Combinatorica (2024) 44:651–657
https://doi.org/10.1007/s00493-024-00087-2

ORIGINAL PAPER

The Number of Topological Types of Trees

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Received: 6 September 2023 / Revised: 21 December 2023 / Accepted: 25 January 2024 /
Published online: 4 April 2024

Abstract
Two graphs are of the same *topological type* if they can be mutually embedded into each other topologically. We show that there are exactly $\aleph_1$ distinct topological types of countable trees. In general, for any infinite cardinal κ there are exactly κ^+ distinct topological types of trees of size κ. This solves a problem of van der Holst from 2005.

Keywords infinte tree · topological minor · better-quasi-ordering

Mathematics Subject Classification 05C05 · 05C63 · 06A07

1 Introduction

A graph-theoretic tree T is a *topological minor* of another tree S, written $T \leq S$, if some subdivision of T embeds as a subgraph into S. Nash-Williams [11] proved in 1965 the seminal result that the class of graph-theoretic trees is *well-quasi-ordered* under $\leq$, i.e. that it is a reflexive and transitive relation without infinite strictly decreasing sequences or infinite antichains.

However, this embedding relation $\leq$ is not anti-symmetric: Two distinct trees T and S may well be topological minors of each other, i.e. $T \leq S$ and $S \leq T$. In this case, we say they are of the same *topological type*, written $T \equiv S$. Describing the hierarchy of graph-theoretic trees under the quasi-ordering $\leq$ means understanding the partial order that $\leq$ induces on the topological types of trees. By Nash-Williams's theorem, this is a well-partial-order. But determining its most fundamental characteristic, namely its cardinality, has been an open problem until now.

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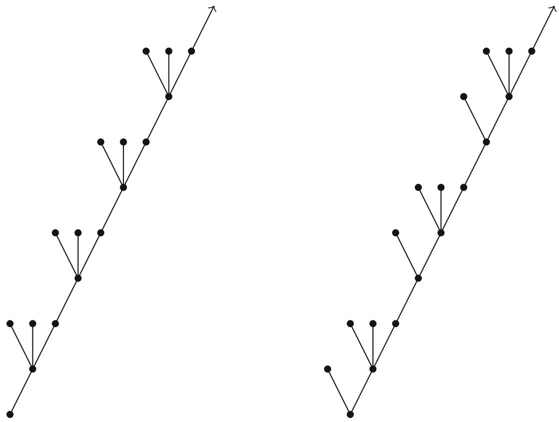

Fig. 1 Distinct trees of the same topological type

Indeed, while up to isomorphism there are exactly $2^{\aleph_0}$ countable trees, determining the exact number of topological types of countable trees is an open problem posed by van der Holst at the Oberwolfach graph theory workshop in 2005 (as reported by Matthiesen [10]). Building on the trees in Fig. 1, we get examples of $2^{\aleph_0}$ non-isomorphic trees of the same topological type. So a priori, it would have been conceivable that there are only countably many topological types of countable trees. However, Matthiesen [10] showed in 2006, by an indirect proof building on Nash-Williams's theorem, that there are uncountably many topological types of countable trees. Bruno [2] in 2017 gave an explicit construction of uncountably many topological types of subtrees of the binary tree. Recently, Bruno and Szeptycki [3] gave the first indication that this bound could be sharp by establishing that there are exactly $\aleph_1$ many topological types of locally finite trees with only countably many rays.

Our main result confirms this pattern for all trees and all cardinalities. We write κ^+ for the successor cardinal of κ. The size of a tree is the cardinality of its vertex set.

Theorem *For any infinite cardinal κ there are exactly κ^+ distinct topological types of trees of size κ.*

Our proof uses a new rank function for trees inspired by Nash-Williams's work on the better-quasi-ordering of infinite trees.

2 The Lower Bound

For common concepts in graph theory and set theory see the textbooks by Diestel [4] and Jech [7].

