exists a constant $c>0$ such that for every $\delta\in(0,1)$, we have*

$$\mathcal{K}_{\Phi_0^\perp}(T,\alpha,\beta,\mu) \le \frac{cM(T,\alpha,\nu,\delta)T^{1/2}}{(\nu+1)\kappa_\alpha^{5/2}} \exp\left(-\frac{T}{2}\kappa_\alpha^2 j_{\nu+1,1}^2\right),$$

   where

$$\begin{aligned}\kappa_\alpha &:= \frac{2-\alpha}{2},\\ \nu = \nu(\alpha,\beta,\mu) &:= \sqrt{\mu(\alpha+\beta)-\mu}/\kappa_\alpha,\\ \Phi_0(x) &= \sqrt{2(\nu+1)\kappa_\alpha}\,x^{-a},\end{aligned} \tag{5}$$

   $j_{\nu+1,1}$ is the first positive zero of the Bessel function $J_{\nu+1}$ (defined in the Appendix), and

$$M(T,\alpha,\nu,\delta) = \left(1+\frac{1}{(1-\delta)\kappa_\alpha^2 T}\right)\left[\exp\left(\frac{1}{\sqrt{2}\kappa_\alpha}\right)+\frac{1}{\delta^5}\exp\left(\frac{3}{(1-\delta)\kappa_\alpha^2 T}\right)\right] \times \exp\left(-\frac{(1-\delta)^{3/2}T^{3/2}}{8(1+T)^{1/2}}\kappa_\alpha^3 j_{\nu+1,1}^2\right).$$

3. *Lower bound of the cost. There exists a constant $c>0$ such that*

$$
\begin{aligned}
& c\left(1+\frac{j_{\nu+1,2}^2}{j_{\nu+1,1}^2}\right) \frac{2^\nu |J_\nu(j_{\nu+1,1})| \exp\left(\left(\frac{1}{2}-\frac{\log 2}{\pi}\right) j_{\nu+1,2}\right)}{\Gamma(\nu+1)^{-1}\,(2T\kappa_\alpha)^{1/2}\,(j_{\nu+1,1})^\nu} \\
& \quad \times \exp\left(-\left(j_{\nu+1,1}^2+\frac{j_{\nu+1,2}^2}{2}\right)\kappa_\alpha^2 T\right) \\
& \quad \leq \mathcal{K}_{L^2_\beta}(T,\alpha,\beta,\mu),
\end{aligned}
$$

where $j_{\nu+1,2}$ is the second positive zero of the Bessel function $J_{\nu+1}$.

We also analyze the null controllability of a similar system but the control acting at the right endpoint,

$$
\begin{cases}
u_t-(x^\alpha u_x)_x-\beta x^{\alpha-1}u_x-\dfrac{\mu}{x^{2-\alpha}}u=0 & \text{in } Q_T, \\
\left[u(\cdot,t),x^{-a}\right](0)=0, \quad a u(1,t)+u_x(1,t)=f(t) & \text{on } (0,T), \\
u(x,0)=u_0(x) & \text{in } (0,1).
\end{cases} \tag{6}
$$

Consider the corresponding set of admissible controls

$$
\begin{aligned}
\widetilde{U}(T,\alpha,\beta,\mu,u_0)=\{f\in L^2(0,T) : u \text{ is solution of the system (6) that satisfies} \\
u(\cdot,T)\equiv 0\},
\end{aligned}
$$

and the cost of the controllability given by

$$
\widetilde{\mathcal{K}}_X(T,\alpha,\beta,\mu):=\sup_{u_0\in X,\|u_0\|_\beta=1} \inf\{\|f\|_{L^2(0,T)} : f\in \widetilde{U}(T,\alpha,\beta,\mu,u_0)\},
$$

where X is a subspace in $L^2_\beta(0,1)$.

Theorem 2 *Let $T>0$, $\beta\in\mathbb{R}$, $0\leq\alpha<2$, and μ satisfying (2). The next statements hold.*

1. *Existence of a control. For any $u_0\in L^2_\beta(0,1)$ there exists a control $f\in L^2(0,T)$ such that the solution u to (6) satisfies $u(\cdot,T)\equiv 0$.*
2. *Upper bound of the cost. There exists a constant $c>0$ such that for every $\delta\in(0,1)$, we have*

$$
\widetilde{\mathcal{K}}_{\Phi_0^\perp}(T,\alpha,\beta,\mu)\leq \frac{cM(T,\alpha,\nu,\delta)T^{1/2}}{\kappa_\alpha^{\nu+1}\Gamma(\upsilon+2)}\left(\frac{2\nu+1}{4T\mathrm{e}}\right)^{(2\nu+1)/4}\exp\left(-\frac{T}{4}\kappa_\alpha^2 j_{\nu+1,1}^2\right).
$$

3. *Lower bound of the cost. There exists a constant $c>0$ such that*

$$
\begin{aligned}
& c\left(1+\frac{j_{\nu+1,2}^2}{j_{\nu+1,1}^2}\right)\frac{\exp\left(\left(\frac{1}{2}-\frac{\log 2}{\pi}\right)j_{\nu+1,2}\right)}{(2T\kappa_\alpha)^{1/2}}\exp\left(-\left(j_{\nu+1,1}^2+\frac{j_{\nu,2}^2}{2}\right)\kappa_\alpha^2 T\right) \\
& \leq \widetilde{\mathcal{K}}_{L^2_\beta}(T,\alpha,\beta,\mu).
\end{aligned}
$$

2 Previous work

In the last twenty years, there has been extensive research activity on the controllability of degenerate/singular parabolic equations with appropriate boundary conditions, due to both theoretical interest and their interesting applications in engineering, physics, biology, and economics. Currently, there are well-known methods to solve this kind of problems: the use of global Carleman inequalities, the flatness approach, the moment method, the transmutation method. We refer to [7, 9], whose authors obtain Carleman inequalities for degenerate/singular parabolic equations on the unit interval or on a non-empty subset in $\mathbb{R}^2$, and as application they prove null controllability by means of controls acting at the boundary or at an interior point in the domain.

Throughout this section consider the differential operator

$$\mathbb{A}_\lambda u := -(au_x)_x - \frac{\lambda}{b(x)}u \quad \text{or} \quad \mathbb{A}_\lambda u := -au_{xx} - \frac{\lambda}{b(x)}u, \quad \lambda \in \mathbb{R}, \tag{7}$$

on the unit interval, where $a, b \geq 0$ can degenerate somewhere. If $a = 0$ somewhere in $[0, 1]$, the problem becomes degenerate, while if $b = 0$, it is singular. We also assume that ω is a non-empty subinterval in $(0, 1)$.

Consider the (weighted) boundary operator

$$B_i u(t) := \lim_{x \to 0^+} a(x)^i \partial_x^i u(x, t), \quad i = 0, 1, \quad t > 0,$$

provided the limit exists. Notice that B_0 is a Dirichlet boundary operator at $x = 0$, and B_1 is a weighted Neumann boundary operator at $x = 0$.

In [5, 6], the authors first demonstrated the null controllability, at the time $T > 0$, of the following system,

$$\begin{cases} u_t + \mathbb{A}_0 u = f\chi_w, & (x, t) \in Q_T, \\ u(1, t) = 0, & t \in (0, T), \\ B_i u(t) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \tag{8}$$

where $\mathbb{A}_0$ is the operator given in (7) in divergence form with $a(x) := x^\alpha$, $f \in L^2(Q_T)$, $u_0 \in L^2(0, 1)$, $i = 0$ in the weak degenerate case $0 \leq \alpha < 1$, $i = 1$ in the strong degenerate case $1 \leq \alpha < 2$.

In [5, 6], the authors build weights related to the degeneracy of the diffusion coefficient a to get Carleman estimates. The authors combine these estimates with Hardy-type inequalities to prove observability for the adjoint system. It can be proved that their Carleman estimates [6, Theorem 2.2] imply a boundary null controllability result with a control acting at $x = 1$. In this case, our differential operator $\mathcal{A}$, given in (10) and considering $\beta = \mu = 0$, generalizes the operator $\mathbb{A}_0$ in the divergence form. In [13], the author solves the weak degenerate case (in homogeneous divergence form) by using a Dirichlet boundary control at $x = 0$. There the author uses the transmutation method: First, it proves an observability inequality for the degenerate wave equation $v_{tt} - (x^\alpha v_x)_x$=0 considering the usual boundary conditions, uses a transmutation to pass from heat processes to waves; thus, it gets an observability inequality for the heat equation which implies the null controllability.

The next step was to consider coefficients with degeneracy at an interior point or non-smooth coefficients. In [8], the authors analyze the null controllability of the system (8) with homogenous Dirichlet boundary conditions at the endpoints, where $\mathbb{A}_0$ is the operator given in the both forms in (7), the initial data u_0 is in X (where $X = L^2(0, 1)$ in the divergence case and $X = L^2_{1/a}(0, 1)$ in the non-divergence case), and the control $f \in L^2(0, T; X)$ is supported in $\omega \subset (0, 1)$, which can contain the degenerate point x_0. In this work, the diffusion coefficient a is a non-smooth function. When a degenerates at an interior point x_0, the authors distinguish between the so-called weakly degenerate case and the strong degenerate case.

Then, the authors give two versions of Carleman estimates for the adjoint system. In the first one, a is globally non-smooth and does not degenerate; in the second one, a is non-smooth and degenerates at x_0. They prove a weighted Hardy–Poincare inequality for functions which may not be globally absolutely continuous in the domain, but whose irregularity point is compensated by the fact that the weight degenerates exactly there. Then, observability inequalities are obtained from the Carleman estimates, thus they get the null controllability. In the divergence case, the degeneracy point x_0 can be outside as well as inside ω. In the non-divergence case, only the case in which the degeneracy point lies outside the control region is considered.

An open problem is to obtain a Carleman estimate for the adjoint system (with homogeneous weighted Robin boundary conditions) of the system (4), and try to get a distributed control on ω (which could include the degeneracy point) for the system (4).

Another useful tool to prove boundary null controllability of degenerate systems is the so-called flatness method. In [18], the author considers the system (8) with the homogeneous PDE in divergence form, boundary operator B_1, $a(x) = x^\alpha$, $\alpha \in [1, 2)$, $u_0 \in L^2(0, 1)$, and a control h acting at the right endpoint, i.e., $u(1, t) = h(t)$.

In [18], the author uses the flatness approach to construct explicit (smooth) controls h in some Gevrey classes. To do this, the author uses that $\mathbb{A}_0$ is a diagonalizable self-adjoint positive operator, whose corresponding orthogonal basis can be written as a composition of powers of the variable x with a Bessel function of the first kind (and involving its positive zeros), to construct a flat output in a Gevrey class. We think the flatness method could be adapted to prove the boundary null controllability of our system (4), by using Proposition A.1 to construct the corresponding flat output.

