



Rainbow Cycles in Properly Edge-Colored Graphs

Jaehoon Kim¹ · Joonkyung Lee² · Hong Liu³ · Tuan Tran⁴

Received: 6 November 2022 / Revised: 25 January 2024 / Accepted: 25 February 2024
© The Author(s) 2024

Abstract

We prove that every properly edge-colored n -vertex graph with average degree at least $32(\log 5n)^2$ contains a rainbow cycle, improving upon the $(\log n)^{2+o(1)}$ bound due to Tomon. We also prove that every properly edge-colored n -vertex graph with at least $10^5 k^3 n^{1+1/k}$ edges contains a rainbow $2k$ -cycle, which improves the previous bound $2ck^2 n^{1+1/k}$ obtained by Janzer. Our method using homomorphism inequalities and a lopsided regularization lemma also provides a simple way to prove the Erdős–Simonovits supersaturation theorem for even cycles, which may be of independent interest.

Keywords Rainbow cycles · Turan problem · Extremal number

J. Kim: Supported by the POSCO Science Fellowship of POSCO TJ Park Foundation and by the National Research Foundation of Korea (NRF) grant (MSIT) No. RS-2023-00210430.

J. Lee: Supported by Samsung STF Grant SSTF-BA2201-02 and the National Research Foundation of Korea (NRF) grant MSIT NRF-2022R1C1C1010300.

H. Liu: Supported by IBS-R029-C4.

T. Tran: Supported by National Key Research and Development Program of China 2023YFA1010201, and the Excellent Young Talents Program (Overseas) of the National Natural Science Foundation of China.

✉ Jaehoon Kim
jaehoon.kim@kaist.ac.kr

Joonkyung Lee
joonkyunglee@yonsei.ac.kr

Hong Liu
hongliu@ibs.re.kr

Tuan Tran
trantuan@ustc.edu.cn

¹ Department of Mathematical Sciences, KAIST, Daejeon, South Korea

² Department of Mathematics, Yonsei University, Seoul, South Korea

³ Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS), Daejeon, South Korea

⁴ School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, China

1 Introduction

In an edge-colored graph, a copy of a graph H is called *rainbow* if every edge in the copy receives a unique color. If we forbid rainbow copies of some graphs in properly edge-colored graphs G , what is the maximum number of edges in G ? This extremal question was first investigated by Keevash, Mubayi, Sudakov, and Verstraëte [8], where they defined the *rainbow Turán number* $\text{ex}^*(n, \mathcal{H})$ for a family of graphs \mathcal{H} . Formally, $\text{ex}^*(n, \mathcal{H})$ denotes the maximum number of edges in a properly edge-colored n -vertex graph with no rainbow copies of any $H \in \mathcal{H}$. If \mathcal{H} consists of a single graph H , then we simply write $\text{ex}^*(n, H)$ instead of $\text{ex}^*(n, \{H\})$.

To quote [8], there are two questions that are the most important among the several ones raised therein. The first one is to determine $\text{ex}^*(n, \mathcal{C})$, where \mathcal{C} is the class of all cycles. It is shown that $\text{ex}^*(n, \mathcal{C}) = \Omega(n \log n)$ in [8] and Das, Lee, and Sudakov [3] obtained an upper bound $O(ne^{(\log n)^{\frac{1}{2}+o(1)}})$. This was improved by Janzer [5] to $O(n \log^4 n)$ and the current best one is $O(n(\log n)^{2+o(1)})$ due to Tomon [9]. We improve Tomon's bound to $O(n \log^2 n)$.

Theorem 1.1 *A properly edge-colored n -vertex graph G with at least $32n \log^2(5n)$ edges always contains a rainbow cycle.*

The second question in [8] concerns $\text{ex}^*(n, C_{2k})$, where C_{2k} is the even cycle of length $2k$. In [8], a general lower bound $\text{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k})$ is obtained, whereas the matching upper bounds were only verified for $k = 2, 3$. This upper bound was subsequently improved by Das, Lee, and Sudakov [3] to $O(n^{1+(1+o_k(1)) \log k/k})$ and by Janzer [5] to $O(n^{1+1/k})$. While Janzer's bound matches the lower bound given in [8], the implicit constant is exponential in k . We improve it to a polynomial one as follows.

Theorem 1.2 *A properly edge-colored n -vertex graph G with at least $10^5 k^3 n^{1+1/k}$ edges always contains a rainbow $2k$ -cycle.*

Whereas the recent improvement in [9] took a different approach to that of Janzer [5], our proof is in spirit closer to that of [5] using homomorphism counts and improves it in two ways. First, we use a more efficient lopsided regularization lemma than the Jiang–Seiver lemma [7] used by Janzer. Second, the main proof after regularization is conceptually simpler and more intuitive in the sense that it only relies on repeated applications of the Cauchy–Schwarz inequality.