We recall Schmidt's rank function for rayless graphs [13], see also [6] for an English account: We say that a graph G has *rank* 0 if G is finite. Given an ordinal $\alpha > 0$, we assign *rank* α to G if G does not already have a rank $< \alpha$ and there exists a finite set of vertices X in G such that all components of $G - X$ have a rank $< \alpha$. Schmidt proved that a graph has a rank if and only if it is rayless. Moreover, a routine induction on the rank shows that Schmidt's rank function is non-decreasing with respect to the (topological) minor relation, see e.g. [6, Proposition 4.4]:

Lemma 2.1 *If a rayless graph H is a (topological) minor of a rayless graph G, then the rank of H is at most the rank of G.*

We can now give the argument for the lower bound in our main theorem:

Lemma 2.2 *For any infinite cardinal κ there are at least κ^+ distinct topological types of (rayless) trees of size κ.*

Proof We show that for all ordinals $0 < \alpha < \kappa^+$, there exists a rayless tree T_α of size κ and rank α. Then it follows from Lemma 2.1 that all T_α belong to different topological types, establishing the assertion of the lemma.

We construct the T_α by recursion on α, beginning with T_1 as the κ-star. For successor steps, take countably many disjoint copies T_n ($n \in \mathbb{N}$) of T_α and obtain $T_{\alpha+1}$ by adding a new vertex v to $\bigsqcup_{n\in\mathbb{N}} T_n$ and connecting it to the root of every T_n. Then deleting $\{v\}$ witnesses that $T_{\alpha+1}$ has rank at most $\alpha + 1$. On the other hand, every finite set of vertices X of $T_{\alpha+1}$ avoids infinitely many copies of T_α, so there are components of $T_{\alpha+1} - X$ containing copies of T_α. Every such component has rank at least α by Lemma 2.1, showing that $T_{\alpha+1}$ has rank at least $\alpha + 1$.

For limit steps, obtain T_ℓ by adding a new vertex v to $\bigsqcup_{\alpha<\ell} T_\alpha$ and connecting it to the root of every T_α for $\alpha < \ell$. Then deleting $\{v\}$ witnesses that T_ℓ has rank at most ℓ; on the other hand, every finite set of vertices X of T_ℓ avoids almost all T_α copies for $\alpha < \ell$, so $T_\ell - X$ contains components of arbitrarily large rank below ℓ by Lemma 2.1, showing that T_ℓ has rank at least ℓ. □

3 Better-Quasi-Orderings

A *quasi-ordering* is a binary relation that is reflexive and transitive. A quasi-ordering $\leq$ on set Q is a *well-quasi-ordering* if for every sequence $q_1, q_2, q_3, \dots$ of elements in Q there are indices $n < m \in \mathbb{N}$ such that $q_n \leq q_m$. We define an equivalence relation $\equiv$ on Q by $q \equiv q'$ if both $q \leq q'$ and $q' \leq q$. We abbreviate $|Q|_\equiv := |Q/{\equiv}|$.

Let Q be quasi-ordered and κ an infinite cardinal. We say that $q \in Q$ is κ-*embeddable in Q* if there exist at least κ many elements $q' \in Q$ with $q \leq q'$. We need the following routine result, a proof of which can be found e.g. in [1, Lemma 3.3]:

Lemma 3.1 *For any well-quasi-order Q and infinite cardinal κ, the number of elements of Q which are not κ-embeddable in Q is less than κ.*

Let $(Q, \leq_Q)$ be a quasi-order. Following Nash-Williams [11], we consider the quasi-ordering on the power set $\mathcal{P}(Q)$ where for $A, B \subseteq Q$ we have $A \leq B$ if there is an injective function $f : A \to B$ such that $a \leq_Q f(a)$ for all $a \in A$. Recall that $\mathcal{P}(Q)$ is not necessarily well-quasi-ordered if Q is well-quasi-ordered (see [12]). This is remedied by the introduction of the concept of a *better-quasi-ordering*. We shall not define this concept precisely; we only use as a blackbox that every better-quasi-ordered set is also well-quasi-ordered, that $\mathcal{P}(Q)$ is better-quasi-ordered if Q is better-quasi-ordered, and that the class of all trees is better-quasi-ordered under the topological minor relation [11] (also see [8, 9]).

