Invariance under discretization for positive systems

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Abstract

Positive dynamical or control systems have all their variables nonnegative. Euler discretization transforms a continuous-time system into a system on a discrete time scale. Some structural properties of the system may be preserved by discretization, while other may be lost. Four fundamental properties of positive systems are studied in the context of discretization: positivity, positive stability, positive reachability and positive observability. Both linear and nonlinear systems are investigated.

Keywords Positive system · Discretization · Time scale · Invariance

Mathematics Subject Classification 93D20 · 93B05 · 93C55

1 Introduction

Positive systems have all the variables nonnegative. They appear in biology, chemistry and economics (see, e.g., [20,25] for example). A continuous-time positive control system is described by a differential equation of the form $\dot{x} = f(x, u)$, where x is the vector of state variables, u is the vector of control (input) variables, and $\dot{x} = \frac{dx}{dt}$. If the control does not appear on the right-hand side, then the system is an autonomous dynamical system. In many situations, we want to discretize the time: the continuous time is replaced by a discrete one. The main reason for discretization is of computational nature: we can find trajectories of the discretized system, and they are approximations of trajectories of the continuous-time system. However, such a procedure may destroy certain properties of the original continuous-time system. In this work, we study several qualitative properties of positive systems and check if they are invariant under discretization.

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We restrict ourselves to Euler discretization, where the time derivative of $\dot{x}(t)$ is replaced by a difference quotient $(x(t+h) - x(t))/h$. However, contrary to a standard approach, we allow the step h to depend on time t . This means that the discretized system becomes a dynamical system on some discrete time scale $\mathbb{T}$. Thus, we can use calculus on time scales and theory of dynamical systems on time scales in our study on discretization [12]. Theory of systems on time scales contains as particular cases theory of continuous-time systems and theory of discrete-time systems. However, it is more than just mere unification of two theories, since it admits systems on hybrid time scales, which are partly continuous and partly discrete, and on nonhomogeneous discrete time scales. Besides nonuniform discretization also nonuniform sampling leads to systems on discrete time scales that are not homogeneous.

Invariance of certain structural properties under discretization has been studied since the beginning of the numerical methods for solving differential equations. Overviews of this topic can be found in [13,19]. Much effort has been put to the problem of preserving stability under discretization (see, e.g., [1,19,21]). Positivity and discretization was a topic of [14], with the emphasis put on constraints. Nonstandard discretization, with a variable discretization step, was studied in [16]. Though the language of time scales has not been used there, this approach is close in the spirit to the one admitted here. Positivity of the system means that the nonnegative cone is invariant with respect to the dynamics of the system. In [23,24], the authors considered systems for which instead of the nonnegative cone other sets, like polyhedrons or ellipsoids, were used. Both continuous-time and discrete-time systems were studied, as well as constant step discretization. Still another approach to discretization can be found in [26]. The authors of this paper generate a discrete time scale in a stochastic way.

First, we attack the basic property of positive systems, i.e., positivity itself. We give conditions on the system and the graininess function of the discrete time scale under which the discretized system is positive. For linear systems small graininess guarantee positivity of the discretized system, but for nonlinear systems such discretization may not exist.

Another important property that we investigate is positive uniform exponential stability. We show that if discretization preserves positivity than it also preserves positive uniform exponential stability. This is especially simple for linear systems as for any time scale with a bounded graininess necessary and sufficient condition for positive uniform exponential stability is the same. Though preservation of stability under discretization was studied before for a more general class of systems (not necessarily positive), the result of this paper cannot be deduced from these studies, since we exploit the specific structure of positive systems.

Finally, we study positive reachability and observability. These are very demanding properties for positive continuous-time systems, so discretization cannot spoil too much. Thus, a discretized system is positively reachable once it is positive and the original continuous-time system is positively reachable. Similarly, a discretized system is positively observable if it is positive and the original continuous-time system is positively observable.

Thus, the contribution of this paper consists of presenting conditions under which positivity, positive uniform exponential stability, positive reachability and positive observability are preserved during discretization. To our knowledge, these results are

new. Most of them are specific to positive systems. They do not have their counterparts for other classes of systems. For example, Proposition 12 states that any discretization of a continuous-time positive system that preserves positivity, preserves also positive uniform exponential stability. Preservation of stability under discretization has been studied by many authors, but not for positive systems. Also, the technique based on calculus on time scales is new for this type of problems. It allows for natural treatment of discretizations with variable step.

We rely here on our previous results concerning positivity, positive stability, positive reachability and positive observability of systems on various time scales [2–4,6–10]. We recall these results without proofs. They allow for easy switching from one time scale to another, so in particular we can easily derive conditions under which certain properties are preserved during discretization. Without these earlier results, the proofs would be much more complicated and might involve recovering some of the facts proved in these papers.