We stress that the repeated Cauchy–Schwarz method may be of independent interest. There are various problems in extremal combinatorics, from the classical Mantel's theorem and the Kővári–Sós–Turán theorem to the recent developments on Sidorenko's conjecture, where the Cauchy–Schwarz inequality and its variants have been extremely useful; however, the convexity inequalities have seen less success in determining extremal numbers of bipartite graphs other than complete bipartite graphs.

To the best of our knowledge, it has been unknown whether even the Bondy–Simonovits theorem [2], a weaker statement than Theorem 1.2, has a proof that only uses the Cauchy–Schwarz inequality. Our proof method answers this natural question by giving a simple proof of the Bondy–Simonovits theorem and moreover, obtains the Erdős–Simonovits supersaturation for even cycles [4] as follows.

Corollary 1.3 Any n -vertex graph G with average degree $d \geq 2 \cdot 10^5 k^3 n^{1/k}$ contains at least $\frac{1}{2}(2^{12}k)^{-k} d^{2k}$ copies of the $2k$ -cycle.

2 The Key Homomorphism Inequality

In what follows, a *coloring* always means an edge coloring and likewise, a *colored graph* is an edge-colored graph. Let $\text{Hom}(H, G)$ be the set of homomorphisms from H to G . In particular, we fully label vertices of H .

Each homomorphism in $\text{Hom}(C_{2k}, G)$ can be seen as a closed walk $v_0 v_1 \dots v_{2k}$ of length $2k$ (or closed $2k$ -walk for short) in G with $v_0 = v_{2k}$. Let ϕ be a proper coloring of a graph G . A closed $2k$ -walk is *degenerate* if, for distinct i and j with $\{i, j\} \neq \{0, 2k\}$, $v_i = v_j$ or $\phi(v_i v_{i+1}) = \phi(v_j v_{j+1})$, where the index addition is taken modulo $2k$. That is, the walk revisits a vertex in the middle or repeats a color. The former case is said to be *vertex-degenerate at (i, j)* and the latter is called *color-degenerate at (i, j)* . If a $2k$ -walk is vertex-degenerate (resp. color-degenerate) at (i, j) , then it is of type $|i - j| - 1$ (resp. $|i - j|$).

One degenerate walk may have multiple types, so the types of degeneracy are not disjoint in general. As ϕ is a proper coloring, a closed walk is vertex-degenerate of type 1 if and only if it is color-degenerate of type 1. Note that a non-degenerate $2k$ -walk is a rainbow cycle of length $2k$. Let $\mathcal{D}(C_{2k}, G)$ be the set of all degenerate closed walks of length $2k$ and let S_k be the star with k leaves. If G has no rainbow $2k$ -cycle, then $\text{Hom}(C_{2k}, G) = \mathcal{D}(C_{2k}, G)$. Our strategy is to bound $|\mathcal{D}(C_{2k}, G)|$ from above to obtain an upper bound for $|\text{Hom}(C_{2k}, G)|$.

Lemma 2.1 Given a colored graph G , let \mathcal{U}_1 be the set of closed $2k$ -walks in G that are vertex-degenerate at $(0, 2)$. Then

$$|\text{Hom}(C_{2k}, G)| \leq \left(\frac{|\text{Hom}(C_{2k}, G)|}{|\mathcal{U}_1|} \right)^k \cdot |\text{Hom}(S_k, G)| \quad \text{and} \quad |\mathcal{D}(C_{2k}, G)| \leq 4k^2 |\mathcal{U}_1|.$$

In particular, if G has no rainbow $2k$ -cycle, then $|\text{Hom}(C_{2k}, G)| \leq (2k)^{2k} |\text{Hom}(S_k, G)|$.

Proof Let \mathcal{U}_s and \mathcal{F}_s be the set of closed $2k$ -walks that are vertex-degenerate at $(0, s+1)$ and color-degenerate at $(0, s)$, respectively, i.e., they consist of those walks of type s . Note that $\mathcal{U}_1 = \mathcal{F}_1$. □

Claim 2.2 For $1 \leq s \leq k - 1$ and $1 \leq t \leq k$,

$$|\mathcal{U}_s|^2 \leq |\mathcal{U}_1| \cdot |\mathcal{U}_{2s-1}| \quad \text{and} \quad |\mathcal{F}_t|^2 \leq |\mathcal{F}_1| \cdot |\mathcal{F}_{2t-1}|. \tag{1}$$

In particular, $|\mathcal{U}_s| \leq |\mathcal{U}_1|$ and $|\mathcal{F}_t| \leq |\mathcal{F}_1|$ for all $1 \leq s \leq 2k - 3$ and $1 \leq t \leq 2k - 1$.