In [19], the authors also use the flatness approach to prove the boundary null controllability of the following system:

$$\begin{aligned} (a(x)u_x)_x + b(x)u_x + c(x)u - \rho(x)u_t &= 0, \quad x \in (0, 1), t \in (0, T), \\ r_0 u(0, t) + s_0(au_x)(0, t) &= 0, \quad t \in (0, T), \\ r_1 u(1, t) + s_1 (au_x)(1, t) &= h(t), \quad t \in (0, T), \\ u(x, 0) &= u_0(x), \quad x \in (0, 1), \end{aligned} \tag{9}$$

where $r_0, s_0, r_1, s_1 \in \mathbb{R}$, $r_j^2 + s_j^2 > 0$, $u_0 \in L^2(0, 1)$ y $h \in L^2(0, T)$.

They assume that $a(x) > 0$ and $\rho(x) > 0$ for a.e $x \in (0, 1)$, $1/a, b/a, c, \rho \in L^1(0, 1)$,

$$\exists K \geq 0, \frac{c(x)}{\rho(x)} \leq K \text{ for a.e } x \in (0, 1), \quad \exists p \geq (1, \infty], \; a^{1-1/p}\rho \in L^p(0, 1).$$

If we multiply the PDE in (4) by x^β, we obtain the PDE in (9) with $a(x) = x^{\alpha+\beta}$, $b \equiv 0$, $c(x) = \mu/x^{2-\alpha-\beta}$, $\rho(x) = x^\beta$. Thus, $1/a \in L^1(0, 1)$ iff $\alpha + \beta < 1$, and $c \in L^1(0, 1)$ iff $\alpha + \beta > 1$. Therefore, our problem does not fit in the scheme of [19]. Moreover, we consider a suitable weighted Robin boundary condition at $x = 0$, where the degeneracy/singularity arises, and the control acts at this point.

The condition $1/a \in L^1(0, 1)$ in [19] implies that the PDE in (9) is a weakly degenerate parabolic equation. In [2], the authors use the flatness approach to show the null controllability of the degenerate parabolic equation without drift ($b \equiv 0$) in (9), with the boundary conditions corresponding to $r_0 = 0, s_0 = 1$. The main assumption is that the function $x/a(x)$ is in $L^p(0, 1)$ for some $p > 1$, which implies that $1/a \notin L^1(0, 1)$. Thus, a may vanish strongly at $x = 0$, and the potential c may be singular at the same point, but in [2] the control acts at $x = 1$; by contrast, our control acts at $x = 0$, and we have a drift, provided that $\beta \neq 0$.

In [21], the author proves some global Carleman estimates for the degenerate/singular parabolic operator $w_t - \mathbb{A}_\lambda w$ with $a(x) = x^\alpha$, $b(x) = x^{\widetilde{\beta}}$, and boundary conditions (depending on α) as in (8). The author gets an improved Hardy–Poincaré inequality and obtains an observability result that implies the null controllability of the system (8), with $\mathbb{A}_\lambda$ (instead of $\mathbb{A}_0$) in divergence form, by means of a distributed control f. In the case $\widetilde{\beta} = 2 - \alpha$, $\lambda < \mu(\alpha)$, the corresponding PDE coincides with the PDE in (4) with $\beta = 0$, $\mu < \mu(\alpha)$.

In [4, 11, 12, 14], the authors use the moment method to prove the boundary null controllability of systems like (9). In [14], the authors consider $a(x) = \varepsilon x^{\alpha+1}$, $b(x) = -x^\alpha$, $\varepsilon, \alpha \in (0, 1)$. They consider $r_0 = r_1 = 1, s_0 = s_1 = 0$, so their control acts at the left endpoint. This is a strongly degenerate parabolic problem, but at present, we know this kind of degeneracy is related to a Neumann weighted boundary condition, see [12].

In [11], the authors prove the null controllability of the equation (1) with a weighted Dirichlet boundary condition at the left endpoint, provided that $\alpha + \beta < 1$. In the case $\alpha + \beta > 1$, in [12], they get the null controllability of the equation (1) with a weighted Neumann boundary condition at the left endpoint. They consider initial data in $L^2_\beta(0, 1)$ in both cases. In these works, the authors prove suitable versions of a Hardy inequality to assure the well-posedness of their systems, but in the case $\alpha + \beta = 1$ is necessary to consider some results from the singular Sturm–Liouville theory, see [12]. Here, we use that approach to show the well-posedness of our system.

Unfortunately, for this paper, we could not prove a suitable weighted Hardy–Poincaré considering the (weighted) homogeneous Robin boundary conditions in (4). This fact motivate us to use the singular Sturm–Liouville theory, which shows that the operator $(\mathcal{A}, D(\mathcal{A}))$ given in (10) is self-adjoint.

This paper is organized as follows. Section 3 uses some results from the singular Sturm–Liouville theory to show that the operator $\mathcal{A}$ given in (10) is self-adjoint. There, we also use Fourier–Dini expansions to show that $\mathcal{A}$ is diagonalizable, this allows us to consider initial data in some interpolation spaces. Next, we introduce a notion of a weak solution for both systems and then show the well-posedness of these systems.

In Sect. 4, we prove Theorem 1 by using the moment method introduced by Fattorini & Russell. Here, the idea is to construct a biorthogonal sequence to a family of exponentials involving the eigenvalues of $\mathcal{A}$. To do this, we use some results from complex analysis to construct a suitable complex multiplier. As a consequence, we get an upper estimate of the cost of the controllability. Finally, we use a representation theorem, Theorem 13, to obtain a lower estimate of the cost of the controllability.

In Sect. 5, we proceed as before to solve the case when the control acts at the right endpoint.

3 Functional setting and well-posedness

Consider the differential expression M defined by

$$Mu = -(pu_x)_x + qu$$

where $p(x) = x^{\alpha+\beta}$, $q(x) = -\mu x^{-2+\alpha+\beta}$, $w(x) = x^{\beta}$.

Clearly,

$$1/p, q, w \in L_{\text{loc}}(0,1), \quad p, w > 0 \text{ on } (0,1),$$

thus Mu is defined a.e. for functions u such that $u, pu_x \in AC_{\text{loc}}(0,1)$, where $AC_{\text{loc}}(0,1)$ is the space of all locally absolutely continuous functions in $(0,1)$.

Now, we introduce the operator $\mathcal{A}$ given by

$$\mathcal{A}u := w^{-1}Mu = -(x^{\alpha}u_x)_x - \beta x^{\alpha-1}u_x - \frac{\mu}{x^{2-\alpha}}u. \tag{10}$$

From the theory developed in [23], we can build a self-adjoint domain $D(\mathcal{A})$ for the operator $\mathcal{A}$.

For μ satisfying (2), $0 \le \alpha < 2$, and $\beta \in \mathbb{R}$, we set

$$D_{\max} := \left\{ u \in AC_{\text{loc}}(0,1) \mid pu_x \in AC_{\text{loc}}(0,1),\ u, \mathcal{A}u \in L^2_{\beta}(0,1) \right\}, \quad \text{and}$$

$$D(\mathcal{A}) := \begin{cases} \{u \in D_{\max} \mid \lim_{x\to 0^+} x^{(\alpha+\beta-1)/2+\sqrt{\mu(\alpha+\beta)-\mu}}u(x) \\ \quad = (au+u_x)(1) = 0\} & \text{if } \sqrt{\mu(\alpha+\beta)-\mu} < \kappa_{\alpha}, \\ \{u \in D_{\max} \mid (au+u_x)(1) = 0\} & \text{if } \sqrt{\mu(\alpha+\beta)-\mu} \ge \kappa_{\alpha}. \end{cases}$$

Recall that the Lagrange form associated with M is defined as follows:

$$[u, v] := upv' - vpu', \quad \text{for all} \quad u, v \in D_{\max}.$$

The next result shows that $\mathcal{A}$ is a diagonalizable operator whose Hilbert basis of eigenfunctions can be written in terms of the function $x^{1/2+\nu}$, the Bessel function of the first kind J_ν and the corresponding positive zeros $j_{\nu+1,k}$, $k \geq 1$, of the Bessel function $J_{\nu+1}$, see the proof of Proposition A.1. In the appendix, we give some properties of Bessel functions of the first kind and their zeros.

Proposition 3 *Let* $0 \leq \alpha < 2$, $\beta \in \mathbb{R}$, $\mu < \mu(\alpha+\beta)$, *and* κ_α, ν *given in (5). Then,* $\mathcal{A} : D(\mathcal{A}) \subset L^2_\beta(0,1) \to L^2_\beta(0,1)$ *is a self-adjoint operator. Furthermore, the family* $\{\Phi_k\}_{k\geq 0}$ *given by*

$$\begin{aligned} \Phi_0(x) &:= \sqrt{2-\alpha+2\sqrt{\mu(\alpha+\beta)-\mu}}\; x^{(1-\alpha-\beta)/2+\sqrt{\mu(\alpha+\beta)-\mu}}, \\ \Phi_k(x) &:= \frac{\sqrt{2\kappa_\alpha}}{|J_\nu(j_{\nu+1,k})|} x^{(1-\alpha-\beta)/2} J_\nu\left(j_{\nu+1,k} x^{\kappa_\alpha}\right), \quad k \geq 1, \end{aligned} \tag{11}$$

is an orthonormal basis for $L^2_\beta(0,1)$ *such that*

$$\mathcal{A}\Phi_k = \lambda_k \Phi_k, \quad k \geq 0,$$

where $\lambda_0 := 0$ *and* $\lambda_k := \kappa_\alpha^2 (j_{\nu+1,k})^2$, $k \geq 1$.

Proof Since $1/p, q, w \in L^1(1/2,1)$ we have that $x = 1$ is a regular point.

Case i) Assume $\sqrt{\mu(\alpha+\beta)-\mu} < \kappa_\alpha$.

First, we will build a (BC) basis $\{y_0, z_0\}$ at $x = 0$ and a (BC) basis $\{y_1, z_1\}$ at $x = 1$, see [23, Definition 10.4.3].

Consider the functions given by

$$y_0(x) := x^{(1-\alpha-\beta)/2+\sqrt{\mu(\alpha+\beta)-\mu}}, \;\; z_0(x) := \frac{x^{(1-\alpha-\beta)/2-\sqrt{\mu(\alpha+\beta)-\mu}}}{2\sqrt{\mu(\alpha+\beta)-\mu}}, \quad x \in (0,1). \tag{12}$$

Notice the assumption implies that $y_0, z_0 \in D_{\max}$. Clearly, $[z_0, y_0](0) = 1$, thus $\{y_0, z_0\}$ is a (BC) basis at $x = 0$.

Since $y_0, z_0 \in L^2_\beta(0,1)$ are linearly independent solutions of $Mu = 0u$ it follows that $x = 0$ is limit-circle (LC), see [23, Definition 7.3.1, Theorem 7.2.2].

