



Edge-Connectivity and Pairwise Disjoint Perfect Matchings in Regular Graphs

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Abstract

For $0 \leq t \leq r$ let $m(t, r)$ be the maximum number s such that every t -edge-connected r -graph has s pairwise disjoint perfect matchings. There are only a few values of $m(t, r)$ known, for instance $m(3, 3) = m(4, r) = 1$, and $m(t, r) \leq r - 2$ for all $t \neq 5$, and $m(t, r) \leq r - 3$ if r is even. We prove that $m(2l, r) \leq 3l - 6$ for every $l \geq 3$ and $r \geq 2l$.

Keywords Perfect matchings · Regular graphs · Factors · r -Graphs · Edge-colorings · Class 2 graphs

Mathematics Subject Classification 05C40 · 05C70

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1 Introduction and Motivation

This paper studies regular graphs which may have parallel edges but no loops. All graphs and all multisets considered in this paper are finite. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. The number of parallel edges connecting two vertices u, v of G is denoted by $\mu_G(u, v)$ and $\mu(G) = \max\{\mu_G(u, v) : u, v \in V(G)\}$. For any two disjoint subsets X and Y of $V(G)$ we denote by $E_G(X, Y)$ the set of edges with one end in X and the other end in Y . If $Y = X^c$ we denote $E_G(X, Y)$ by $\partial_G(X)$. The graph induced by X is denoted by $G[X]$. The *edge-connectivity* of G , denoted by $\lambda(G)$, is the maximum number k such that $|\partial_G(X)| \geq k$ for every non-empty $X \subset V(G)$. Similarly, the *odd edge-connectivity* $\lambda_o(G)$ is defined as the maximum number t such that $|\partial_G(Y)| \geq t$ for every $Y \subseteq V(G)$ of odd cardinality. Clearly, $\lambda(G) \leq \lambda_o(G)$ if G is of even order. An r -regular graph G is an r -graph if $\lambda_o(G) = r$. Note that an r -graph can have small edge-cuts which separate sets of even cardinality.

An r -graph is *class 1*, if its edge set can be partitioned into r perfect matchings and it is *class 2* otherwise. Consequently, an r -graph is class 2 if and only if it has at most $r - 2$ pairwise disjoint perfect matchings. There are many hard problems with regard to perfect matchings in r -graphs. For instance Seymour's exact conjecture [6] states that every planar r -graph is class 1.

If G has a set of k pairwise disjoint perfect matchings we say that it has a k -PDPM. For $0 \leq t \leq r$ let $m(t, r)$ be the maximum number s such that every t -edge-connected r -graph has an s -PDPM. In addition to its exact determination, lower and upper bounds for this parameter are of great interest. The function $m(t, r)$ is monotone increasing in t , in other words $m(t, r) \leq m(t', r)$ for $t \leq t'$. In particular we have that $m(t, r) \leq m(r, r)$ for all $t \in \{2, \dots, r\}$.

For all $r \geq 3$ and $r \neq 5$, class 2 r -edge-connected r -graphs are known, [2, 4]. Thus, $m(r, r) \leq r - 2$ for these r . Surprisingly, no such graphs seem to be known for $r = 5$. The question whether $m(5, 5) = 5$ is raised in [2] where also some consequences (if true) are discussed. Thomassen [7] asked whether $m(r, r) = r - 2$. This is not true if r is even. In [2, 3] it is proved that $m(r, r) \leq r - 3$ if r is even and $m(r - 1, r) \leq r - 3$ if r is odd.

There is also not much known with regard to lower bounds for $m(t, r)$. Trivially we have $m(t, r) \geq 1$. It is an open question which r -graphs have two disjoint perfect matchings. Rizzi [5] constructed (non-planar) r -graphs where any two perfect matchings intersect. These r -graphs are called poorly matchable and so far all known poorly matchable r -graphs have a 4-edge-cut. We do not know any 5-edge-connected poorly matchable r -graph. It might be that high edge-connectivity (instead of odd edge-connectivity) enforces the existence of pairwise disjoint perfect matchings in r -graphs. Thomassen [7] conjectured this to be true. Precisely, he conjectured that there is an integer r_0 such that there is no poorly matchable r -graph for $r \geq r_0$. Seymour's exact conjecture implies that there is no poorly matchable planar r -graph. However, even this seemingly weaker statement is unproved so far. Up to now, we have $m(3, 3) = 1$ for cubic graphs and $m(4, r) = 1$ by Rizzi's result.

In this paper, we improve upper bounds for $m(t, r)$ which only depend on the edge-connectivity parameter. Our main result is that $m(2l, r) \leq 3l - 6$ for every $l \geq 3$ and $r \geq 2l$.

2 Basic Definitions and Results

In this paper, we make extensive use of the Petersen graph, denoted by P , and of the properties of its perfect matchings. Rizzi [5] observed that every two distinct 1-factors of the Petersen graph have precisely one edge in common, and proved that there is a one-to-one correspondence between edges and pairs of distinct 1-factors in the Petersen graph. Then we have the following proposition immediately.