Proof of the claim For $s = 1$, the inequality becomes a trivial equality, so we may assume $s > 1$. The key idea is to analyze \mathcal{U}_s by counting the closed walks therein in the following way:

- fix vertices $x = v_0 = v_{s+1}$, $y = v_{s+k}$, and $z = v_s$ with $xz \in E(G)$;
- choose a walk $v_{s+1}v_{s+2} \dots v_{s+k}$ of length $k - 1$ from x to y ;
- choose a walk $v_{s+k}v_{s+k+1} \dots v_{2k}$ of length $k - s$ from y to x ; and
- choose a walk $v_0v_1 \dots v_s$ of length s from x to z .

Let $w_\ell(u, v)$ be the number of walks of length ℓ from u to v and let $g(u, v)$ be the edge indicator function of G , i.e., $g(u, v) = 1$ if $uv \in E(G)$ and 0 otherwise. Then

$$|\mathcal{U}_s| = \sum_{x,y,z} w_{k-1}(x, y)w_{k-s}(y, x)w_s(x, z)g(x, z).$$

The Cauchy–Schwarz inequality now gives the following bound:

$$\begin{aligned} |\mathcal{U}_s|^2 &\leq \left(\sum_{x,y,z} w_{k-1}(x, y)^2 g(x, z) \right) \left(\sum_{x,y,z} w_{k-s}(y, x)^2 w_s(x, z)^2 g(x, z) \right) \\ &\leq \left(\sum_{x,y,z} w_{k-1}(x, y)^2 g(x, z) \right) \left(\sum_{x,y,z} w_{k-s}(y, x)^2 w_s(x, z)^2 \right), \end{aligned}$$

where the second inequality follows from $g(x, y) \leq 1$. The sum $\sum_{x,y,z} w_{k-1}(x, y)^2 g(x, z)$ counts the number of closed $(2k - 2)$ -walks plus a pendant edge that corresponds to xz , which are exactly the $2k$ -walks in \mathcal{U}_1 . Thus, $\sum_{x,y,z} w_{k-1}(x, y)^2 g(x, z) = |\mathcal{U}_1|$. Analogously, the term $w_{k-s}(y, x)^2$ and $w_s(x, z)^2$ count the number of closed walks of length $2(k - s)$ and $2s$ when summed over the choices of y and z , respectively, both starting at x . Thus, summing $w_{k-s}(y, x)^2 w_s(x, z)^2$ over $x, y, z \in V(G)$ counts the number of closed $2k$ -walks that are vertex-degenerate at $(0, 2s)$, which is exactly $|\mathcal{U}_{2s-1}|$. Therefore, $|\mathcal{U}_s|^2 \leq |\mathcal{U}_1| \cdot |\mathcal{U}_{2s-1}|$.

Let \mathcal{U}_j be the largest set among $\mathcal{U}_1, \dots, \mathcal{U}_{2k-3}$. By symmetry, $|\mathcal{U}_j| = |\mathcal{U}_{2k-2-j}|$, which allows us to assume $j \leq k - 1$. Then $|\mathcal{U}_j| \leq |\mathcal{U}_1|^{1/2} |\mathcal{U}_{2j-1}|^{1/2} \leq |\mathcal{U}_1|^{1/2} |\mathcal{U}_j|^{1/2}$, so $|\mathcal{U}_j| = |\mathcal{U}_1|$. This proves $|\mathcal{U}_s| \leq |\mathcal{U}_1|$ for all $s = 1, 2, \dots, 2k - 3$.

The second inequality for \mathcal{F}_t can also be obtained by using essentially the same technique. As each closed $2k$ -walk $v_0 \dots v_{2k}$ in \mathcal{F}_t satisfies $\phi(v_0v_1) = \phi(v_tv_{t+1})$, we count $|\mathcal{F}_t|$ as follows:

- fix a color c that repeats at v_0v_1 and v_tv_{t+1} ;
- fix vertices $x = v_0 = v_{2k}$ and $y = v_k$;
- choose a walk $v_0v_1 \dots v_k$ of length k from x to y where $\phi(v_0v_1) = c$; and
- choose a walk $v_kv_{k+1} \dots v_{2k}$ of length k from y to x where $\phi(v_tv_{t+1}) = c$.