Consider also the functions given by

$$\begin{aligned} y_1(x) &:= -x^{(1-\alpha-\beta)/2+\sqrt{\mu(\alpha+\beta)-\mu}}, \\ z_1(x) &:= \frac{x^{(1-\alpha-\beta)/2+\sqrt{\mu(\alpha+\beta)-\mu}} - x^{(1-\alpha-\beta)/2-\sqrt{\mu(\alpha+\beta)-\mu}}}{2\sqrt{\mu(\alpha+\beta)-\mu}}. \end{aligned}$$

Since $y_1, z_1 \in D_{\max}$ and $[z_1, y_1](1) = 1$, it follows that $\{y_1, z_1\}$ is a (BC) basis at $x = 1$.

Now, we fix $c, d \in (0, 1)$ with $c < d$. From the Patching Lemma, Lemma 10.4.1 in [23], there exist functions $g_1, g_2 \in D_{\max}$ such that

$$\begin{cases} g_1(c) = y_0(c), & g_1(d) = y_1(d), \\ (pg_1')(c) = (py_0')(c), & (pg_1')(d) = (py_1')(d), \end{cases}$$
$$\begin{cases} g_2(c) = z_0(c), & g_2(d) = z_1(d), \\ (pg_2')(c) = (pz_0')(c), & (pg_2')(d) = (pz_1')(d). \end{cases}$$

Thus, the pair $\{y_+, y_-\}$ is a (BC) basis on $(0, 1)$, see [23, Definition 10.4.3], where

$$y_+(x) := \begin{cases} y_0(x) & \text{if } x \in (0, c), \\ g_1(x) & \text{if } x \in [c, d], \\ y_1(x) & \text{if } x \in (d, 1), \end{cases} \qquad y_-(x) := \begin{cases} z_0(x) & \text{if } x \in (0, c), \\ g_2(x) & \text{if } x \in [c, d], \\ z_1(x) & \text{if } x \in (d, 1). \end{cases}$$

The matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfy the hypothesis in [23, Proposition 10.4.2], then

$$\begin{aligned} D(\mathcal{A}) :&= \left\{ u \in D_{\max} : A \left(\begin{bmatrix} u, y_+ \end{bmatrix}(0) \atop \begin{bmatrix} u, y_- \end{bmatrix}(0) \right) + B \left(\begin{bmatrix} u, y_+ \end{bmatrix}(1) \atop \begin{bmatrix} u, y_- \end{bmatrix}(1) \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ u \in D_{\max} : \left[u, y_+\right](0) = \left[u, y_+\right](1) = 0 \right\} \\ &= \left\{ u \in D_{\max} : \left[u, y_+\right](0) = (au + u_x)(1) = 0 \right\} \end{aligned}$$

is a self-adjoint domain, therefore the operator $\mathcal{A} : D(\mathcal{A}) \subset L^2_\beta(0, 1) \to L^2_\beta(0, 1)$ is self-adjoint.

Finally, we have that

$$\begin{aligned} \left[u, y_+\right](0) &= \lim_{x \to 0^+} [u, y_0](x) = \lim_{x \to 0^+} \left\{ \frac{u}{z_0}(x)[z_0, y_0](x) + [u, z_0](x) \frac{y_0}{z_0}(x) \right\} \\ &= \lim_{x \to 0^+} \frac{u}{z_0}(x), \end{aligned}$$

because $[z_0, y_0](0) = 1$, $[u, z_0](0)$ is finite (see [23, Lemma 10.2.3]), and $\lim_{x \to 0^+} y_0/z_0(x) = 0$. Hence, the result follows.

Case ii) Assume $\sqrt{\mu(\alpha + \beta) - \mu} \geq \kappa_\alpha$.

The assumption implies that $z_0 \notin L^2_\beta(0, 1)$, then $x = 0$ is limit point (LP). Theorem 10.4.4 in [23] with $A_1 = a$, $A_2 = 1$ implies that $D(\mathcal{A}) = \{u \in D_{\max} \,|(au + u_x)(1) = 0\}$ is a self-adjoint domain.

This concludes the first part of the proof.

Clearly, $\Phi_k \in C^{\infty}(0,1)$ and (61) implies that $\Phi_k \in L^2_\beta(0,1)$ for all $k \geq 0$. Moreover,

$$\lim_{x\to 0^+} x^{(\alpha+\beta-1)/2+\sqrt{\mu(\alpha+\beta)-\mu}}\Phi_k(x) = C_{\alpha,\beta,\mu} \lim_{x\to 0^+} x^{2\sqrt{\mu(\alpha+\beta)-\mu}} = 0, \quad k \geq 0.$$

By using (63), we obtain

$$\begin{aligned}\frac{|J_\nu(j_{\nu+1,k})|}{\sqrt{2\kappa_\alpha}}\Phi_k'(1) &= \frac{1-\alpha-\beta}{2}J_\nu(j_{\nu+1,k}) + \kappa_\alpha j_{\nu+1,k}J_\nu'(j_{\nu+1,k})\\ &= \left(\frac{1-\alpha-\beta}{2}+\kappa_\alpha\nu\right)J_\nu(j_{\nu+1,k}) = -a\frac{|J_\nu(j_{\nu+1,k})|}{\sqrt{2\kappa_\alpha}}\Phi_k(1),\end{aligned}$$

therefore $(a\Phi_k + \Phi_k')(1) = 0$ for all $k \geq 1$. Clearly, $(a\Phi_0 + \Phi_0')(1) = 0$. Therefore, $\Phi_k \in D(\mathcal{A})$ for all $k \geq 0$.

We set $v(x) = x^b J_\nu(cx^r)$ with $r, c > 0$ and $b \in \mathbb{R}$. The proof of Proposition 11 in [12] was shown that

$$x^{2-2r}\frac{d^2v}{dx^2} + (1-2b)x^{1-2r}\frac{dv}{dx} + (b^2 - r^2\nu^2)x^{-2r}v = -r^2c^2v.$$

By taking $r = \kappa_\alpha$, $b = (1-\alpha-\beta)/2$, and $c = j_{\nu+1,k}$, we get $\mathcal{A}\Phi_k = \lambda_k\Phi_k$ for all $k \geq 1$. Clearly, $\mathcal{A}\Phi_0 = 0$. The result follows by Proposition A.1. □

Remark 4 If $\sqrt{\mu(\alpha+\beta)-\mu} \geq \kappa_\alpha$, from Lemma 10.4.1(b) in [23], we have that $[u, y_0](0) = 0$ for all $u \in D(\mathcal{A})$. When $\sqrt{\mu(\alpha+\beta)-\mu} < \kappa_\alpha$, in the proof of the last proposition was shown that $[u, y_0](0) = 0$ for all $u \in D(\mathcal{A})$, where y_0 is given in (12).

Remark 5 The family $\{\Theta_k\}_{k\geq 0}$ given in (67) is the so-called Fourier–Dini basis for $L^2(0,1)$.

Then, $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of a diagonalizable self-adjoint semigroup in $L^2_\beta(0,1)$. Thus, we can consider interpolation spaces for the initial data. For any $s \geq 0$, we define

$$\mathcal{H}^s = \mathcal{H}^s(0,1) := D(\mathcal{A}^{s/2}) = \left\{u = \sum_{k=0}^{\infty} a_k\Phi_k : \|u\|^2_{\mathcal{H}^s} = |a_0|^2 + \sum_{k=1}^{\infty}|a_k|^2\lambda_k^s < \infty\right\},$$

and we also consider the corresponding dual spaces

$$\mathcal{H}^{-s} := \left[\mathcal{H}^s(0,1)\right]'.$$

It is well known that $\mathcal{H}^{-s}$ is the dual space of $\mathcal{H}^s$ with respect to the pivot space $L^2_\beta(0,1)$, i.e.,

$$\mathcal{H}^s \hookrightarrow \mathcal{H}^0 = L^2_\beta(0,1) = \left(L^2_\beta(0,1)\right)' \hookrightarrow \mathcal{H}^{-s}, \quad s > 0.$$

Equivalently, $\mathcal{H}^{-s}$ is the completion of $L^2_\beta(0,1)$ with respect to the norm

$$\|u\|^2_{-s} := |\langle u, \Phi_0\rangle_\beta|^2 + \sum_{k=1}^{\infty} \lambda_k^{-s} |\langle u, \Phi_k\rangle_\beta|^2.$$

It is well known that the linear mapping given by

$$S(t)u_0 = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_k t} a_k \Phi_k \quad \text{if} \quad u_0 = \sum_{k=0}^{\infty} a_k \Phi_k \in \mathcal{H}^s,$$

defines a self-adjoint semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{H}^s$ for all $s \in \mathbb{R}$.

For $\delta \in \mathbb{R}$ and a function $h : (0,1) \to \mathbb{R}$, we introduce the notion of δ-generalized limit of h at $x = 0$ as follows:

$$\mathcal{O}_\delta(h) := \lim_{x\to 0^+} x^\delta h(x).$$

Notation: Let $t > 0$ fixed. If $z \in \mathcal{H}^s$ then $S(t)z \in \mathcal{H}^s$, so we write $\lim_{x\to 1^-} S(t)z$ instead of $\lim_{x\to 1^-}(S(t)z)(x)$.

3.1 Notion of weak solutions for both systems

Now, we consider a convenient definition of a weak solution for the system (4). Let $\tau > 0$ be fixed. We multiply the equation in (4) by $x^\beta \varphi(x,t) = x^\beta S(\tau - t)z^\tau$, $0 \leq t \leq \tau$, integrate by parts (formally), and by using the boundary conditions for u, φ, see Remark 4, we get

$$\begin{aligned}\langle u(\tau), z^\tau\rangle_\beta - \langle u_0, S(\tau)z^\tau\rangle_\beta &= \int_0^T [u(\cdot,t), S(\tau - t)z^\tau](0)\mathrm{d}t \\ &= \int_0^T [u(\cdot,t), x^{-a}](0)\mathcal{O}_a(S(\tau - t)z^\tau)\mathrm{d}t \\ &= \int_0^T f(t)\mathcal{O}_a(S(\tau - t)z^\tau)\mathrm{d}t.\end{aligned}$$

Definition 6 Let $T > 0$, $0 \leq \alpha < 2$, $\beta \in \mathbb{R}$, $\mu < \mu(\alpha + \beta)$, and a given by (3). Let $f \in L^2(0,T)$ and $u_0 \in \mathcal{H}^{-s}$ for some $s > 0$. A weak solution of (4) is a function $u \in C^0([0,T];\mathcal{H}^{-s})$ such that for every $\tau \in (0,T]$ and for every $z^\tau \in \mathcal{H}^s$, we have

$$\left\langle u(\tau), z^\tau\right\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \left\langle u_0, S(\tau)z^\tau\right\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} + \int_0^\tau f(t)\mathcal{O}_a\left(S(\tau - t)z^\tau\right)\mathrm{d}t. \quad (13)$$

The next result shows the existence of weak solutions for the system (4) under suitable conditions on the parameters $\alpha, \beta, \mu,$ and s, and its proof is similar to the proof of Proposition 2.9 in [11].