2 Preliminaries

Let us first fix notation and terminology. By $\mathbb{R}_+$, we denote the set of all nonnegative real numbers. Similarly, $\mathbb{R}_+^n$ and $\mathbb{R}_+^{n \times m}$ denote the sets of real column vectors and $n \times m$ matrices with nonnegative elements. Such a vector or matrix will be called *nonnegative*. A column or row vector is called *i-monomial* if its i th component is positive and the others are 0. It is called *monomial*, if it is *i-monomial* for some i . The *i-monomial* column vector whose i th component is equal 1 is denoted by e_i . An $n \times n$ real matrix is *monomial*, if all its rows and columns are monomial. Then such a matrix is invertible and its inverse is also monomial. If X is an arbitrary set, then a map $f : X \rightarrow \mathbb{R}_+^n$ is *i-monomial*, if for every $x \in X$ the vector $f(x)$ is *i-monomial*.

By a cone in $\mathbb{R}^n$, we mean a subset K of $\mathbb{R}^n$ such that if $x \in K$ then for every $\alpha \in \mathbb{R}_+$, $\alpha x \in K$. The set $\mathbb{R}_+^n$ is a cone, called the *nonnegative cone*. It has n faces. Each face is the intersections of $\mathbb{R}_+^n$ with some hyperplane $x_i = 0$, where $i \in \{1, \dots, n\}$. The faces are cones as well. For a nonempty subset I of $\{1, \dots, n\}$ let K_I mean the intersection of $\mathbb{R}_+^n$ with all the hyperplanes $x_i = 0$ with $i \notin I$. Each K_I is a cone. Occasionally, it will be called a subface of $\mathbb{R}_+^n$. If $I = \{i\}$, then K_I is a nonnegative half-axis $x_i \geq 0$. It will be shortly denoted by K_i . Let U be an open neighborhood of 0. We say that a vector field f on $\mathbb{R}^n$ is *tangent to $K_I \cap U$* if for all $x \in K_I \cap U$, $f_j(x) = 0$ for $j \notin I$. For $I = \{1, \dots, n\}$, $K_I = \mathbb{R}_+^n$.

By a *nonnegative neighborhood of 0 in $\mathbb{R}^n$* , we mean the intersection of some open neighborhood of 0 with $\mathbb{R}_+^n$.

Let U and V be open subsets of $\mathbb{R}^n$ and $f : U \rightarrow V$. The map f is a *diffeomorphism*, if it is bijective, differentiable and its inverse is also differentiable.

We need some basic information about calculus on time scales. More can be found in [12].

Definition 1 A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.

Example 1 The following sets are examples of time scales: $\mathbb{R}$, $h\mathbb{Z}$ for $h > 0$ and $q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ for $q > 1$.

A time scale is a topological space with the topology induced from $\mathbb{R}$.

Assumption 1 We assume that $\sup \mathbb{T} = +\infty$.

The *forward jump operator* $\sigma_{\mathbb{T}} : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma_{\mathbb{T}}(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the *graininess function* by $\mu_{\mathbb{T}}(t) := \sigma_{\mathbb{T}}(t) - t$.

Definition 2 $\mathbb{T}$ is a *homogeneous time scale*, if $\mu_{\mathbb{T}}$ is constant. $\mathbb{T}$ is a *discrete time scale*, if $\mu_{\mathbb{T}}(t) > 0$ for all $t \in \mathbb{T}$.

Observe that a homogeneous time scale satisfying Assumption 1 has one of the forms: $\mathbb{R}$, $[a, +\infty)$, $h\mathbb{Z}$ and $\{a + hk, k \in \mathbb{N}\}$ for some $a \in \mathbb{R}$ and $h > 0$.

If $t_0, t_1 \in \mathbb{T}$, then $(t_0, t_1)_{\mathbb{T}}$ denotes the intersection of the ordinary interval (t_0, t_1) with $\mathbb{T}$. Similarly for other types of intervals.

Definition 3 Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$. The *delta derivative of f* at t , denoted by $f^{\Delta}(t)$, is the real number with the property that given any $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ such that

$$|(f(\sigma_{\mathbb{T}}(t)) - f(s)) - f^{\Delta}(t)(\sigma_{\mathbb{T}}(t) - s)| \leq \varepsilon |\sigma_{\mathbb{T}}(t) - s|$$

for all $s \in U$.

Example 2 If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$.

If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}$.

If $\mathbb{T} = q^{\mathbb{N}}$, then $f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}$.

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Let $a, b \in \mathbb{T}$. Then, the *delta integral* of f on the interval $[a, b)_{\mathbb{T}}$ is defined by

$$\int_a^b f(\tau) \Delta\tau := \int_{[a,b)} f(\tau) \Delta\tau := F(b) - F(a).$$

It is more convenient to consider the half-open interval $[a, b)_{\mathbb{T}}$ than the closed interval $[a, b]_{\mathbb{T}}$ in the definition of the integral. If b is a left-dense point, then the value of f at b would not affect the integral. On the other hand, if b is left-scattered, the value of f at b is not essential for the integral (see Example 3). This is caused by the fact that we use delta integral, corresponding to the forward jump function.