Let $\tilde{w}_{k,\ell}(u, v, c)$ be the number of k -walks from u to v such that the ℓ -th edge has color c . The Cauchy–Schwarz inequality then gives

$$|\mathcal{F}_t|^2 = \left(\sum_{x,y,c} \tilde{w}_{k,1}(x, y, c) \tilde{w}_{k,t-k+1}(y, x, c) \right)^2$$

$$\leq \left(\sum_{x,y,c} \tilde{w}_{k,1}(x, y, c)^2 \right) \left(\sum_{x,y,c} \tilde{w}_{k,t-k+1}(y, x, c)^2 \right).$$

The sum $\sum_{x,y,c} \tilde{w}_{k,1}(x, y, c)^2$ counts the number of closed $2k$ -walks from x that repeat c at the first and the last edges, which is exactly $|\mathcal{F}_1|$ by rotational symmetry. Similarly, the second sum corresponds to $|\mathcal{F}_{2t-2k+1}| = |\mathcal{F}_{2t-1}|$. Therefore, we obtain the inequality $|\mathcal{F}_t|^2 \leq |\mathcal{F}_1| \cdot |\mathcal{F}_{2t-1}|$. Finally, $|\mathcal{F}_t| \leq |\mathcal{F}_1|$ follows from the symmetry that yields $|\mathcal{F}_t| = |\mathcal{F}_{2k-t}|$ and verbatim the same argument used for showing $|\mathcal{U}_s| \leq |\mathcal{U}_1|$. \square

Claim 2.3 $|\mathcal{D}(C_{2k}, G)| \leq 4k^2|\mathcal{U}_1|$.

Proof of the claim Recall that both \mathcal{U}_s and \mathcal{F}_s specify the labels of the vertices where degeneracy occurs. By rotational symmetry, the number of closed $2k$ -walks that are vertex-degenerate at $(i, i + s + 1)$ is exactly $|\mathcal{U}_s|$ for each $i = 0, 1, \dots, 2k - 1$. Likewise, the number of closed $2k$ -walks that are color-degenerate at $(i, i + s)$ is $|\mathcal{F}_s|$ for each $i = 1, 2, \dots, 2k$. Thus, the number of degenerate C_{2k} -homomorphisms of type s is at most $2k(|\mathcal{U}_s| + |\mathcal{F}_s|)$. Taking the union bound over all possible types, we get

$$|\mathcal{D}(C_{2k}, G)| \leq 2k \left(\sum_{s=1}^{k-1} |\mathcal{U}_s| + \sum_{s=1}^k |\mathcal{F}_s| \right). \tag{2}$$

Together with Claim 2.2 this yields $|\mathcal{D}(C_{2k}, G)| \leq 4k^2|\mathcal{U}_1|$ as desired. \square

Let \mathcal{O}_s be the set of all closed $2k$ -walks $v_0 \dots v_{2k}$ with $v_0 = v_{2k}$ such that $v_0 = v_2 = \dots = v_{2s}$. In particular, $\mathcal{O}_1 = \mathcal{U}_1 = \mathcal{F}_1$ and $\mathcal{O}_k = \mathcal{O}_{k-1} \subseteq \mathcal{O}_{k-2} \subseteq \dots \subseteq \mathcal{O}_1$. We also use $\mathcal{O}_0 = \text{Hom}(C_{2k}, G)$ for consistency.

Claim 2.4 The sequence $|\mathcal{O}_s|, 0 \leq s \leq k$, is log-convex, i.e., for each $s = 1, \dots, k - 1$,

$$|\mathcal{O}_s|^2 \leq |\mathcal{O}_{s-1}| \cdot |\mathcal{O}_{s+1}|.$$

Proof of the claim A star-walk of length ℓ is a walk $u_0 u_1 \dots u_\ell$ of length ℓ such that every even-indexed vertex is the same one, i.e., $u_0 = u_2 = \dots = u_{2t}$ where $t = \lfloor \frac{\ell}{2} \rfloor$. If $s = k - 1$, then the inequality $|\mathcal{O}_{k-1}|^2 \leq |\mathcal{O}_{k-2}| \cdot |\mathcal{O}_k|$ trivially follows from the fact $\mathcal{O}_k = \mathcal{O}_{k-1}$ and $\mathcal{O}_{k-1} \subseteq \mathcal{O}_{k-2}$.