We write $\mathcal{P}_\kappa(Q)$ for the set of subsets of Q of size exactly κ and $\mathcal{P}_{\leq\kappa}(Q)$ for the set of subsets of Q of size at most κ. Extending [3, Theorem 3], we prove the following result on the number of equivalence classes in $\mathcal{P}_\kappa(Q)$:

Lemma 3.2 *Let μ be an infinite cardinal and Q a better-quasi-ordered set with $|Q|_\equiv = \mu$. Then $|\mathcal{P}_\kappa(Q)|_\equiv = \mu$ for all cardinals $\kappa < \aleph_{\mu^+}$.*

Proof By induction on κ. Suppose for a contradiction that $|\mathcal{P}_\kappa(Q)|_\equiv \geq \mu^+$. By the Erdős-Dushnik-Miller theorem, every partial order $(P, \leq)$ contains an infinite antichain or a chain of size $|P|$, see [5, Theorem 5.25]. As $(Q, \leq_Q)$ is better-quasi-ordered, $(\mathcal{P}_\kappa(Q), \leq)$ is well-quasi-ordered [11, Corollary 28A]. So the partial order $\mathcal{P}_\kappa(Q)/\equiv$ contains no infinite antichains and thus contains a chain of size μ^+. Since $\mathcal{P}_\kappa(Q)/\equiv$ is well-founded, this chain is well-ordered. Hence, there is a strictly increasing chain $\mathcal{A} = (A_\alpha : \alpha < \mu^+)$ in $\mathcal{P}_\kappa(Q)$.

By applying Lemma 3.1 to each induced suborder $(A_\alpha, \leq)$ of $(Q, \leq_Q)$, we obtain for every $A_\alpha \in \mathcal{A}$ a subset $X_\alpha \subseteq A_\alpha$ with $|X_\alpha| < \kappa$ such that all elements of $A_\alpha \setminus X_\alpha$ are κ-embeddable in $(A_\alpha, \leq)$. Since $|X_\alpha| < \kappa < \aleph_{\mu^+}$ for all $\alpha < \mu^+$, there are at most μ different possible cardinalities for the sets X_α (since $\kappa = \aleph_i < \aleph_{\mu^+}$ has at most $|i| \leq \mu$ many $\aleph$'s preceding it).

Since μ^+ is a successor cardinal and thus regular (i.e. any union of fewer than μ^+ sets each containing fewer than μ^+ elements has size less than μ^+), there is a cardinal $\nu < \kappa$ and a μ^+-sized subchain of $\mathcal{A}$ for which the respective $|X_\alpha|$ all take the same value ν. Hence, by restricting to that subchain and relabelling, we may assume that $|X_\alpha| = \nu$ for all $\alpha < \mu^+$ and some cardinal $\nu < \kappa$. Furthermore, by similar considerations, we may assume that all sets X_α for $\alpha < \mu^+$ are pairwise equivalent with respect to $\equiv$, since $|\mathcal{P}_\nu(Q)|_\equiv = \mu$ by induction.

Next, let $\{q_\beta : \beta < \mu\}$ be a representation system for the equivalence classes of $Q/\equiv$. For every q_β that is κ-embeddable in some $A \in \mathcal{A}$, we pick a suitable $A_{\alpha(\beta)} \in \mathcal{A}$ witnessing this. Let $\alpha^* := \sup\{\alpha(\beta) : \beta < \mu\} < \mu^+$. We arrive at the desired contradiction once we have proved that $A_\alpha \equiv A_{\alpha^*}$ for all $\alpha > \alpha^*$. Since $X_\alpha \equiv X_{\alpha^*}$ already, it suffices to show that

$$A_\alpha \setminus X_\alpha \leq A_{\alpha^*} \setminus X_{\alpha^*}$$

for all $\alpha > \alpha^*$. For this, we need an injective function $f : A_\alpha \setminus X_\alpha \to A_{\alpha^*} \setminus X_{\alpha^*}$ that satisfies $a \leq_Q f(a)$ for all $a \in A_\alpha \setminus X_\alpha$. Enumerate $A_\alpha \setminus X_\alpha = \{a_i : i < \kappa\}$, let $i < \kappa$, and suppose that f has been defined on a_j for all $j < i$. Since a_i is κ-embeddable in A_α, it is also κ-embeddable in $A_{\alpha'}$ for some $\alpha' \leq \alpha^*$ by the definition of α^*. Since $A_{\alpha'} \leq A_{\alpha^*}$, the element a_i is also κ-embeddable in A_{α^*}. Hence we can find an element $b \in A_{\alpha^*} \setminus X_{\alpha^*}$ such that $a_i \leq_Q b$ and b is distinct from all values of f that have already been defined. We set $f(a_i) := b$, which completes the construction of f. □