Proposition 7 *Let $T > 0$, $0 \leq \alpha < 2$, $\beta \in \mathbb{R}$, $\mu < \mu(\alpha+\beta)$, a given in (3). Let $f \in L^2(0,T)$ and $u_0 \in \mathcal{H}^{-s}$ such that $s > \nu$, with ν given in (5). Then, formula (13) defines for each $\tau \in [0,T]$ a unique element $u(\tau) \in \mathcal{H}^{-s}$ that can be written as*

$$u(\tau) = S(\tau)u_0 + B(\tau)f, \quad \tau \in (0,T],$$

where $B(\tau)$ is the strongly continuous family of bounded operators $B(\tau) : L^2(0,T) \to \mathcal{H}^{-s}$ given by

$$\langle B(\tau)f, z^\tau \rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \int_0^\tau f(t)\mathcal{O}_a\left(S(\tau-t)z^\tau\right)\mathrm{d}t, \quad \textit{for all } z^\tau \in \mathcal{H}^s.$$

Furthermore, the unique weak solution u on $[0,T]$ to (4) (in the sense of (13)) belongs to $C^0\left([0,T];\mathcal{H}^{-s}\right)$ and fulfills

$$\|u\|_{L^\infty([0,T];\mathcal{H}^{-s})} \leq C\left(\|u_0\|_{\mathcal{H}^{-s}} + \|f\|_{L^2(0,T)}\right).$$

Proof Fix $\tau > 0$. Let $u(\tau) \in H^{-s}$ be determined by the condition (13), hence

$$u(\tau) - S(\tau)u_0 = \zeta(\tau)f,$$

where

$$\langle \zeta(\tau)f, z^\tau \rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \int_0^\tau f(t)\mathcal{O}_a\left(S(\tau-t)z^\tau\right)\mathrm{d}t, \quad \text{for all } z^\tau \in \mathcal{H}^s.$$

We claim that $\zeta(\tau)$ is a bounded operator from $L^2(0,T)$ into $\mathcal{H}^{-s}$: consider $z^\tau \in \mathcal{H}^s$ given by

$$z^\tau = \sum_{k=0}^{\infty} b_k \Phi_k, \tag{14}$$

therefore

$$S(\tau-t)z^\tau = \sum_{k=0}^{\infty} \mathrm{e}^{\lambda_k(t-\tau)} b_k \Phi_k, \quad \text{for all } t \in [0,\tau].$$

By using Lemma A.3 and (70), we obtain that there exists a constant $C = C(\alpha,\beta,\mu) > 0$ such that

$$|\mathcal{O}_a(\Phi_k)| \leq C|j_{\nu+1,k}|^{\nu+1/2}, \quad k \geq 1,$$

hence (69) implies that there exists a constant $C = C(\alpha,\beta,\mu,\tau) > 0$ such that

$$
\begin{aligned}
&\left(\int_0^\tau \left|\mathcal{O}_a\left(S(\tau-t)z^\tau\right)\right|^2\,\mathrm{d}t\right)^{1/2} \le \sum_{k=0}^{\infty}|b_k||\mathcal{O}_a(\Phi_k)|\left(\int_0^\tau \mathrm{e}^{2\lambda_k(t-\tau)}\,\mathrm{d}t\right)^{1/2}\\
&\le C\left(\tau^{1/2}|b_0|+\left(\sum_{k=1}^{\infty}|b_k|^2\lambda_k^s\right)^{1/2}\left(\sum_{k=1}^{\infty}|\lambda_k|^{\nu-1/2-s}\left(1-\mathrm{e}^{-2\lambda_k\tau}\right)\right)^{1/2}\right)\\
&\le C\left(\tau^{1/2}|b_0|+\left(\sum_{k=1}^{\infty}|b_k|^2\lambda_k^s\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2(s-\nu+1/2)}}\right)^{1/2}\right)\\
&\le C\left\|z^\tau\right\|_{\mathcal{H}^s}.
\end{aligned}
$$

Therefore, $\|\zeta(\tau)f\|_{\mathcal{H}^{-s}} \le C\|f\|_{L^2(0,T)}$ for all $f \in L^2(0,T)$, $\tau \in (0,T]$.

Finally, we fix $f \in L^2(0,T)$ and show that the mapping $\tau \mapsto \zeta(\tau)f$ is right-continuous on $[0,T)$. Let $h > 0$ small enough and $z \in \mathcal{H}^s$ given as in (14). Thus, proceeding as in the last inequalities, we have

$$
\begin{aligned}
&|\langle \zeta(\tau+h)f-\zeta(\tau)f, z\rangle_{\mathcal{H}^{-s},\mathcal{H}^s}|\\
&\le C\|f\|_{L^2(0,T)}\left(|b_0|h+\left(\sum_{k=1}^{\infty}|b_k|^2\lambda_k^s\right)^{1/2}\right.\\
&\left.\times\left[\left(\sum_{k=1}^{\infty}\frac{I(\tau,k,h)}{k^{2(s-\nu+1/2)}}\right)^{1/2}+\left(\sum_{k=1}^{\infty}\frac{1-\mathrm{e}^{-2\lambda_k h}}{k^{2(s-\nu+1/2)}}\right)^{1/2}\right]\right),
\end{aligned}
$$

where

$$
\begin{aligned}
I(\tau,k,h) &= \lambda_k\int_0^\tau\left(\mathrm{e}^{\lambda_k(t-\tau-h)}-\mathrm{e}^{\lambda_k(t-\tau)}\right)^2\,\mathrm{d}t\\
&= \frac{1}{2}(1-\mathrm{e}^{-\lambda_k h})^2(1-\mathrm{e}^{-2\lambda_k\tau}) \to 0 \quad \text{as} \quad h\to 0^+. \qquad (15)
\end{aligned}
$$

Since $0 \le I(\tau,k,h) \le 1/2$ uniformly for $\tau, h > 0$, $k \ge 1$, the result follows by the dominated convergence theorem. □

Remark 8 In the following section, we will consider initial conditions in $L^2_\beta(0,1)$. Notice that $L^2_\beta(0,1) \subset \mathcal{H}^{-\nu-\delta}$ for all $\delta > 0$, and we can apply Proposition 7 with $s = \nu+\delta$, $\delta > 0$, then the corresponding solutions will be in $C^0([0,T],\mathcal{H}^{-\nu-\delta})$.

As before, we introduce a suitable definition of a weak solution for the system (6).

Definition 9 Let $T > 0$, $\beta \in \mathbb{R}$, $0 \le \alpha < 2$, $\mu < \mu(\alpha+\beta)$ and a given in (3). Let $f \in L^2(0,T)$ and $u_0 \in L^2_\beta(0,1)$. A weak solution of (6) is a function $u \in C^0\left([0,T]; L^2_\beta(0,1)\right)$ such that for every $\tau \in (0,T]$ and for every $z^\tau \in L^2_\beta(0,1)$, we have

$$
\left\langle u(\tau), z^\tau\right\rangle_\beta = \left\langle u_0, S(\tau)z^\tau\right\rangle_\beta + \int_0^\tau f(t)\lim_{x\to 1^-} S(\tau-t)z^\tau\mathrm{d}t. \qquad (16)
$$

The next result shows the existence of weak solutions for the system (6) under certain conditions on the parameters α, β, μ and a, and its proof is similar to the proof of Proposition 18 in [12].

Proposition 10 *Let $T > 0$, $\beta \in \mathbb{R}$, $0 \leq \alpha < 2$, $\mu < \mu(\alpha + \beta)$ and a given in (3). Let $f \in L^2(0, T)$ and $u_0 \in L^2_\beta(0, 1)$. Then, formula (16) defines for each $\tau \in [0, T]$ a unique element $u(\tau) \in L^2_\beta(0, 1)$ that can be written as*

$$u(\tau) - S(\tau)u_0 = \mathcal{B}(\tau)f, \quad \tau \in (0, T],$$

where $B(\tau)$ is the strongly continuous family of bounded operators $\mathcal{B}(\tau) : L^2(0, T) \to L^2_\beta(0, 1)$ given by

$$\left\langle \mathcal{B}(\tau)f, z^\tau \right\rangle_\beta = \int_0^\tau f(t) \lim_{x \to 1^-} S(\tau - t)z^\tau \mathrm{d}t, \quad \textit{for all } z^\tau \in L^2_\beta(0, 1).$$

Furthermore, the unique weak solution u on $[0, T]$ to (6) (in the sense of (16)) belongs to $C^0\left([0, T]; L^2_\beta(0, 1)\right)$ and fulfills

$$\|u\|_{L^\infty\left([0,T];L^2_\beta(0,1)\right)} \leq C\left(\|u_0\|_\beta + \|f\|_{L^2(0,T)}\right).$$

Proof Fix $\tau > 0$. Let $u(\tau) \in L^2_\beta(0, 1)$ be determined by the condition (16), hence

$$u(\tau) - S(\tau)u_0 = \zeta(\tau)f,$$

where

$$\left\langle \zeta(\tau)f, z^\tau \right\rangle_\beta = \int_0^\tau f(t) \lim_{x \to 1^-} S(\tau - t)z^\tau \mathrm{d}t \quad \text{for all } z^\tau \in L^2_\beta(0, 1).$$

Let $z^\tau \in L^2_\beta(0, 1)$ written as

$$z^\tau = \sum_{k=0}^\infty b_k \Phi_k, \tag{17}$$

therefore

$$\lim_{x \to 1^-} S(\tau - t)z^\tau = \sum_{k=0}^\infty \mathrm{e}^{\lambda_k(t-\tau)} b_k \Phi_k(1) \quad \text{for all } t \in [0, \tau].$$

By (11), we get

$$|\Phi_0(1)| = \sqrt{2 - \alpha + 2\sqrt{\mu(\alpha + \beta) - \mu}}, \qquad |\Phi_k(1)| = \sqrt{2\kappa_\alpha}, \quad k \geq 1, \tag{18}$$

hence there exists a constant $C = C(\alpha, \beta, \mu, \tau) > 0$ such that

$$\left(\int_0^\tau \left|\lim_{x\to 1^-} S(\tau - t)z^\tau\right|^2 \mathrm{d}t\right)^{1/2} \le \sum_{k=0}^{\infty} |b_k||\Phi_k(1)| \left(\int_0^\tau \mathrm{e}^{2\lambda_k(t-\tau)}\,\mathrm{d}t\right)^{1/2}$$
$$\le C\|z^\tau\|_\beta \left(\sum_{k=0}^{\infty}\int_0^\tau \mathrm{e}^{2\lambda_k(t-\tau)}\,\mathrm{d}t\right)^{1/2} = C\|z^\tau\|_\beta \left(\tau + \sum_{k=1}^{\infty}\frac{1-\mathrm{e}^{-2\lambda_k\tau}}{2\lambda_k}\right)^{1/2}$$
$$\le C\|z^\tau\|_\beta \left(\tau + \sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{1/2}.$$

Therefore, $\|\zeta(\tau)f\|_\beta \le C\|f\|_{L^2(0,T)}$ for all $f \in L^2(0,T)$, $\tau \in (0,T]$.