For $1 \leq s \leq k - 2$, the walks in \mathcal{O}_s can be counted as follows:

- fix vertices $x = v_0 = v_2 = \dots = v_{2s}, y = v_{s+1}, z = v_{k+s+1}$, where y is either x or a neighbor of x depending on the parity of s ;
- choose a walk $v_{k+s+1} v_{k+s+2} \dots v_{2k}$ of length $k - s - 1$ from z to x ;
- choose a star-walk $v_0 v_1 \dots v_{s+1}$ of length $s + 1$ from x to y ;
- choose a walk $v_{k+s+1} v_{k+s} \dots v_{2s}$ of length $k - s + 1$ from z to x ; and
- choose a star-walk $v_{2s} v_{2s-1} \dots v_{s+1}$ of length $s - 1$ from x to y .

As in the proof of Claim 2.2, $w_\ell(u, v)$ denotes the number of ℓ -walks from u to v . Let $\sigma_\ell(u, v)$ be the number of star-walks of length ℓ from u to v . Note that, unlike $w_\ell(\cdot, \cdot)$, $\sigma_\ell(\cdot, \cdot)$ may not be a symmetric function in general. Now we can compute $|\mathcal{O}_s|$ as

$$|\mathcal{O}_s| = \sum_{x,y,z} w_{k-s-1}(z, x)\sigma_{s+1}(x, y)w_{k-s+1}(z, x)\sigma_{s-1}(x, y).$$

The Cauchy–Schwarz inequality then yields

$$|\mathcal{O}_s|^2 \leq \left(\sum_{x,y,z} w_{k-s-1}(z, x)^2\sigma_{s+1}(x, y)^2 \right) \left(\sum_{x,y,z} w_{k-s+1}(z, x)^2\sigma_{s-1}(x, y)^2 \right).$$

In the first sum, $\sigma_{s+1}(x, y)^2$ counts either a closed star-walk of length $2(s + 1)$ (if s is odd and $x = y$) from x or a star-walk of length $2s$ together with an edge xy (if s is even and $xy \in E(G)$). By considering xy and yx as the $(s + 1)$ -th and $(s + 2)$ -th edge of the star-walk, summing the latter case over all choices of y counts the closed star-walks of length $2(s + 1)$. Hence the first sum counts the number of walks in \mathcal{O}_{s+1} by summing the number of ways to augment each closed walk of length $2(k - s - 1)$ by a closed star-walk of length $2(s + 1)$ at x . By the same reason with replacing $s + 1$ by $s - 1$, the second sum counts the number of walks in \mathcal{O}_{s-1} , which proves the claim. \square

We are now ready to finish the proof of the lemma. By Claim 2.4,

$$\frac{|\text{Hom}(C_{2k}, G)|}{|\mathcal{U}_1|} = \frac{|\mathcal{O}_0|}{|\mathcal{O}_1|} \geq \frac{|\mathcal{O}_1|}{|\mathcal{O}_2|} \geq \dots \geq \frac{|\mathcal{O}_{k-1}|}{|\mathcal{O}_k|}.$$

Therefore,

$$\frac{|\mathcal{O}_0|}{|\mathcal{O}_k|} = \frac{|\mathcal{O}_0|}{|\mathcal{O}_1|} \cdot \frac{|\mathcal{O}_1|}{|\mathcal{O}_2|} \dots \frac{|\mathcal{O}_{k-1}|}{|\mathcal{O}_k|} \leq \left(\frac{|\mathcal{O}_0|}{|\mathcal{O}_1|} \right)^k.$$

Each closed $2k$ -walk in \mathcal{O}_k is of the form $vu_1vu_2 \dots vu_kv$, which naturally corresponds to a homomorphism from S_k to G that maps the central vertex to v and the i -th leaf to u_i . Therefore,

$$\frac{|\text{Hom}(C_{2k}, G)|}{|\text{Hom}(S_k, G)|} = \frac{|\mathcal{O}_0|}{|\mathcal{O}_k|} \leq \left(\frac{|\mathcal{O}_0|}{|\mathcal{O}_1|} \right)^k = \left(\frac{|\text{Hom}(C_{2k}, G)|}{|\mathcal{U}_1|} \right)^k,$$

which concludes the proof. \square

3 Regularization

Suppose that the n -vertex graph G is “close” to being regular, e.g., $|\text{Hom}(S_k, G)| \leq 10n^{k+1}p^k$, where $p = 2e(G)/n^2$ is the edge density of G . Here the constant 10 is

arbitrarily chosen to illustrate. If G contains no rainbow $2k$ -cycles, then Lemma 2.1 gives

$$n^{2k} p^{2k} \leq |\text{Hom}(C_{2k}, G)| \leq (2k)^{2k} |\text{Hom}(S_k, G)| \leq 10 \cdot (2k)^{2k} n^{k+1} p^k,$$

where the first inequality follows from the fact that even cycles satisfy Sidorenko’s conjecture. As a corollary, $e(G) \leq 2 \cdot 10^{1/k} k^2 n^{1+1/k}$, which is stronger than Theorem 1.2.