It can be shown that every continuous function has an antiderivative. Moreover,

$$\int_a^{\sigma_{\mathbb{T}}(a)} f(\tau) \Delta\tau = f(a) \mu_{\mathbb{T}}(a).$$

Example 3 a) If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau) \Delta\tau = \int_a^b f(\tau) d\tau$, where the integral on the right is the usual Riemann integral.

b) If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau) \Delta\tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

Consider now a nonlinear control system on the time scale $\mathbb{T}$

$$x^\Delta(t) = f(x(t)) + \sum_{j=1}^m g^j(x(t))u_j(t), \quad (1)$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u_j(t) \in \mathbb{R}$ for $j = 1, \dots, m$, and f and g^j , $j = 1, \dots, m$, are maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will assume that the control $u = (u_1, \dots, u_m)^T$ is a piecewise constant function defined on $[0, T_u]_{\mathbb{T}}$ with values in $\mathbb{R}^m$, where T_u depends on u . See [6] for a precise definition of piecewise constant controls. For each initial point $x(0) = x_0 \in \mathbb{R}^n$, there exists a nonempty set of admissible controls, such that for each u in this set there exists a unique forward solution of (1), defined on $[0, T_u]_{\mathbb{T}}$. When we set $u = 0$, system (1) becomes a dynamical system on the time scale $\mathbb{T}$, given by

$$x^\Delta(t) = f(x(t)). \quad (2)$$

Let $G(x) = (g^1(x), \dots, g^m(x))$.

Remark 1 If $\mu_{\mathbb{T}} \equiv 0$, then (1) is a standard differential equation $\dot{x} = f(x) + G(x)u$.

If $\mu_{\mathbb{T}}(t) > 0$ for all $t \in \mathbb{T}$, then (1) may be rewritten as

$$x(t + \mu_{\mathbb{T}}(t)) = x(t) + \mu_{\mathbb{T}}(t)(f(x(t)) + G(x(t))u(t)).$$

This is a discrete-time system in the shift form, but, in general, with a nonconstant time shift. For $\mathbb{T} = \mathbb{Z}$, one gets the classical discrete-time system.

Proposition 1 ([12]) *For every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, there exists a unique forward solution (trajectory) $x : [t_0, T)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ of (2) for some $T > t_0$, $T \in \mathbb{T}$.*

Note that for discrete time scales backward trajectories of (2) may not exist.

Assumption 2 We assume that the forward trajectories of (2) are defined for all $t \geq t_0$, $t \in \mathbb{T}$.

Assumption 2 holds, for example, for linear systems $x^\Delta(t) = Ax(t)$, where A is a real $n \times n$ matrix.

Theorem 1 ([12]) *Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then, the linear system $x^\Delta(t) = Ax(t)$ with the initial condition $x(t_0) = x_0$ has a unique solution x defined on $[t_0, +\infty) \cap \mathbb{T}$.*

This result can be extended to matrix-valued solutions of the equation $X^\Delta(t) = AX(t)$, where $X(t)$ is an $n \times n$ matrix, which leads to the following definition.

Definition 4 Let $t_0 \in \mathbb{T}$. A function $X : [t_0, +\infty) \cap \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ that satisfies the matrix delta differential equation

$$X^\Delta(t) = AX(t) \quad (3)$$

and the initial condition $X(t_0) = I$, where I is the $n \times n$ identity matrix, is called *the matrix exponential function* (corresponding to A) *initialized at t_0* . Its value at $t \in \mathbb{T}$, $t \geq t_0$, is denoted by $e_A(t, t_0)$.

Then, the solution of the initial value problem

$$x^\Delta = Ax, \quad x(t_0) = x_0$$

can be written as

$$x(t) = e_A(t, t_0)x_0.$$

The exponential function e_A can also be used to express the forward solution to the linear control system $x^\Delta = Ax + Bu$, corresponding to the initial condition $x(t_0) = x_0$:

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^t e_A(t, \sigma_{\mathbb{T}}(\tau))Bu(\tau)d\tau,$$

where $t \in \mathbb{T}$, $t \geq t_0$ (see, e.g., [12]).

3 Discretization and positivity

We introduce here two main concepts of this paper: discretization and positivity, and study relations between them.

Definition 5 Let $\mathbb{T}$ be a discrete time scale. By $\mathbb{T}$ -discretization of the continuous-time system $\dot{x} = f(x) + G(x)u$, we mean the system

$$x^\Delta = f(x) + G(x)u$$

on the time scale $\mathbb{T}$, where x^Δ denotes the delta derivative of x on $\mathbb{T}$.

Thus, in $\mathbb{T}$ -discretization we replace $\dot{x}(t)$ by $\frac{x(t+\mu_{\mathbb{T}}(t)) - x(t)}{\mu_{\mathbb{T}}(t)}$. This corresponds to the classical Euler discretization, but with possibly variable step.