Finally, we fix $f \in L^2(0,T)$ and show that the mapping $\tau \mapsto \zeta(\tau)f$ is right-continuous on $[0,T)$. Let $h > 0$ small enough and $z \in L^2_\beta(0,1)$ given as in (17). Then, we have

$$|\langle \zeta(\tau+h)f - \zeta(\tau)f, z\rangle_\beta|$$
$$\le \int_0^\tau |f(t)| \left|\lim_{x\to 1^-} (S(\tau+h-t) - S(\tau-t))z\right| \mathrm{d}t$$
$$+ \int_\tau^{\tau+h} |f(t)| \left|\lim_{x\to 1^-} S(\tau+h-t)z\right| \mathrm{d}t$$
$$\le C\|z^\tau\|_\beta \|f\|_{L^2(0,T)} \left[\left(\sum_{k=1}^{\infty}\frac{I(\tau,k,h)}{k^2}\right)^{1/2} + \left(h + \sum_{k=1}^{\infty}\frac{1-\mathrm{e}^{-2\lambda_k h}}{k^2}\right)^{1/2}\right],$$

where $I(\tau,k,h) \to 0$ as $h \to 0^+$, see (15). □

4 Control at the left endpoint

4.1 Upper estimate of the cost of the null controllability

Here, we use the moment method, introduced by Fattorini & Russell in [10], to prove the null controllability of the system (4). The first step is to construct a biorthogonal family $\{\psi_k\}_{k\ge 0} \subset L^2(0,T)$ to the family of exponential functions $\left\{\mathrm{e}^{-\lambda_k(T-t)}\right\}_{k\ge 0}$ on $[0,T]$, i.e., that satisfies

$$\int_0^T \psi_k(t)\mathrm{e}^{-\lambda_l(T-t)}dt = \delta_{kl}, \quad \text{for all} \quad k, l \ge 0.$$

This construction will help us to get an upper bound for the cost of the null controllability of the system (4).

Assume that for each $k \geq 0$ there exists an entire function F_k of exponential type $T/2$ such that $F_k(x) \in L^2(\mathbb{R})$, and

$$F_k(i\lambda_l) = \delta_{kl}, \quad \text{for all} \quad k, l \geq 0. \tag{19}$$

The L^2-version of the Paley-Wiener theorem implies that there exists $\eta_k \in L^2(\mathbb{R})$ with support in $[-T/2, T/2]$ such that $F_k(z)$ is the analytic extension of the Fourier transform of η_k. Then, we have that

$$\psi_k(t) := \mathrm{e}^{\lambda_k T/2}\eta_k(t - T/2), \quad t \in [0, T],\ k \geq 0, \tag{20}$$

is the family we are looking for.

Now, we proceed to construct the family $F_k, k \geq 0$. Consider the Weierstrass infinite product

$$\Lambda(z) := z\prod_{k=1}^{\infty}\left(1 + \frac{iz}{(\kappa_\alpha j_{\nu+1,k})^2}\right). \tag{21}$$

From (68), we have that $j_{\nu+1,k} = O(k)$ for k large, thus the infinite product converges absolutely in $\mathbb{C}$. Hence, $\Lambda(z)$ is an entire function with simple zeros at $i\lambda_k$, $k \geq 0$.

From [22, Chap. XV, p. 498, eq. (3)], we have for $\nu > -1$ that

$$\Lambda(z) = z\Gamma(\nu + 2)\left(\frac{2\kappa_\alpha}{\sqrt{-iz}}\right)^{\nu+1} J_{\nu+1}\left(\frac{\sqrt{-iz}}{\kappa_\alpha}\right). \tag{22}$$

[11] proved that

$$|J_\nu(z)| \leq \frac{|z|^\nu \mathrm{e}^{|\Im(z)|}}{2^\nu\Gamma(\nu + 1)}, \quad z \in \mathbb{C}.$$

Therefore,

$$|\Lambda(z)| \leq |z| \exp\left(\frac{|\Im(\sqrt{-iz})|}{\kappa_\alpha}\right), \quad z \in \mathbb{C}.$$

In particular,

$$|\Lambda(z)| \leq |z| \exp\left(\frac{|z|^{1/2}}{\kappa_\alpha}\right), \quad z \in \mathbb{C}, \quad |\Lambda(x)| \leq |x| \exp\left(\frac{|x|^{1/2}}{\sqrt{2}\kappa_\alpha}\right), \quad x \in \mathbb{R}. \tag{23}$$

It follows that

$$\Psi_k(z) := \frac{\Lambda(z)}{\Lambda'(i\lambda_k)(z - i\lambda_k)}, \quad k \geq 0, \tag{24}$$

is a family of entire functions that satisfy (19). Since $\Psi_k(x)$ is not in $L^2(\mathbb{R})$, we need to fix this by using a suitable "complex multiplier", thus we follow the approach introduced in [20].

For $\theta, \omega > 0$, we define

$$\sigma_\theta(t) := \exp\left(-\frac{\theta}{1-t^2}\right), \quad t \in (-1,1),$$

and extended by 0 outside of $(-1,1)$. Clearly σ_θ is analytic on $(-1,1)$. Set $C_\theta^{-1} := \int_{-1}^{1} \sigma_\theta(t)\mathrm{d}t$ and define

$$H_{\omega,\theta}(z) = C_\theta \int_{-1}^{1} \sigma_\theta(t) \exp(-i\omega t z)\,\mathrm{d}t. \tag{25}$$

$H_{\omega,\theta}(z)$ is an entire function, and the next result provides additional properties of $H_{\omega,\theta}(z)$.

Lemma 11 *The function $H_{\omega,\theta}$ fulfills the following inequalities:*

$$H_{\omega,\theta}(ix) \geq \frac{\exp\left(\omega|x|/\left(2\sqrt{\theta+1}\right)\right)}{11\sqrt{\theta+1}}, \quad x \in \mathbb{R}, \tag{26}$$

$$|H_{\omega,\theta}(z)| \leq \exp\left(\omega|\Im(z)|\right), \quad z \in \mathbb{C}, \tag{27}$$

$$|H_{\omega,\theta}(x)| \leq \chi_{|x|\leq 1}(x) + c\sqrt{\theta+1}\sqrt{\omega\theta\,|x|}\exp\left(3\theta/4 - \sqrt{\omega\theta\,|x|}\right)\chi_{|x|>1}(x), \; x \in \mathbb{R}, \tag{28}$$

where $c > 0$ does not depend on ω and θ.

We refer to [20, pp. 85–86] for the details.

For $k \geq 0$, consider the entire function F_k given as

$$F_k(z) := \Psi_k(z)\frac{H_{\omega,\theta}(z)}{H_{\omega,\theta}(i\lambda_k)}, \quad z \in \mathbb{C}. \tag{29}$$

For $\delta \in (0,1)$, we set

$$\omega := \frac{T(1-\delta)}{2} > 0, \quad \text{and} \quad \theta := \frac{(1+\delta)^2}{\kappa_\alpha^2 T(1-\delta)} > 0. \tag{30}$$

Lemma 12 *The function $F_k(z)$, $k \geq 0$, has the following properties:*
(i) F_k is of exponential type $T/2$.
(ii) $F_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
(iii) F_k satisfies (19).
(iv) Furthermore, there exists a constant $c > 0$, independent of T, α and δ, such that

$$\|F_0\|_{L^1(\mathbb{R})} \leq C(T,\alpha,\delta) \quad \textit{and} \tag{31}$$

$$\|F_k\|_{L^1(\mathbb{R})} \leq \frac{C(T,\alpha,\delta)}{\lambda_k\,|\Lambda'(i\lambda_k)|}\exp\left(-\frac{\omega\lambda_k}{2\sqrt{\theta+1}}\right), \quad k \geq 1, \tag{32}$$

where

$$C(T,\alpha,\delta) = c\sqrt{\theta+1}\left[\exp\left(\frac{1}{\sqrt{2}\kappa_\alpha}\right) + \sqrt{\theta+1}\frac{\kappa_\alpha^2}{\delta^5}\exp\left(\frac{3\theta}{4}\right)\right]. \tag{33}$$

Proof By using (23), (27), (29) and (30), we get that F_k is of exponential type $T/2$ for all $k \geq 0$. Moreover, by using (24) and (29), we can see that F_k fulfills (19).

Now, we use (23), (26), (28), (29), and (30) to get

$$\begin{aligned}|F_k(x)| &\leq c\exp\left(-\frac{\omega\lambda_k}{2\sqrt{\theta+1}}\right)\frac{\sqrt{\theta+1}|x|}{|\Lambda'(i\lambda_k)|\,|x^2+\lambda_k^2|^{1/2}}|H_{\omega,\theta}(x)|\exp\left(\frac{|x|^{1/2}}{\sqrt{2}\kappa_\alpha}\right)\\ &\leq c\exp\left(-\frac{\omega\lambda_k}{2\sqrt{\theta+1}}\right)\frac{\sqrt{\theta+1}}{\lambda_k|\Lambda'(i\lambda_k)|}\\ &\quad\times\left[\mathrm{e}^{\frac{1}{\sqrt{2}\kappa_\alpha}}\chi_{|x|\leq 1}(x) + \sqrt{\theta+1}\sqrt{\omega\theta}\,|x|^{3/2}\exp\left(\frac{3\theta}{4}-\frac{\delta|x|^{1/2}}{\sqrt{2}\kappa_\alpha}\right)\chi_{|x|>1}(x)\right],\end{aligned}$$

for all $k \geq 1$. Since the function on the right-hand side is rapidly decreasing in $\mathbb{R}$, we have $F_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Finally, the change of variable $y = (\kappa_\alpha)^{-1}\delta|x|^{1/2}/\sqrt{2}$ implies (32).