However, this ideal assumption is not guaranteed in general. Instead, the following two lemmas will enable us to “regularize” the graph G .

Lemma 3.1 *Let G be a properly edge-colored graph with minimum degree $\delta(G) \geq 1$. Then there exists a properly edge-colored graph G' with the following properties:*

- (1) G' satisfies $|V(G)| \leq |V(G')| \leq 4e(G)/\delta(G)$ and every vertex of G' has degree between $\delta(G)/2$ and $\delta(G)$;
- (2) There is a color-preserving homomorphism ψ from G' to G . In particular, if G' contains a rainbow cycle, then so does G .

Proof Let $\delta := \delta(G)$ for brevity. We construct G' by iterating the following process. Fix an ordering the vertices of G . At each step, take a vertex $v \in V(G)$ according to the ordering and let $s := \lceil d_G(v)/\delta \rceil$. We then split v into new vertices v_1, \dots, v_s so that the neighbor sets $N(v_1), \dots, N(v_s)$ form a partition of the neighbor set of v in G and $\delta/2 \leq |N(v_i)| \leq \delta$ for every $1 \leq i \leq s$. This is possible since $d_G(v) \geq \delta$ and we can make the sizes of $N(v_1), N(v_2), \dots, N(v_s)$ as equal as possible. We color the edges in such a way that each edge uv_i for $u \in N_G(v)$ inherits the same color as uv .

Let ψ be a map from G' to G so that each vertex maps to the original vertex of G before splitting. Then ψ is a color-preserving homomorphism and every vertex of G' has degree between $\delta/2$ and δ . Since the number of edges is preserved throughout the whole process,

$$|V(G')| \cdot \delta/2 \leq 2e(G') = 2e(G),$$

which implies $|V(G')| \leq 4e(G)/\delta$. Finally, the color-preserving homomorphism ψ maps a rainbow cycle in G' to a rainbow circuit in G which contains a rainbow cycle. This proves the “in particular” part. □

Lemma 3.2 *Let $k \geq 2$ and let G be an n -vertex bipartite graph with average degree $d > 0$. Suppose that G contains no proper subgraph with larger average degree. Then G contains a subgraph G' with bipartition (A, B) satisfying the following for some $i \in \mathbb{N}$:*

- (1) $|A| \geq \frac{1}{k} 2^{-\frac{ki}{k-1}} n, |B| \geq \frac{n}{64}$;
- (2) $d_{G'}(a) \in [2^{i-6}d, 2^{i-5}d]$ for all $a \in A$;
- (3) $d_{G'}(b) \leq 4d$ for all $b \in B$.

Proof Denote by (X, Y) a bipartition of G . Let X_0 and Y_0 be the set of vertices in X and Y , respectively, of degree at least $4d$ and let $X_1 := X \setminus X_0$ and $Y_1 := Y \setminus Y_0$. Then $|X_0|, |Y_0| \leq e(G)/(4d) = n/8$.

Since there is no proper subgraph of G with average degree larger than d ,

$$e(G[X_0, Y_0]) \leq \frac{1}{2}(|X_0| + |Y_0|)d \leq \frac{nd}{8} = \frac{1}{4}e(G).$$

Hence, one of $G[X_0, Y_1]$, $G[X_1, Y_0]$ and $G[X_1, Y_1]$ has at least $e(G)/4$ edges. By symmetry, we can assume that for some $X' \in \{X_0, X_1\}$, the graph $G[X', Y_1]$ has at least $e(G)/4$ edges.

As $G[X', Y_1]$ has at least $e(G)/4$ edges and average degree at least $d/4$, we can delete some vertices of $G[X', Y_1]$ to obtain a graph $G_1 = G[X^*, Y^*]$ with minimum degree at least $d/16$ and $e(G_1) \geq e(G)/8$. Since the vertices in Y^* have degree at most $4d$,

$$|Y^*| \geq \frac{e(G_1)}{4d} \geq \frac{e(G)}{32d} = \frac{n}{64}.$$

We partition X^* into the following sets

$$Z_i = \left\{ v \in X^* : d_{G_1}(v) \in [2^{i-6}d, 2^{i-5}d) \right\}, \quad i \in \mathbb{N}.$$