Corollary 3.3 *Let μ be an infinite cardinal and Q a better-quasi-ordered set with $|Q|_\equiv = \mu$. Then $|\mathcal{P}_{\leq\kappa}(Q)|_\equiv = \mu$ for all cardinals $\kappa < \aleph_{\mu^+}$.*

Proof Since $\kappa < \aleph_{\mu^+}$, there exist at most μ cardinals $\leq \kappa$. Hence $|\mathcal{P}_{\leq\kappa}(Q)|_{\equiv} \leq \mu \cdot \mu = \mu$ by Theorem 3.2 applied to $\mathcal{P}_\nu(Q)$ for all cardinals $\nu \leq \kappa$. □

4 The Upper Bound

We consider rooted, graph theoretic trees and tree-order preserving topological minors. For this, we introduce a minimal amount of notation, cf. [4, §12.2]. Recall that fixing a root r of a graph-theoretic tree T yields a natural tree-order $\leq_r$ on T where $t \leq_r s$ if t lies on the unique path from r to s in T. Given a rooted tree, write $\lfloor t \rfloor$ for the subtree of T induced by the set $\{t' \in T : t \leq_r t'\}$ with root t. The neighbours of t in $\lfloor t \rfloor$ are the *successors* of t, denoted by the set $\mathrm{succ}(t)$. Given rooted trees T and S, we write $T \leq S$ if there exists a topological minor embedding $\varphi \colon T \to S$ that preserves the tree-order: If $x \leq y$ in T then $\varphi(x) \leq \varphi(y)$ in S.

We now introduce a new rank function inspired by the proof methods of the better-quasi-ordering of trees due to Nash-Williams:

Definition 4.1 We say that a tree T has rank 0 if $\lfloor t \rfloor \equiv T$ holds for all $t \in T$. Given an ordinal $\alpha > 0$, we assign rank α to T if T does not already have a rank $< \alpha$ and for all $t \in T$, we have either $\lfloor t \rfloor \equiv T$ or $\lfloor t \rfloor$ has rank $< \alpha$. We also write $\mathrm{rank}(T)$ for the rank of T.

We remark that this rank function is not monotone with respect to subtrees. For example, a path on $n + 1$ vertices rooted in one of its endvertices has rank n for all $n \in \mathbb{N}$. However, a ray rooted in its unique vertex of degree 1 has rank 0. Similarly, also the infinite rooted binary tree has rank 0. By taking disjoint paths of all finite lengths and adding a root which we connect to an endvertex of each path, we obtain a graph of infinite rank ω.

Lemma 4.2 *Every tree of size at most κ has a rank $< \kappa^+$.*

Proof Suppose for a contradiction that there is a tree of size at most κ which does not have a rank $< \kappa^+$. Since rooted trees are well-quasi-ordered under $\leq$ by Nash-Williams's theorem [11], there exists a $\leq$-minimal such tree T. Then for every $t \in T$ with $\lfloor t \rfloor \not\equiv T$ we have $\mathrm{rank}(\lfloor t \rfloor) < \kappa^+$ by minimality of T. However, the rank of T is at most

$$\sup\{\mathrm{rank}(\lfloor t \rfloor) \colon t \in T, \lfloor t \rfloor \not\equiv T\} + 1,$$

which is an ordinal $< \kappa^+$ since $|T| \leq \kappa$. This contradicts the choice of T. □

For the remainder of this section, let κ be a fixed infinite cardinal. We write $\mathcal{T}$ for the class of rooted trees of size at most κ, and $\mathcal{C}$ for the set of cardinals of size at most κ.