When $k = 0$, we have

$$\begin{aligned}|F_0(x)| \leq \exp\left(\frac{|x|^{1/2}}{\sqrt{2}\kappa_\alpha}\right)|H_{\omega,\theta}(x)| &\leq \mathrm{e}^{\frac{1}{\sqrt{2}\kappa_\alpha}}\chi_{|x|\leq 1}(x)\\ &\quad+\sqrt{\theta+1}\sqrt{\omega\theta\,|x|}\exp\left(\frac{3\theta}{4}-\frac{\delta|x|^{1/2}}{\sqrt{2}\kappa_\alpha}\right)\chi_{|x|>1}(x),\end{aligned}$$

then we integrate on $\mathbb{R}$ and the result follows. □

Since $\eta_k, F_k \in L^1(\mathbb{R})$, the inverse Fourier theorem yields

$$\eta_k(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{it\tau}F_k(\tau)\mathrm{d}\tau, \quad t \in \mathbb{R}, k \geq 0,$$

hence (20) implies that $\psi_k \in C([0,T])$. From (31) and (32), we have $\|\psi_0\|_\infty \leq C(T,\alpha,\delta)$ and

$$\|\psi_k\|_\infty \leq \frac{C(T,\alpha,\delta)}{\lambda_k\,|\Lambda'(i\lambda_k)|}\exp\left(\frac{T\lambda_k}{2}-\frac{\omega\lambda_k}{2\sqrt{\theta+1}}\right), \quad k \geq 1. \tag{34}$$

Now, we are ready to prove the null controllability of the system (4). Let $u_0 \in L^2_\beta(0,1)$. Then, consider its (generalized) Fourier–Dini series with respect to the orthonormal basis $\{\Phi_k\}_{k\geq 0}$,

$$u_0(x) = \sum_{k=0}^{\infty} b_k \Phi_k(x). \tag{35}$$

We set

$$f(t) := -\sum_{k=0}^{\infty} \frac{b_k \mathrm{e}^{-\lambda_k T}}{\mathcal{O}_a(\Phi_k)} \psi_k(t). \tag{36}$$

Since $\{\psi_k\}_{k\geq 0}$ is biorthogonal to $\{\mathrm{e}^{-\lambda_k(T-t)}\}_{k\geq 0}$, we have

$$\begin{aligned}\int_0^T f(t)\mathcal{O}_a(\Phi_k)\,\mathrm{e}^{-\lambda_k(T-t)}\mathrm{d}t &= -b_k \mathrm{e}^{-\lambda_k T} = -\left\langle u_0, \mathrm{e}^{-\lambda_k T}\Phi_k\right\rangle_\beta \\ &= -\left\langle u_0, \mathrm{e}^{-\lambda_k T}\Phi_k\right\rangle_{\mathcal{H}^{-s},\mathcal{H}^s}, \quad k \geq 0.\end{aligned}$$

Let $u \in C([0,T]; H^{-s})$ that satisfies (13) for all $\tau \in (0,T]$, $z^\tau \in H^s$. In particular, for $\tau = T$, we take $z^T = \Phi_k$, $k \geq 0$, then the last equality implies that

$$\langle u(\cdot,T), \Phi_k\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = 0 \quad \text{for all} \quad k \geq 0,$$

hence $u(\cdot,T) = 0$.

It just remains to estimate the norm of the control f. From (34) and (36), we get

$$\begin{aligned}C(T,\alpha,\delta)^{-1}\|f\|_\infty \leq \frac{|b_0|}{|\mathcal{O}_a(\Phi_0)|} &+ \sum_{k=1}^{\infty} \frac{|b_k|}{|\mathcal{O}_a(\Phi_k)|} \frac{1}{\lambda_k |\Lambda'(i\lambda_k)|} \\ &\times \exp\left(-\frac{T\lambda_k}{2} - \frac{\omega\lambda_k}{2\sqrt{\theta+1}}\right).\end{aligned} \tag{37}$$

From (21), (22), and (64) (with $\nu+1$ instead of ν), we get that

$$\Lambda'(i\lambda_k) = i\lambda_k \frac{2^{\nu+1}\Gamma(\nu+2)}{(j_{\nu+1,k})^{\nu+2}} \frac{-i}{2\kappa_\alpha^2} J'_{\nu+1}(j_{\nu+1,k}) = \frac{2^\nu \Gamma(\nu+2)}{(j_{\nu+1,k})^\nu} J_\nu(j_{\nu+1,k}), \quad k \geq 1, \tag{38}$$

and by using (70), we get

$$\left|\mathcal{O}_a(\Phi_k)\,\Lambda'(i\lambda_k)\right| = \frac{\Gamma(\nu+2)}{\Gamma(\nu+1)}\sqrt{2\kappa_\alpha} = (\nu+1)\sqrt{2\kappa_\alpha}, \quad k \geq 1.$$

From (37), (69), and using that $\lambda_k \geq \lambda_1$, it follows that

$$\begin{aligned} &C(T,\alpha,\delta)^{-1}\|f\|_\infty \\ &\leq \frac{|b_0|}{|\mathcal{O}_a(\Phi_0)|} + \frac{1}{\sqrt{2}(\nu+1)\kappa_\alpha^{5/2}} \exp\left(-\frac{T\lambda_1}{2} - \frac{\omega\lambda_1}{2\sqrt{\theta+1}}\right) \sum_{k=1}^{\infty} \frac{|b_k|}{(j_{\nu+1,k})^2} \\ &\leq \frac{|b_0|}{|\mathcal{O}_a(\Phi_0)|} + \frac{c}{(\nu+1)\kappa_\alpha^{5/2}} \exp\left(-\frac{T\lambda_1}{2} - \frac{\omega\lambda_1}{2\sqrt{\theta+1}}\right) \left(\sum_{k=1}^{\infty} |b_k|^2\right)^{1/2}. \end{aligned}$$

Using the expression of ω, θ given in (30) and the facts $\theta > 0$, $\delta \in (0,1)$, and $0 < \kappa_\alpha \leq 1$, we get that

$$\theta \leq \frac{4}{(1-\delta)\kappa_\alpha^2 T}, \quad \sqrt{\theta+1} \leq \frac{2(1+T)^{1/2}}{(1-\delta)^{1/2}\kappa_\alpha T^{1/2}}, \quad \sqrt{\theta+1} \leq \theta+1,$$

therefore

$$\begin{aligned} &\frac{\omega}{\sqrt{\theta+1}} \geq \frac{\kappa_\alpha(1-\delta)^{3/2}T^{3/2}}{4(1+T)^{1/2}}, \\ &C(T,\alpha,\delta) \leq c\left(1+\frac{1}{(1-\delta)\kappa_\alpha^2 T}\right)\left[\exp\left(\frac{1}{\sqrt{2}\kappa_\alpha}\right) + \frac{1}{\delta^5}\exp\left(\frac{3}{(1-\delta)\kappa_\alpha^2 T}\right)\right]. \end{aligned} \tag{39}$$

By using the definition of λ_1, and setting $b_0 = 0$, we get the estimate for $\mathcal{K}_{\Phi_0^\perp}$.

4.2 Lower estimate of the cost of the null controllability

In this section, we get a lower estimate of the cost $\mathcal{K} = \mathcal{K}_{L^2_\beta}(T, \alpha, \beta, \mu)$.
We set

$$u_0(x) := \frac{|J_\nu(j_{\nu+1,1})|}{(2\kappa_\alpha)^{1/2}}\Phi_1(x),\ x \in (0,1), \quad \text{hence} \quad \|u_0\|_\beta^2 = \frac{|J_\nu(j_{\nu+1,1})|^2}{2\kappa_\alpha}. \tag{40}$$

For $\varepsilon > 0$ small enough, there exists $f \in U(\alpha, \beta, \mu, T, u_0)$ such that

$$u(\cdot, T) \equiv 0, \quad \text{and} \quad \|f\|_{L^2(0,T)} \leq (\mathcal{K}+\varepsilon)\|u_0\|_\beta. \tag{41}$$

Then, in (13), we set $\tau = T$ and take $z^\tau = \Phi_k$, $k \geq 0$, to obtain

$$\begin{aligned} e^{-\lambda_k T}\langle u_0, \Phi_k\rangle_\beta = \langle u_0, S(T)\Phi_k\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} &= -\int_0^T f(t)\mathcal{O}_a\left(S(T-t)\Phi_k\right)\mathrm{d}t \\ &= -e^{-\lambda_k T}\mathcal{O}_a\left(\Phi_k\right)\int_0^T f(t)e^{\lambda_k t}\mathrm{d}t, \end{aligned}$$

from (40) and (70), it follows that

$$\int_0^T f(t)\mathrm{e}^{\lambda_k t}\mathrm{d}t = -\frac{2^\nu \Gamma(\nu+1)|J_\nu(j_{\nu+1,1})|^2}{2\kappa_\alpha (j_{\nu+1,1})^\nu}\delta_{1,k}, \quad k \geq 0. \tag{42}$$

Now, consider the function $v : \mathbb{C} \to \mathbb{C}$ given by

$$v(s) := \int_{-T/2}^{T/2} f\left(t+\frac{T}{2}\right)\mathrm{e}^{-ist}\,\mathrm{d}t, \quad s \in \mathbb{C}.$$

Fubini and Morera's theorems imply that $v(s)$ is an entire function. Moreover, (42) implies that

$$v(i\lambda_k) = 0 \quad \text{for all } k \geq 0, k \neq 1, \text{ and } v(i\lambda_1) = -\frac{2^\nu \Gamma(\nu+1)|J_\nu(j_{\nu+1,1})|^2}{2\kappa_\alpha (j_{\nu+1,1})^\nu}\mathrm{e}^{-\lambda_1 T/2}.$$

We also have that

$$|v(s)| \leq \mathrm{e}^{T|\Im(s)|/2}\int_0^T |f(t)|\mathrm{d}t \leq (\mathcal{K}+\varepsilon)T^{1/2}\mathrm{e}^{T|\Im(s)|/2}\,\|u_0\|_\beta\,. \tag{43}$$

Consider the entire function $F(z)$ given by

$$F(s) := v\left(s - i\delta\right), \quad s \in \mathbb{C}, \tag{44}$$

for some $\delta > 0$ that will be chosen later on. Clearly,

$$F\left(b_k\right) = 0, \quad k \geq 0, k \neq 1, \quad \text{where} \quad b_k := i\left(\lambda_k + \delta\right), \quad k \geq 0, \quad \text{and}$$

$$F(b_1) = -\frac{2^\nu \Gamma(\nu+1)|J_\nu(j_{\nu+1,1})|^2}{2\kappa_\alpha (j_{\nu+1,1})^\nu}\mathrm{e}^{-\lambda_1 T/2}. \tag{45}$$

From (40), (43) and (44), we obtain

$$\log|F(s)| \leq \frac{T}{2}|\Im(s)-\delta| + \log\left((\mathcal{K}+\varepsilon)T^{1/2}\frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{(2\kappa_\alpha)^{1/2}}\right), \quad s \in \mathbb{C}. \tag{46}$$

We recall the following representation theorem, see [17, p. 56].