If there exists i such that $|Z_i| \geq \frac{1}{k}2^{-\frac{ki}{k-1}}n$, then take A to be Z_i and $B = Y^*$, and we are done. If not, then we have

$$\frac{e(G)}{8} \leq e(G_1) \leq \sum_{i=1}^{\infty} |Z_i| \cdot 2^{i-5}d < \frac{dn}{32k} \cdot \sum_{i=1}^{\infty} 2^{-\frac{i}{k-1}} < \frac{dn}{32k} \cdot \frac{1}{1 - 2^{-\frac{1}{k-1}}} < \frac{dn}{16}.$$

In the last inequality we used the facts that $2^{-x} \leq 1 - x/2$ for $0 \leq x \leq 1$ and that $0 < \frac{1}{k-1} \leq 1$ for $k \geq 2$. This is a contradiction as $e(G) = dn/2$. This proves the lemma. □

The following simple lemma essentially proves Sidorenko’s conjecture for even cycles in the ‘asymmetric’ bipartite setting. We include a short proof for completeness.

Lemma 3.3 *Let G be a bipartite graph with vertex partition (A, B) . Suppose that the average of degrees of the vertices in A is d_A and the average of degrees of the vertices in B is d_B . Then for every $k \in \mathbb{N}$ we have*

$$|\text{Hom}(C_{2k}, G)| \geq d_A^k \cdot d_B^k.$$

Proof Consider the bipartite adjacency matrix $M = (m_{ij})$ of G , where the rows and columns of M are indexed by the elements of A and B , respectively, with

$$m_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$|\text{Hom}(C_{2k}, G)| = \text{trace}((M^T M)^k) = \lambda_1^k + \dots + \lambda_n^k,$$

where $n := |B|$ and $\lambda_1 \geq \dots \geq \lambda_n$ are (nonnegative) eigenvalues of the positive semidefinite matrix $M^T M$.

It is well-known that $\lambda_1 = \max_{x \in \mathbb{R}^n} \frac{x^T M^T M x}{x^T x}$. Taking $x = (1, \dots, 1)^T$ and using the Cauchy–Schwarz inequality, we obtain

$$\lambda_1 \geq \frac{\sum_{a \in A} \text{deg}^2(a)}{|B|} \geq \frac{(\sum_{a \in A} \text{deg}(a))^2}{|A||B|} = d_A d_B.$$

Hence $|\text{Hom}(C_{2k}, G)| \geq \lambda_1^k \geq (d_A d_B)^k$. □

4 Proofs of the Main Results

Now we are ready to prove our main results.

Proof of Theorem 1.2 By taking a subgraph, assume that G is a bipartite graph with at least $50000k^3 n^{1+1/k}$ edges and let $d \geq 10^5 k^3 n^{1/k}$ be the average degree of G . We further assume that G has no subgraph with larger average degree, as otherwise we can just take that subgraph to be our graph. Also assume that G has no rainbow $2k$ -cycle. We apply Lemma 3.2 to obtain a graph G' and some $i \in \mathbb{N}$ such that

- $|A| = \frac{1}{k} 2^{-\frac{ki}{k-1}} n$;
- $d_{G'}(a) \in [2^{i-6}d, 2^{i-5}d]$ for all $a \in A$;
- $d_{G'}(b) \leq 4d$ for all $b \in B$.

Indeed, we can obtain the equality in the first bullet point by deleting some vertices if necessary. Note that the first two conditions ensures that the average of degrees of vertices in B is at least

$$\frac{2^{i-6}d|A|}{n} \geq \frac{d}{64k} 2^{-\frac{i}{k-1}}.$$

Apply Lemma 3.3 to obtain that

$$\text{Hom}(C_{2k}, G') \geq 2^{k(i-6)} d^k \left(\frac{d}{64k}\right)^k 2^{-\frac{ki}{k-1}}. \tag{3}$$

As $d_{G'}(b) \leq 4d$ for all $b \in B$ and $\sum_{b \in B} d_{G'}(b) = e(G')$, the convexity of the function $f(x) = x^k$ yields that

$$\sum_{b \in B} d_{G'}(b)^k \leq \frac{e(G')}{4d} \cdot (4d)^k \leq (4d)^{k-1} e(G').$$