Later we will study a continuous-time system with output:

$$\dot{x} = f(x) + G(x)u, \quad y = h(x), \quad (4)$$

where the output y belongs to $\mathbb{R}^p$. As the output equation does not involve time derivative, the $\mathbb{T}$ -discretization of (4) will consist of $\mathbb{T}$ -discretization of the dynamic part together with the same output equation. On the other hand, setting $u = 0$ we get a dynamical system without control, so Definition 5 says also what is $\mathbb{T}$ -discretization of the system $x^\Delta = f(x)$.

Positivity will be defined for a control system with output on an arbitrary time scale $\mathbb{T}$.

Definition 6 The system

$$x^\Delta = f(x) + G(x)u, \quad y = h(x) \quad (5)$$

is *positive* if for all $t_0 \in \mathbb{T}$ the trajectory starting from any $x(t_0) = x_0 \in \mathbb{R}_+^n$ and corresponding to control $u(t) \in \mathbb{R}_+^m$ stays in $\mathbb{R}_+^n$ for all $t \geq t_0, t \in \mathbb{T}$, and $y(t) \in \mathbb{R}_+^p$, for all $t \geq t_0, t \in \mathbb{T}$.

Let f_i be the i th component of f and g_i^j be the i th component of g^j . The following characterization of positivity is an extension to systems on time scales of known characterizations for continuous-time and discrete-time systems

Proposition 2 ([5]) *The system (5) is positive if and only if the following conditions are satisfied:*

- (i) *for all $i = 1, \dots, n$, all $j = 1, \dots, m$, all $x \in \mathbb{R}_+^n$ and all $t \in \mathbb{T}$: $x_i + \mu_{\mathbb{T}}(t)f_i(x) \geq 0$ and $g_i^j(x) \geq 0$,*
- (ii) *if there is $t \in \mathbb{T}$ such that $\mu_{\mathbb{T}}(t) = 0$, then for all $i = 1, \dots, n$, all $j = 1, \dots, m$, and all $x \in \mathbb{R}_+^n$ such that $x_i = 0$, $f_i(x) \geq 0$ and $g_i^j(x) \geq 0$,*
- (iii) *for all $i = 1, \dots, p$ and for all $x \in \mathbb{R}_+^n$, $h_i(x) \geq 0$.*

For the linear system $x^\Delta = Ax + Bu, \quad y = Cx$, positivity may be characterized with the aid of the matrices A, B and C . An $n \times n$ matrix A is called a *Metzler matrix* if $a_{ij} \geq 0$ for $i \neq j$. For a Metzler matrix A , let $c(A) := \min\{a \geq 0 : A + aI \geq 0\}$.

Let $\bar{\mu}_{\mathbb{T}} := \sup\{\mu_{\mathbb{T}}(t) : t \in \mathbb{T}\}$.

Proposition 3 ([3, 18]) *The system $x^\Delta = Ax + Bu, \quad y = Cx$, is positive if and only if $B \in \mathbb{R}_+^{n \times m}$, $C \in \mathbb{R}_+^{p \times n}$, A is Metzler and $c(A) \leq 1/\bar{\mu}_{\mathbb{T}}$, where $1/0 := +\infty$ and $1/+\infty := 0$.*

Directly from the definition of discretization and Proposition 2, we get the following:

Proposition 4 *Let the system $\dot{x} = f(x) + G(x)u, \quad y = h(x)$, be positive and let $\mathbb{T}$ be a discrete time scale. $\mathbb{T}$ -discretization of this system is positive if and only if for every $t \in \mathbb{T}$ and for every $x \in \mathbb{R}_+^n$, $x + \mu_{\mathbb{T}}(t)f(x) \in \mathbb{R}_+^n$ and $g^j(x) \in \mathbb{R}_+^m$.*

Remark 2 It may happen that $\mathbb{T}$ -discretization of a positive system $\dot{x}(t) = f(x(t)) + \sum_{j=1}^m g^j(x(t))u_j(t)$ is not positive for any discrete time scale $\mathbb{T}$, even locally around 0. Consider, for example, the system: $\dot{x}(t) = -x(t)u(t)$, where $x(t) \in \mathbb{R}$. Then, the discretized system takes the form: $x(t + \mu_{\mathbb{T}}(t)) = x(t)(1 - \mu_{\mathbb{T}}(t)u(t))$. For $u(t)$ sufficiently large, the right-hand side becomes negative for positive $x(t)$.

For linear systems, we can express this using the matrices of the system. The following proposition is a simple consequence of Proposition 3.

Proposition 5 *Let the system $\dot{x} = Ax + Bu, \quad y = Cx$, be positive and let $\mathbb{T}$ be a discrete time scale. $\mathbb{T}$ -discretization of the system is positive if and only if $c(A) \leq 1/\bar{\mu}_{\mathbb{T}}$.*

Remark 3 If $\bar{\mu}_{\mathbb{T}}$ is sufficiently small, then $\mathbb{T}$ -discretization of a positive linear system $\dot{x} = Ax$ is again a positive system. This is not true for nonlinear systems. For example, the system $\dot{x} = -x^2$, where $x \in \mathbb{R}$, is positive, but its $\mathbb{T}$ -discretization given by $x(t + \mu(t)) = x - \mu(t)x^2$ is not positive for any discrete time scale $\mathbb{T}$. However, when we restrict a positive nonlinear system to a bounded neighborhood of 0 (e.g., a ball), there are discretizations that preserve positivity. Similarly, such discretizations can be found for the system $\dot{x} = f(x)$, with f globally Lipschitz on $\mathbb{R}_+^n$.