Let T be a tree in $\mathcal{T}$. For all $t \in T$, let

$$\Gamma(t) := (|\{s \in \mathrm{succ}(t) \colon \lfloor s \rfloor \equiv T\}|, \{\lfloor s \rfloor \colon s \in \mathrm{succ}(t), \lfloor s \rfloor \not\equiv T\}) \in \mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}).$$

Furthermore, we define

$$\Theta(T) := \{\Gamma(t) \colon t \in T\} \in \mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T})).$$

Given two quasi-orderings $(Q, \leq)$ and $(R, \leq)$, we define a quasi-ordering on $Q \times R$ by letting $(q, r) \leq (q', r')$ if $q \leq q'$ and $r \leq r'$. Together with the quasi-ordering on $\mathcal{P}(Q)$ defined in Sect. 3, this yields a quasi-ordering on the set $\mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}))$ considered in the definition of $\Theta(T)$ above. Nash-Williams showed in [11, Lemma 29]:

Lemma 4.3 *For all rooted trees with $\Theta(T) \leq \Theta(S)$, we have $T \leq S$.*

Finally, we give the argument for the upper bound in our main theorem, in a stronger version for rooted trees:

Theorem 4.4 *For any infinite cardinal κ there are at most κ^+ distinct topological types of rooted trees of size κ.*

To see that Theorem 4.4 yields the same bound for unrooted trees, consider the map f mapping any rooted tree of size κ to its corresponding unrooted tree. Since the images under f of two equivalent rooted trees are also equivalent as unrooted trees, the map f induces a surjection from the topological types of κ-sized rooted trees to the topological types of κ-sized unrooted trees.

Proof For all ordinals $\alpha < \kappa^+$, we write $\mathcal{T}_\alpha$ for the class of all trees of size at most κ and rank α and $\mathcal{T}_{<\alpha}$ for the class of all trees of size at most κ and rank $< \alpha$. We show by induction on α that $|\mathcal{T}_\alpha|_\equiv \leq \kappa$ holds for all $\alpha < \kappa^+$. Then it follows from Lemma 4.2 that

$$|\mathcal{T}|_\equiv = \left| \bigcup_{\alpha<\kappa^+} \mathcal{T}_\alpha \right|_\equiv \leq \kappa^+,$$

completing the proof.

Let $\alpha < \kappa^+$ and suppose $|\mathcal{T}_\beta|_\equiv \leq \kappa$ for all $\beta < \alpha$. Consider the function

$$\mathcal{T}_\alpha \to \mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha})),\ T \mapsto \Theta(T).$$

If $T, S \in \mathcal{T}_\alpha$ belong to different twin classes, then also $\Theta(T)$ and $\Theta(S)$ belong to different twin classes of $\mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha}))$ by Lemma 4.3. We conclude that

$$|\mathcal{T}_\alpha|_\equiv \leq |\mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha}))|_\equiv.$$

Thus it suffices to show

$$|\mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha}))|_\equiv \leq \kappa.$$

First, we argue that $|\mathcal{T}_{<\alpha}|_{\equiv} \leq \kappa$: This is clear if $\alpha = 0$. If $\alpha > 0$, we have $|\mathcal{T}_{\beta}|_{\equiv} \leq \kappa$ for all $\beta < \alpha$ and hence $|\mathcal{T}_{<\alpha}|_{\equiv} = |\bigcup_{\beta<\alpha} \mathcal{T}_{\beta}|_{\equiv} \leq \kappa$ since $\alpha < \kappa^{+}$. Next, $\mathcal{T}$ and therefore $\mathcal{T}_{<\alpha}$ is better-quasi-ordered by [11] and thus we have $|\mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha})|_{\equiv} \leq \kappa$ by Corollary 3.3. Then it follows from cardinal arithmetic that also $|\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha})|_{\equiv} \leq \kappa$. Finally, since $\mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha})$ is better-quasi-ordered by [11, Corollary 28A] and hence $\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha})$ is better-quasi-ordered by [11, Corollary 22A], applying Corollary 3.3 once more yields $|\mathcal{P}_{\leq\kappa}(\mathcal{C} \times \mathcal{P}_{\leq\kappa}(\mathcal{T}_{<\alpha}))|_{\equiv} \leq \kappa$. □

Funding Open Access funding enabled and organized by Projekt DEAL.

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