Hence, Lemma 2.1 implies that

$$\begin{aligned}
 |\text{Hom}(C_{2k}, G')| &\leq (2k)^{2k} |\text{Hom}(S_k, G')| \leq (2k)^{2k} \left(\sum_{a \in A} d_{G'}(a)^k + \sum_{b \in B} d_{G'}(b)^k \right) \\
 &\leq (2k)^{2k} (|A| 2^{k(i-5)} d^k + (4d)^{k-1} e(G')) \\
 &\leq (2k)^{2k} (2^{k(i-5)} d^k + (4d)^{k-1} \cdot 2^{i-5} d) |A| \\
 &\leq (2k)^{2k} d^k (2^{k(i-5)} + 4^{k-1} 2^{i-5}) \cdot \frac{1}{k} 2^{-\frac{ki}{k-1}} n.
 \end{aligned}$$

Here, the penultimate inequality holds as $e(G') \leq 2^{i-5} d |A|$. Combining this with (3), we obtain

$$d^k < (10^5 k^3)^k n.$$

However, we assume that $d \geq 10^5 k^3 n^{1/k}$, a contradiction. This proves the theorem. \square

Proof of Corollary 1.3 The proof proceeds as in Theorem 1.2. If at least half of the $\text{Hom}(C_{2k}, G)$ is non-degenerate, then we can bound it from below using (3); otherwise we reach a similar contradiction as now $|\text{Hom}(C_{2k}, G)| \leq (8k^2)^k |\text{Hom}(S_k, G)|$ by Lemma 2.1. \square

Proof of Theorem 1.1 Suppose that G does not have a rainbow cycle. By iteratively deleting low degree vertices, we may assume that the minimum degree of G is $\delta \geq d(G)/2 \geq 32 \log^2(5n)$. Apply Lemma 3.1 to obtain a graph G' on $n' \leq 4n$ vertices such that G' doesn't contain a rainbow cycle and every vertex of G' has degree between $\delta/2$ and δ .

Let $k = \log n'$. Because G' does not contain any rainbow $2k$ -cycle, Lemma 2.1 implies that

$$|\text{Hom}(C_{2k}, G')| \leq (2k)^{2k} |\text{Hom}(S_k, G')| \leq (2k)^{2k} \delta^k n'.$$

On the other hand, every even cycle satisfies Sidorenko's conjecture, so we know that

$$(2k)^{2k} \delta^k n' \geq (\delta/2)^{2k}.$$

As $n' = 2^k$, this yields that

$$(4k^2 \cdot \delta \cdot 2)^k \geq (\delta^2/4)^k,$$

which is a contradiction as $\delta \geq 32 \log^2(5n) > 32k^2$. Hence G must contain a rainbow cycle. \square

Acknowledgements While writing this note, Janzer and Sudakov [6] obtained asymptotically the same bound $O(n \log^2 n)$ as our Theorem 1.1 independently. Instead of regularization, they did a weighted homomorphism count. Their homomorphism inequalities are analogous to ours, and they are able to apply it to a much larger class of reflective graphs, including e.g. hypercubes. Very recently, Alon, Bucić, Saueremann, Zakharov and Zamir [1] have improved the bound $O(n \log^2 n)$ to $O(n \log n \log \log n)$.

Funding Open Access funding enabled and organized by KAIST.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Alon, N., Bucić, M., Saueremann, L., Zakharov, D., Zamir, O.: Essentially tight bounds for rainbow cycles in proper edge-colourings. [arXiv:2309.04460](https://arxiv.org/abs/2309.04460)
2. Bondy, J.A., Simonovits, M.: Cycles of even length in graphs. *J. Combin. Theory Ser. B* **16**, 97–105 (1974)
3. Das, S., Lee, C., Sudakov, B.: Rainbow Turán problem for even cycles. *Eur. J. Combin.* **34**(5), 905–915 (2013)
4. Erdős, P., and Simonovits, M.: Cube-supersaturated graphs and related problems. In: *Progress in Graph Theory*, pp. 203–218. Academic Press, Toronto (1984)
5. Janzer, O.: Rainbow Turán number of even cycles, repeated patterns and blow-ups of cycles. *Israel J. Math.* **253**(2), 813–840 (2023)
6. Janzer, O., Sudakov, B.: On the Turán number of the hypercube. *Forum Math. Sigma* **12**, e38 (2024)
7. Jiang, T., Seiver, R.: Turán numbers of subdivided graphs. *SIAM J. Discrete Math.* **26**, 1238–1255 (2012)
8. Keevash, P., Mubayi, D., Sudakov, B., Verstraëte, J.: Rainbow Turán problems. *Combin. Probab. Comput.* **16**(1), 109–126 (2007)
9. Tomon, I.: Robust (rainbow) subdivisions and simplicial cycles. *Advances in Combinatorics*. <https://doi.org/10.19086/aic.2024.1> (2024)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.