4 Positive stability and discretization

We study here positive uniform exponential stability of positive systems and its invariance under discretization.

Definition 7 Assume that system $x^\Delta = f(x)$ is positive. We say that this system is (positively) *uniformly exponentially stable* if there are constants $K \geq 1$ and $\alpha > 0$, and an open neighborhood V of 0 in $\mathbb{R}^n$ such that for every $t_0, t \in \mathbb{T}$ with $t \geq t_0$ and every $x_0 \in V$ ($x_0 \in \mathbb{R}_+^n \cap V$), the forward trajectory x of the system, corresponding to the initial condition $x(t_0) = x_0$, satisfies $\|x(t)\| \leq K \exp(-\alpha(t - t_0))\|x_0\|$ for all $t \in \mathbb{T}$ such that $t \geq t_0$.

If a positive system is uniformly exponentially stable, then it is positively uniformly exponentially stable. If the time scale is homogeneous, (positive) uniform exponential stability is equivalent to (positive) exponential stability, which is defined as (positive) uniform exponential stability, but the constant K may depend on the initial time t_0 .

Example 4 The positive system $\dot{x} = -|x|, x \in \mathbb{R}$, is positively uniformly exponentially stable, but it is not uniformly exponentially stable.

It is known that the condition $\sup\{\mu_{\mathbb{T}}(t) : t \in \mathbb{T}\} < +\infty$ is necessary for uniform exponential stability of the system $x^\Delta = f(x)$ (see, e.g., [11]). For example, on $\mathbb{T} = q^{\mathbb{N}}, q > 1$, there are no uniformly exponentially stable systems of the form $x^\Delta = f(x)$.

Therefore, in this section we shall assume that $\bar{\mu}_{\mathbb{T}} := \sup\{\mu_{\mathbb{T}}(t) : t \in \mathbb{T}\} < +\infty$. Below we recall basic results on positive stability that will be used later to show invariance under discretization.

For linear systems, uniform exponential stability and positive uniform exponential stability coincide.

Proposition 6 ([9]) *A positive system $x^\Delta = Ax$ is positively uniformly exponentially stable if and only if it is uniformly exponentially stable.*

For linear systems, there is a simple characterization of positive uniform exponential stability. It does not depend on the time scale as long as its graininess is bounded.

Proposition 7 ([10]) *Assume that $\bar{\mu}_{\mathbb{T}} < +\infty$. A positive system $x^\Delta = Ax$ is positively uniformly exponentially stable if and only if all the coefficients of the characteristic polynomial of A , $\chi_A(\lambda) = \det(\lambda I - A)$, are positive.*

Remark 4 This characterization has long been known for continuous-time systems ($\mathbb{T} = \mathbb{R}$) and in a similar form for discrete-time systems of the form $x(k+1) = Fx(k)$ as a consequence of Perron–Frobenius theorem (see, e.g., [20,25]). For the proof of this fact on an arbitrary time scale $\mathbb{T}$ (with $\bar{\mu}_{\mathbb{T}} < +\infty$), we had to rely on properties of so-called stability sets of time scales shown in [18].

First, we obtain invariance of positive stability under discretization for linear systems. Let $\mathbb{T}$ be a discrete time scale with $\bar{\mu}_{\mathbb{T}} < +\infty$.

Proposition 8 *If the system $\dot{x} = Ax$ is positive and (positively) uniformly exponentially stable, and its $\mathbb{T}$ -discretization $x^\Delta = Ax$ is positive, then $x^\Delta = Ax$ is also (positively) uniformly exponentially stable.*

Proof Since the system $\dot{x} = Ax$ is positive and positively uniformly exponentially stable, by Proposition 7, all the coefficients of the characteristic polynomial of A are positive. The $\mathbb{T}$ -discretization $x^\Delta = Ax$ is positive, so again from Proposition 7 $x^\Delta = Ax$ is positively uniformly exponentially stable. $\square$

Remark 5 Positive uniform exponential stability of a linear system $x^\Delta = Ax$ on a time scale $\mathbb{T}$ may be characterized by the spectrum of the matrix A . Namely, system $x^\Delta = Ax$ is positively uniformly exponentially stable if and only if the spectrum of the matrix A is contained in the stability set $\mathcal{S}_{\mathbb{T}}$, which depends on the time scale $\mathbb{T}$ (see [17,18,27] for the definition of $\mathcal{S}_{\mathbb{T}}$ and the proof of this fact). But since discretization changes the stability set, a proof of Proposition 8 relying on the spectral characterization of stability would be much more complicated.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be now of class C^1 and $A := f'(0)$ be the Jacobian matrix of f at 0. We assume that $f(0) = 0$. We need two important facts concerning nonlinear systems.