Theorem 13 *Let $g(z)$ be an entire function of exponential type and assume that*

$$\int_{-\infty}^{\infty}\frac{\log^+|g(x)|}{1+x^2}\mathrm{d}x < \infty.$$

Let $\{d_\ell\}_{\ell\geq 1}$ *be the set of zeros of* $g(z)$ *in the upper half plane* $\Im(z) > 0$ *(each zero being repeated as many times as its multiplicity). Then,*

$$\log|g(z)| = A\Im(z) + \sum_{\ell=1}^{\infty} \log\left|\frac{z-d_\ell}{z-\bar{d}_\ell}\right| + \frac{\Im(z)}{\pi}\int_{-\infty}^{\infty} \frac{\log|g(s)|}{|s-z|^2}\mathrm{d}s, \quad \Im(z) > 0,$$

where

$$A = \limsup_{y\to\infty} \frac{\log|g(iy)|}{y}.$$

We apply the last result to the function $F(z)$ given in (44). In this case, (43) implies that $A \leq T/2$. Also notice that $\Im(b_k) > 0$, $k \geq 0$, to get

$$\log|F(b_1)| \leq (\lambda_1+\delta)\frac{T}{2} + \sum_{k=0,k\neq 1}^{\infty} \log\left|\frac{b_1-b_k}{b_1-\bar{b}_k}\right| + \frac{\Im(b_1)}{\pi}\int_{-\infty}^{\infty} \frac{\log|F(s)|}{|s-b_1|^2}\,\mathrm{d}s. \tag{47}$$

By using the definition of the constants b_k's, we have

$$\begin{aligned}
&\sum_{k=0,k\neq 1}^{\infty} \log\left|\frac{b_1-b_k}{b_1-\bar{b}_k}\right| \\
&= \log\left(\frac{j_{\nu+1,1}^2}{2\delta/\kappa_\alpha^2 + j_{\nu+1,1}^2}\right) + \sum_{k=2}^{\infty} \log\left(\frac{\left(j_{\nu+1,k}\right)^2 - \left(j_{\nu+1,1}\right)^2}{2\delta/\kappa_\alpha^2 + \left(j_{\nu+1,1}\right)^2 + \left(j_{\nu+1,k}\right)^2}\right) \\
&\leq \log\left(\frac{j_{\nu+1,1}^2}{2\delta/\kappa_\alpha^2 + j_{\nu+1,1}^2}\right) + \sum_{k=2}^{\infty} \frac{1}{j_{\nu+1,k+1}-j_{\nu+1,k}} \int_{j_{\nu+1,k}}^{j_{\nu+1,k+1}} \log\left(\frac{x^2}{2\delta/\kappa_\alpha^2 + x^2}\right)\mathrm{d}x \\
&\leq \log\left(\frac{j_{\nu+1,1}^2}{2\delta/\kappa_\alpha^2 + j_{\nu+1,1}^2}\right) + \frac{1}{\pi}\int_{j_{\nu+1,2}}^{\infty} \log\left(\frac{x^2}{2\delta/\kappa_\alpha^2 + x^2}\right)\mathrm{d}x, \\
&= \log\left(\frac{j_{\nu+1,1}^2}{2\delta/\kappa_\alpha^2 + j_{\nu+1,1}^2}\right) - \frac{j_{\nu+1,2}}{\pi}\log\left(\frac{1}{1+2\delta/\left(\kappa_\alpha j_{\nu+1,2}\right)^2}\right) \\
&\quad - \frac{2\sqrt{2\delta}}{\pi\kappa_\alpha}\tan^{-1}\left(\frac{\sqrt{2\delta}}{\kappa_\alpha j_{\nu+1,2}}\right),
\end{aligned} \tag{48}$$

where we have used Lemma A.2 and made the change of variables

$$\tau = \frac{\kappa_\alpha}{\sqrt{2\delta}}x.$$

From (46), we get the estimate

$$\frac{\Im(b_1)}{\pi}\int_{-\infty}^{\infty}\frac{\log|F(s)|}{|s-b_1|^2}\,\mathrm{d}s \le \frac{\delta T}{2}+\log\left((\mathcal{K}+\varepsilon)T^{1/2}\frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{(2\kappa_\alpha)^{1/2}}\right). \tag{49}$$

From (45), (47), (48), and (49), we have

$$\begin{aligned}&\frac{2\sqrt{2\delta}}{\pi\kappa_\alpha}\tan^{-1}\left(\frac{\sqrt{2\delta}}{\kappa_\alpha j_{\nu+1,2}}\right)-\frac{j_{\nu+1,2}}{\pi}\log\left(1+\frac{2\delta}{\left(\kappa_\alpha j_{\nu+1,2}\right)^2}\right)-\frac{\lambda_1+\delta}{T^{-1}}\\&\quad\le\log(\mathcal{K}+\varepsilon)+\log\left(\frac{(2\kappa_\alpha T)^{1/2}(j_{\nu+1,1})^\nu}{2^\nu\Gamma(\nu+1)|J_\nu(j_{\nu+1,1})|}\right)\\&\qquad+\log\left(\frac{j_{\nu+1,1}^2}{2\delta/\kappa_\alpha^2+j_{\nu+1,1}^2}\right).\end{aligned}$$

The result follows by taking $\delta=\kappa_\alpha^2\left(j_{\nu+1,2}\right)^2/2$ and then letting $\varepsilon\to0^+$.

5 Control at the right endpoint

5.1 Upper estimate of the cost of the null controllability

Now we show the null controllability of the system (6). Let $u_0\in L^2_\beta(0,1)$ given as in (35). We set

$$f(t):=-\sum_{k=0}^{\infty}\frac{b_k\mathrm{e}^{-\lambda_k T}}{\Phi_k(1)}\psi_k(t). \tag{50}$$

Since the sequence $\{\psi_k\}_{k\ge0}$ is biorthogonal to $\{\mathrm{e}^{-\lambda_k(T-t)}\}_{k\ge0}$, we have

$$\Phi_k(1)\int_0^T f(t)\mathrm{e}^{-\lambda_k(T-t)}\mathrm{d}t=-b_k\mathrm{e}^{-\lambda_k T}=-\left\langle u_0,\mathrm{e}^{-\lambda_k T}\Phi_k\right\rangle_\beta,\quad k\ge0. \tag{51}$$

Let $u\in C\left([0,T];L^2_\beta(0,1)\right)$ be the weak solution of system (6). In particular, for $\tau=T$, we take $z^T=\Phi_k$, $k\ge0$, then (16) and (51) imply that $\langle u(\cdot,T),\Phi_k\rangle_\beta=0$ for all $k\ge0$, therefore $u(\cdot,T)\equiv0$.

Finally, we estimate the norm of the control f. From (18), (34), (38) and (50), we get

$$\begin{aligned}C(T,\alpha,\delta)^{-1}\|f\|_\infty\le\frac{|b_0|}{|\Phi_0(1)|}+\frac{1}{\sqrt{2\kappa_\alpha}2^\nu\Gamma(\nu+2)}\sum_{k=1}^{\infty}\frac{|j_{\nu+1,k}|^\nu}{|J_\nu(j_{\nu+1,k})|}\frac{|b_k|}{\lambda_k}\exp\\\times\left(-\frac{T\lambda_k}{2}-\frac{\omega\lambda_k}{2\sqrt{\theta+1}}\right).\end{aligned}$$

By using that $e^{-x} \leq e^{-r} r^r x^{-r}$ for all $x, r > 0$, the Cauchy–Schwarz inequality, Lemma A.3 and the fact that $j_{\nu,k} \geq (k - 1/4)\pi$ (by (69)), (35) and $\lambda_1 \leq \lambda_k$, $k \geq 1$, we obtain that

$$
\begin{aligned}
C(T, \alpha, \delta)^{-1} \|f\|_\infty \leq & \frac{|b_0|}{|\Phi_0(1)|} + \frac{c\kappa_\alpha^{-\nu-1}}{\Gamma(\nu+2)} \left(\frac{2\nu+1}{4T}\right)^{(2\nu+1)/4} \\
& \times e^{-\frac{2\nu+1}{4}} \exp\left(-\frac{\omega\lambda_1}{2\sqrt{\theta+1}} - \frac{T\lambda_1}{4}\right) \sum_{k=1}^{\infty} \frac{|b_k|}{\lambda_k} \\
\leq & \frac{|b_0|}{|\Phi_0(1)|} + \frac{c\kappa_\alpha^{-\nu-1}}{\Gamma(\nu+2)} \left(\frac{2\nu+1}{4Te}\right)^{(2\nu+1)/4} \\
& \times \exp\left(-\frac{\omega\lambda_1}{2\sqrt{\theta+1}} - \frac{T\lambda_1}{4}\right) \left(\sum_{k=1}^{\infty} |b_k|^2\right)^{1/2},
\end{aligned}
$$

and the result follows by (39).

5.2 Lower estimate of the cost of the null controllability

Once again, we get a lower estimate of the cost $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_{L^2_\beta}(T, \alpha, \beta, \mu)$. We set

$$
u_0(x) := \frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{(2\kappa_\alpha)^{1/2}} \Phi_1(x),\ x \in (0,1), \quad \text{hence} \quad \|u_0\|_\beta^2 = \frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|^2}{2\kappa_\alpha}. \tag{52}
$$

For $\varepsilon > 0$ small enough, there exists $f \in \widetilde{U}(\alpha, \beta, \mu, T, u_0)$ such that

$$
u(\cdot, T) \equiv 0, \quad \text{and} \quad \|f\|_{L^2(0,T)} \leq (\widetilde{\mathcal{K}} + \varepsilon)\|u_0\|_\beta.
$$

Then, in (16), we set $\tau = T$ and take $z^\tau = \Phi_k$, $k \geq 0$, to obtain

$$
\begin{aligned}
e^{-\lambda_k T} \langle u_0, \Phi_k \rangle_\beta = \langle u_0, S(T)\Phi_k \rangle_\beta &= -\int_0^T f(t) \lim_{x \to 1^-} S(T-t)\Phi_k \mathrm{d}t \\
&= -e^{-\lambda_k T} \Phi_k(1) \int_0^T f(t) e^{\lambda_k t} \mathrm{d}t.
\end{aligned}
$$

From (18) and (52), it follows that

$$
\int_0^T f(t) e^{\lambda_k t} \mathrm{d}t = -\frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{2\kappa_\alpha} \delta_{1,k}, \quad k \geq 0. \tag{53}
$$

Consider the entire function $v : \mathbb{C} \to \mathbb{C}$ given by

$$v(s) := \int_{-T/2}^{T/2} f\left(t + \frac{T}{2}\right) \mathrm{e}^{-ist} \, \mathrm{d}t, \quad s \in \mathbb{C}.$$

Therefore,

$$|v(s)| \leq \mathrm{e}^{T|\Im(s)|/2} \int_0^T |f(t)| \mathrm{d}t \leq (\widetilde{\mathcal{K}} + \varepsilon) T^{1/2} \mathrm{e}^{T|\Im(s)|/2} \|u_0\|_\beta \,. \tag{54}$$