Proposition 9 ([9]) *If the system $\dot{x}^\Delta = f(x)$ is positive, then the system $x^\Delta = Ax$ is also positive.*

Proposition 10 ([9]) *If the system $x^\Delta = Ax$ is positively uniformly exponentially stable, then also the system $\dot{x}^\Delta = f(x)$ is positively uniformly exponentially stable.*

Proposition 10 may be reversed if the time scale is homogeneous.

Proposition 11 *Assume that the time scale $\mathbb{T}$ is homogeneous. If the system $\dot{x}^\Delta = f(x)$ is positively uniformly exponentially stable, then the system $x^\Delta = Ax$ is positively uniformly exponentially stable.*

Proof For $\mathbb{T} = \mathbb{R}$, it has been shown in [28]. The proof for $[a, +\infty)$, where $a \in \mathbb{R}$, is the same. For $\mathbb{T} = \mathbb{Z}$ this was shown in [22]. The proof for $\mathbb{T} = \mathbb{Z}$ may be easily adapted to the cases $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = a + h\mathbb{N}$, where $h > 0$ and $a \in \mathbb{R}$. $\square$

It is not known whether Proposition 11 holds for other time scales.

Let $\mathbb{T}$ be a discrete time scale with $\bar{\mu}_{\mathbb{T}} < +\infty$. The following result is an extension of Proposition 8.

Proposition 12 *If the system $\dot{x} = f(x)$ is positive and positively uniformly exponentially stable, and its $\mathbb{T}$ -discretization $x^\Delta = f(x)$ is positive, then $x^\Delta = f(x)$ is also positively uniformly exponentially stable.*

Proof Let $A = f'(0)$. From Proposition 9, the linearized system $\dot{x} = Ax$ is positive and from Proposition 11 it is positively uniformly exponentially stable. Now Proposition 10 implies that the nonlinear system $x^\Delta = f(x)$ is positively uniformly exponentially stable.

Example 5 Let us consider the continuous-time system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + x_1x_2.\end{aligned}$$

From Proposition 2, we immediately get that the system is positive. Its linearization at $x = 0$ is uniformly positively exponentially stable, so, by Proposition 10, the nonlinear system is uniformly positively exponentially stable as well. Let $\mathbb{T}$ be a discrete time scale. Then, the $\mathbb{T}$ -discretization of the original system may be written as

$$\begin{aligned}x_1(t + \mu_{\mathbb{T}}(t)) &= x_1(t)(1 - \mu_{\mathbb{T}}(t)) + \mu_{\mathbb{T}}(t)x_2(t)^2 \\ x_2(t + \mu_{\mathbb{T}}(t)) &= x_2(t)(1 - \mu_{\mathbb{T}}(t)) + \mu_{\mathbb{T}}(t)x_1(t)x_2(t).\end{aligned}$$

From Proposition 2, it easily follows that the discretized system is positive if and only if $\mu_{\mathbb{T}}(t) \leq 1$ for all $t \in \mathbb{T}$. Let $\mathbb{T}$ consists of $t_k = \sum_{i=1}^k 1/i$, where $k \in \mathbb{N}$. It is a nonhomogeneous time scale with $\mu_{\mathbb{T}}(t_k) = 1/(k+1)$, so the graininess is decreasing and is bounded from above by $1/2$. Thus this $\mathbb{T}$ -discretization gives a positive system. By Proposition 12, the discretized system is uniformly positively exponentially stable.

5 Positive reachability and discretization

To study the influence of discretization on positive reachability, we go back to control systems. Let us suppose that we discretize a continuous-time positive system and the discretized system is also positive. We want to know if positive reachability is preserved as well. Let us first consider the case of linear systems.

Definition 8 A positive system

$$x^\Delta = Ax + Bu \tag{6}$$

is *positively reachable (from 0)* if for any $\bar{x} \in \mathbb{R}_+^n$ there is $u : [T_0^u, T_1^u]_{\mathbb{T}} \rightarrow \mathbb{R}_+^m$ such that the trajectory x of (6) starting from 0 at time T_0^u and corresponding to the control u satisfies $x(T_1^u) = \bar{x}$.

For continuous-time positive systems, positive reachability is very demanding property.

Proposition 13 ([15]) *Let $\mathbb{T} = \mathbb{R}$. A positive system (6) is positively reachable if and only if $m \geq n$, A is diagonal and B contains an $n \times n$ monomial submatrix.*

For an arbitrary time scale, the characterization is more complicated.

Definition 9 ([3]) Let $M \subseteq \{1, \dots, m\}$ and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. For each $k \in M$ let S_k be a subset of $[t_0, t_1]_{\mathbb{T}}$ that is a union of finitely many disjoint intervals of $\mathbb{T}$ of the form $[\tau_0, \tau_1]_{\mathbb{T}}$, and let $S_M = \{S_k : k \in M\}$. By the *subGramian of system (6) corresponding to t_0, t_1, M and S_M* we mean the matrix

$$W := W_{t_0}^{t_1}(M, S_M) := \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k b_k^T e_A(t_1, \sigma(\tau))^T \Delta \tau. \quad (7)$$

Then, we have the following characterization:

Proposition 14 ([3]) *System (6) is positively reachable if and only if there are $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$, $M \subseteq \{1, \dots, m\}$ and the family $S_M = \{S_k : k \in M\}$ of subsets of $[t_0, t_1]_{\mathbb{T}}$ such that the subGramian $W = W_{t_0}^{t_1}(M, S_M)$ is monomial.*

Using these facts, we can show the following:

Theorem 2 *If the positive continuous-time system $\dot{x} = Ax + Bu$ is positively reachable and its $\mathbb{T}$ -discretization (6) is positive, then (6) is also positively reachable.*

Proof Assume that $\dot{x} = Ax + Bu$ is positive and positively reachable. Then, by Proposition 13, $m \geq n$, A is diagonal and B contains an $n \times n$ monomial submatrix $\tilde{B}$. This implies that the matrix $e_A(t_1, \sigma(\tau))$ is diagonal for every $\tau < t_1$, $t_1, \tau \in \mathbb{T}$. Choose any $t_0, t_1 \in \mathbb{T}$ such that $t_0 < t_1$ and let M mean the set of indices that correspond to columns of $\tilde{B}$. Set $S_k = [t_0, t_1]$ for every $k \in M$. If $b_k = \alpha e_i$, then $e_A(t_1, \sigma(\tau)) b_k = \gamma(\tau) e_i$ for some scalar function γ , so $\int_{S_k} e_A(t_1, \sigma(\tau)) b_k b_k^T e_A(t_1, \sigma(\tau))^T \Delta \tau$ is a diagonal matrix with the only nonzero entry at i th place on the diagonal. Since each b_k , where $k \in M$, corresponds to different e_i , the subGramian is a diagonal matrix with all nonzero entries on the diagonal. Thus $W_{t_0}^{t_1}(M, S_M)$ is monomial, so the discretized system is positively reachable by Proposition 14. $\square$

Let us now switch to nonlinear systems. The definition of positive reachability from 0 is exactly the same as for linear systems (Definition 8). We shall also need local positive reachability from 0. This property means that we are able to reach from 0 all the states from $\mathbb{R}_+^n \cap U$, where U is some open neighborhood of 0. The key fact here is a characterization of local positive reachability of a positive system of form (1) for $\mathbb{T} = \mathbb{R}$, obtained in [6].

Proposition 15 ([6]) *Assume that $f, g^1, \dots, g^m$ in (1) are analytic, $f(0) = 0$ and $\mathbb{T} = \mathbb{R}$. A positive system (1) is locally positively reachable from 0 if and only if $m \geq n$ and there is a positive neighborhood V of 0 such that for every $I \subset \{1, \dots, n\}$, f is tangent to $K_I \cap V$ and there is $i \in \{1, \dots, m\}$ such that g^i is tangent to $K_I \cap V$ and nonzero at every point of $K_I \cap V$.*

The condition for local positive reachability from 0 for a positive continuous-time system is very strong. In particular, it is required that the number of control variables be greater than or equal to the number of state variables. But steering a positive continuous-time system from a point different from 0 is even harder. See [8] for a detailed study of this problem.

Theorem 3 *Let $\mathbb{T}$ be a discrete time scale. If the continuous-time system $\dot{x} = f(x) + \sum_{j=1}^m g^j(x)u_j$ is positive and locally positively reachable from 0, and its $\mathbb{T}$ -discretization $x^\Delta = f(x) + \sum_{j=1}^m g^j(x)u_j$ is positive, then this $\mathbb{T}$ -discretization is positively reachable from 0.*

Proof Local positive reachability from 0 of the continuous-time system implies that $m \geq n$. Moreover, for every $i = 1, \dots, n$ there is g^{j_i} such $g^{j_i}(0) = \alpha_i e_i$ for $\alpha_i > 0$. Let us choose any $t_0 \in \mathbb{T}$ and let $t_1 = \sigma_{\mathbb{T}}(t_0)$. We shall show that any $\bar{x} \in \mathbb{R}_+^n$ can be reached at time t_1 starting at time t_0 from 0. Observe that the trajectory of $x^\Delta = f(x) + \sum_{j=1}^m g^j(x)u_j$ satisfies

$$x(t_1) = x(\sigma_{\mathbb{T}}(t_0)) = x(t_0) + \mu_{\mathbb{T}}(t_0)f(x(t_0)) + \sum_{j=1}^m \mu_{\mathbb{T}}(t_0)g^j(x(t_0))u_j(t_0).$$