Moreover, (53) implies that

$$v(i\lambda_k) = 0 \quad \text{for all } k \geq 0, k \neq 1, \quad \text{and} \quad v(i\lambda_1) = -\frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{2\kappa_\alpha} \mathrm{e}^{-\lambda_1 T/2}.$$

Consider the entire function $F(z)$ given by

$$F(s) := v\left(s - i\delta\right), \quad s \in \mathbb{C}, \quad \text{with } \delta = \kappa_\alpha^2 \left(j_{\nu+1,2}\right)^2 /2. \tag{55}$$

Clearly,

$$F\left(b_k\right) = 0, \quad k \geq 0, k \neq 1, \quad \text{where} \quad b_k := i\left(\lambda_k + \delta\right), \quad k \geq 0, \quad \text{and}$$

$$F\left(b_1\right) = -\frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{2\kappa_\alpha} \mathrm{e}^{-\lambda_1 T/2}. \tag{56}$$

From (52), (54) and (55) we obtain

$$\log|F(s)| \leq \frac{T}{2}|\Im(s) - \delta| + \log\left((\widetilde{\mathcal{K}} + \varepsilon) T^{1/2} \frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{(2\kappa_\alpha)^{1/2}}\right), \quad s \in \mathbb{C}. \tag{57}$$

We apply Theorem 13 to the function $F(z)$ given in (55). Then, (54) implies that $A \leq T/2$, hence

$$\log\left|F\left(b_1\right)\right| \leq \left(\lambda_1 + \delta\right) \frac{T}{2} + \sum_{k=0, k\neq 1}^{\infty} \log\left|\frac{b_1 - b_k}{b_1 - \bar{b}_k}\right| + \frac{\Im\left(b_1\right)}{\pi} \int_{-\infty}^{\infty} \frac{\log|F(s)|}{|s - b_1|^2} \, \mathrm{d}s. \tag{58}$$

From (57), we get the estimate

$$\frac{\Im\left(b_1\right)}{\pi} \int_{-\infty}^{\infty} \frac{\log|F(s)|}{|s - b_1|^2} \, \mathrm{d}s \leq \frac{T\delta}{2} + \log\left((\widetilde{\mathcal{K}} + \varepsilon) T^{1/2} \frac{\left|J_\nu\left(j_{\nu+1,1}\right)\right|}{(2\kappa_\alpha)^{1/2}}\right). \tag{59}$$

From (48), (56), (58), and (59), we have

$$\log\left(1+\frac{j_{\nu+1,2}^2}{j_{\nu+1,1}^2}\right)+\left(\frac{1}{2}-\frac{\log 2}{\pi}\right)j_{\nu+1,2}-\left(\lambda_1+\frac{\kappa_\alpha^2 j_{\nu+1,2}^2}{2}\right)T$$
$$\leq \log(\widetilde{\mathcal{K}}+\varepsilon)+\log(2\kappa_\alpha T)^{1/2},$$

the result follows by letting $\varepsilon \to 0^+$.

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Appendix A: Bessel functions

We introduce the Bessel function of the first kind J_ν as follows:

$$J_\nu(x)=\sum_{m\geq 0}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu}, \quad x\geq 0, \tag{60}$$

where $\Gamma(\cdot)$ is the Gamma function. In particular, for $\nu > -1$ and $0 < x \leq \sqrt{\nu+1}$, from (60), we have (see [1, 9.1.7, p. 360])

$$J_\nu(x)\sim\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu} \quad \text{as} \quad x\to 0^+. \tag{61}$$

A Bessel function J_ν of the first kind solves the differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \tag{62}$$

Bessel functions of the first kind satisfy the recurrence formulas (see [1, 9.1.27]):

$$x J_\nu'(x) - \nu J_\nu(x) = -x J_{\nu+1}(x), \tag{63}$$

$$x^{1-\nu} \frac{d}{dx}[x^\nu J_\nu(x)] = x J_\nu'(x) + \nu J_\nu(x) = x J_{\nu-1}(x). \tag{64}$$

Recall the asymptotic behavior of the Bessel function J_ν for large x, see [16, Lem. 7.2, p. 129].

Lemma A.1 *For any* $\nu \in \mathbb{R}$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left\{ \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + \mathcal{O}\left(\frac{1}{x} \right) \right\} \quad as \quad x \to \infty.$$

For $\nu > -1$, $\ell, \ell' \in \mathbb{R}$, we have (see [3, p. 101])

$$\int_0^1 x J_\nu(\ell x) J_\nu\left(\ell' x\right) \mathrm{d}x = \frac{\ell' J_\nu(\ell) J_\nu'\left(\ell'\right) - \ell J_\nu\left(\ell'\right) J_\nu'(\ell)}{\ell^2 - \ell'^2}. \tag{65}$$

For $\nu > -1$, the Bessel function J_ν has an infinite number of real zeros $0 < j_{\nu,1} < j_{\nu,2} < \ldots$, all of which are simple, with the possible exception of $x = 0$. In [16, Proposition 7.8], we can find the next information about the location of the zeros of the Bessel functions J_ν:

Lemma A.2 *Let* $\nu \geq 0$.
1. The difference sequence $\left(j_{\nu,k+1} - j_{\nu,k}\right)_k$ *converges to* π *as* $k \to \infty$.
2. The sequence $\left(j_{\nu,k+1} - j_{\nu,k}\right)_k$ *is strictly decreasing if* $|\nu| > \frac{1}{2}$, *strictly increasing if* $|\nu| < \frac{1}{2}$, *and constant if* $|\nu| = \frac{1}{2}$.

Proposition A.1 *Let* $\nu > -1$, $0 \leq \alpha < 2$ *and* $\beta \in \mathbb{R}$. *The family*

$$\Phi_0(x) = \sqrt{2(\nu+1)\kappa_\alpha}\; x^{(1-\alpha-\beta)/2 + \kappa_\alpha \nu},$$
$$\Phi_k(x) = \frac{\sqrt{2\kappa_\alpha}}{|J_\nu(j_{\nu+1,k})|} x^{(1-\alpha-\beta)/2} J_\nu\left(j_{\nu+1,k} x^{\kappa_\alpha}\right), k \geq 1,$$

is an orthonormal basis for $L^2_\beta(0,1)$.

Proof By using (63) and (65) with $\ell' = j_{\nu+1,k}$, we get

$$\int_0^1 x J_\nu(\ell x) J_\nu\left(j_{\nu+1,k} x\right) \mathrm{d}x = \frac{\ell J_{\nu+1}(\ell) J_\nu(j_{\nu+1,k})}{(\ell + j_{\nu+1,k})(\ell - j_{\nu+1,k})}.$$

By taking the limit as ℓ goes to $j_{\nu+1,k}$, and by using (64) (with $\nu+1$ instead of ν), we obtain

$$\int_0^1 x|J_\nu\left(j_{\nu+1,k}x\right)|^2 \mathrm{d}x = \frac{1}{2}J'_{\nu+1}(j_{\nu+1,k})J_\nu(j_{\nu+1,k}) = \frac{|J_\nu(j_{\nu+1,k})|^2}{2}, \quad k \geq 1. \tag{66}$$

Next, we introduce the following family

$$\Theta_0(x) := \sqrt{2(\nu+1)}x^{1/2+\nu}, \quad \Theta_k(x) := \frac{\sqrt{2}}{|J_\nu(j_{\nu+1,k})|}x^{1/2}J_\nu\left(j_{\nu+1,k}x\right), \quad k \geq 1. \tag{67}$$

In [15] was proved that $\{\Theta_k\}_{k\geq 0}$ is a complete system in $L^2(0,1)$.

Then, (63), (65) and (66) imply that $\langle\Theta_k, \Theta_\ell\rangle = \delta_{k,\ell}$ for all $k, \ell \geq 1$. On the other hand, from (64) with $\nu+1$ instead of ν, we obtain that

$$(j_{\nu+1,k})^{\nu+2}\int_0^1 x^{\nu+1}J_\nu(j_{\nu+1,k}x)\mathrm{d}x = y^{\nu+1}J_{\nu+1}(y)|_{y=0}^{y=j_{\nu+1,k}} = 0, \quad k \geq 1.$$

Therefore $\langle\Theta_k, \Theta_0\rangle = 0$ for all $k \geq 1$. In conclusion, $\{\Theta_k\}_{k\geq 0}$ is an orthonormal basis for $L^2(0,1)$.

Let $\mathcal{U}$ be the unitary operator $\mathcal{U} : L^2(0,1) \to L^2_\beta(0,1)$ given by

$$\mathcal{U}u(x) := \kappa_\alpha^{1/2}x^{-\alpha/4-\beta/2}u(x^{\kappa_\alpha}), \quad u \in L^2(0,1).$$

Notice that $\mathcal{U}\Theta_k = \Phi_k$, $k \geq 0$, therefore Φ_k, $k \geq 0$, is an orthonormal basis for $L^2_\beta(0,1)$. □

For $\nu \geq 0$ fixed, we consider the next asymptotic expansion of the zeros of the Bessel function J_ν, see [22, Section 15.53],

$$j_{\nu,k} = \left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \frac{4\nu^2 - 1}{8\left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi} + O\left(\frac{1}{k^3}\right), \quad \text{as } k \to \infty. \tag{68}$$

In particular, we have

$$\begin{aligned} j_{\nu,k} &\geq \left(k - \frac{1}{4}\right)\pi \quad \text{for } \nu \in [0, 1/2], \\ j_{\nu,k} &\geq \left(k - \frac{1}{8}\right)\pi \quad \text{for } \nu \in [1/2, \infty). \end{aligned} \tag{69}$$

Lemma A.3 *For any $\nu > -1$ and any $k \geq 1$, we have*

$$\sqrt{j_{\nu+1,k}}\left|J_\nu\left(j_{\nu+1,k}\right)\right| = \sqrt{\frac{2}{\pi}} + O\left(\frac{1}{j_{\nu+1,k}}\right) \quad as \quad k \to \infty.$$

The proof of this result follows by using (A.1).

Lemma A.4 *Let* $0 \leq \alpha < 2$, $\beta \in \mathbb{R}$, *a and* $\nu = \nu(\alpha, \beta, \mu)$ *given in (3) and (5), respectively, then the following limits are finite*

$$\mathcal{O}_a(\Phi_0) = \sqrt{2 - \alpha + 2\sqrt{\mu(\alpha+\beta)} - \mu},$$
$$\mathcal{O}_a(\Phi_k) = \frac{(2\kappa_\alpha)^{1/2} \left(j_{\nu+1,k}\right)^\nu}{2^\nu \Gamma(\nu+1) \left|J_\nu\left(j_{\nu+1,k}\right)\right|}, \quad k \geq 1. \tag{70}$$

Proof This result follows from (60). □

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