Since we start at time t_0 from 0 and $f(0) = 0$, this gives

$$x(t_1) = \sum_{j=1}^m \mu_{\mathbb{T}}(t_0)g^j(0)u_j(t_0).$$

Let $M = \{j_i : i = 1, \dots, n\}$ and set $u_j(t_0) = 0$ for $j \notin M$. Then,

$$x(t_1) = \sum_{i=1}^n \mu_{\mathbb{T}}(t_0)\alpha_i u_{j_i}(t_0)e_i.$$

Since every $\bar{x} \in \mathbb{R}_+^n$ is a linear combination of $e_1, \dots, e_n$ with nonnegative coefficients, choosing appropriate $u_{j_i}(t_0) \geq 0$ we are able to reach any $\bar{x} \in \mathbb{R}_+^n$ in one step. $\square$

6 Positive observability and discretization

Now we add an observation part to the system and at the same time remove controls:

$$x^\Delta(t) = f(x(t)), \quad y(t) = h(x(t)). \quad (8)$$

As before $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. We assume in this section that f and h are analytic maps.

Definition 10 System (8) is *locally positively observable* at x_0 if there is an open neighborhood U of x_0 , $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$, and a continuous map $\Phi : [t_0, t_1]_{\mathbb{T}} \times V \rightarrow \mathbb{R}_+^n$ for some open set $V \subseteq \mathbb{R}_+^p$ such that for any $\bar{x} \in \mathbb{R}_+^n \cap U$, $\bar{x} = \int_{t_0}^{t_1} \Phi(t, h(x(t, t_0, \bar{x}))) \Delta t$.

We can characterize local positive observability for continuous-time systems.

Proposition 16 ([7]) *Let $\mathbb{T} = \mathbb{R}$. System (8) is locally positively observable at 0 if and only if $p \geq n$ and there is an open neighborhood U of 0 such that f is tangent to every subface of $\mathbb{R}_+^n$ restricted to U , and there are indices $1 \leq j_1 < j_2 < \dots < j_n \leq p$, such that the map $\tilde{h} := (h_{j_1}, \dots, h_{j_n})^T : U \rightarrow \tilde{h}(U)$ is a diffeomorphism and for every $k \geq 0$, $\tilde{h}(S_k \cap U) = S_k \cap \tilde{h}(U)$.*

The condition for local positive observability at 0 for a positive continuous-time system is very strong, similarly as the condition for local positive reachability from 0. In particular, it is required that the number of output variables be greater than or equal to the number of state variables. These restrictive conditions imply that the discretized system is likely to be locally positively observable at 0 as well.

Theorem 4 *Let $\mathbb{T}$ be a discrete time scale. If the continuous-time system $\dot{x} = f(x)$, $y = h(x)$, is positive and locally positively observable at 0, and its $\mathbb{T}$ -discretization $x^\Delta = f(x)$, $y = h(x)$, is positive, then this $\mathbb{T}$ -discretization is locally positively observable at 0.*

Proof Local positive observability at 0 of the continuous-time system implies that $p \geq n$. Without loss of generality, we can assume that $p = n$. Proposition 16 implies that h is a diffeomorphism of some open neighborhood U of 0 onto $h(U)$. Since $h(U \cap \mathbb{R}_+^n) \subseteq \mathbb{R}_+^n$ and $h(S_k \cap U) = S_k \cap h(U)$, we get that $h(U \cap \mathbb{R}_+^n) = V \cap \mathbb{R}_+^n$ for some open neighborhood V of 0. This implies that h^{-1} is positive on V , i.e., $h^{-1}(y) \in \mathbb{R}_+^n$ for $y \in V \cap \mathbb{R}_+^n$. To show that $\mathbb{T}$ -discretization of the continuous-time system is locally positively observable at 0, choose any $t_0 \in \mathbb{T}$ and set $t_1 := \sigma_{\mathbb{T}}(t_0)$. Then, define $\Phi(t_0, y) := h^{-1}(y)/\mu_{\mathbb{T}}(t_0)$ for $y \in V$. Observe that $\Phi(t_0, y) \in \mathbb{R}_+^n$. Moreover, for $\bar{x} \in U$

$$\begin{aligned} \int_{t_0}^{t_1} \Phi(t, h(x(t, t_0, \bar{x}))) \Delta t &= \Phi(t_0, h(x(t_0, t_0, \bar{x}))) \mu_{\mathbb{T}}(t_0) \\ &= h^{-1}(h(\bar{x})) = \bar{x}. \end{aligned}$$

This shows that the discretized system is locally positively observable at 0. □

Remark 6 Theorem 4 holds in particular for a linear positive system $\dot{x} = Ax$, $y = Cx$, but now we can skip the word “local.” Recall that such a system is positively observable if and only if $p \geq n$, C contains an $n \times n$ monomial submatrix and A is diagonal (see [2,7,25]). This is as strong condition as the one for nonlinear systems. And again, it implies positive observability of any positive $\mathbb{T}$ -discretization of the continuous-time system.

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