Proposition 2.1 *The Petersen graph has exactly six perfect matchings, and each edge is contained in exactly two of them.*

We fix a drawing of P as in Fig. 1 left. With reference to Fig. 1, we define M_0 to be the perfect matching consisting of all edges $u_i v_i$, for $i \in \{1, \dots, 5\}$. Moreover, for $i \in \{1, \dots, 5\}$, by Proposition 2.1 we let M_i be the only other perfect matching of P different from M_0 and containing $u_i v_i$, see Fig. 1.

Let G be a graph and let $F_1, \dots, F_t \subseteq E(G)$. The graph $G + F_1 + \dots + F_t$ is the graph obtained from G by adding a copy of every edge in F_j for every $j \in \{1, \dots, t\}$. For a multiset \mathcal{N} of perfect matchings of G and an edge $e \in E(G)$, we say that \mathcal{N} contains (avoids, respectively) e if $e \in \bigcup_{N \in \mathcal{N}} N$ ($e \notin \bigcup_{N \in \mathcal{N}} N$, respectively).

Let \mathcal{M} be a multiset of perfect matchings of P . We denote by $n_{\mathcal{M}}(i)$ the number of copies of M_i appearing in \mathcal{M} . Define $P^{\mathcal{M}}$ to be the graph $P + \sum_{F \in \mathcal{M}} F$. Now, let \mathcal{N} be a multiset of perfect matchings of $P^{\mathcal{M}}$. Note that each perfect matching of \mathcal{N} can be interpreted as a perfect matching of P by caring only about the end-vertices of each edge. Then the multiset \mathcal{N} can be interpreted as a multiset of perfect matchings of P , which is denoted by \mathcal{N}_P . Note that $|\mathcal{N}_P| = |\mathcal{N}|$.

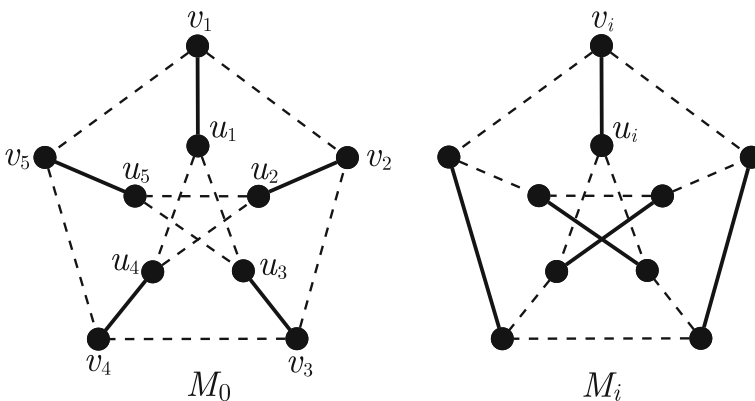


Fig. 1 The Petersen graph P , and its perfect matchings M_0 and M_i

Lemma 2.2 *Let \mathcal{M} be a multiset of perfect matchings of P . Let \mathcal{N} be a set of pairwise disjoint perfect matchings of $P^{\mathcal{M}}$. There is at most one $i \in \{0, \dots, 5\}$ such that $n_{\mathcal{N}_P}(i) > n_{\mathcal{M}}(i)$.*

In particular, there is no triple of different vertices u, v, w in $P^{\mathcal{M}}$, with w adjacent to both v and u , such that \mathcal{N} contains all edges of $E_{P^{\mathcal{M}}}(\{u, v\}, \{w\})$.

Proof First, suppose that there are two indices i and j such that $i \neq j$, $n_{\mathcal{N}_P}(i) > n_{\mathcal{M}}(i)$, and $n_{\mathcal{N}_P}(j) > n_{\mathcal{M}}(j)$. Let uv be the edge of P belonging to both M_i and M_j by Proposition 2.1. Since the perfect matchings of \mathcal{N} are pairwise disjoint, at most $\mu_{P^{\mathcal{M}}}(u, v)$ perfect matchings in \mathcal{N} can contain an edge connecting u and v . This implies $n_{\mathcal{N}_P}(i) + n_{\mathcal{N}_P}(j) \leq \mu_{P^{\mathcal{M}}}(u, v)$. Then the following contradiction arises.

$$\mu_{P^{\mathcal{M}}}(u, v) = n_{\mathcal{M}}(i) + n_{\mathcal{M}}(j) + 1 \leq n_{\mathcal{N}_P}(i) + n_{\mathcal{M}}(j) < n_{\mathcal{N}_P}(i) + n_{\mathcal{N}_P}(j).$$

Next, we prove the second part of the lemma. Let u, v be two different vertices both adjacent to the vertex w in $P^{\mathcal{M}}$. Suppose by contradiction that \mathcal{N} contains all edges of $E_{P^{\mathcal{M}}}(\{u, v\}, \{w\})$. By Proposition 2.1, we may assume without loss of generality that $\{uw\} = M_0 \cap M_1$ and $\{vw\} = M_2 \cap M_3$. Then, since all edges of $E_{P^{\mathcal{M}}}(\{u, v\}, \{w\})$ are contained in \mathcal{N} , we similarly deduce that

- $n_{\mathcal{N}_P}(0) + n_{\mathcal{N}_P}(1) = \mu_{P^{\mathcal{M}}}(u, w) = n_{\mathcal{M}}(0) + n_{\mathcal{M}}(1) + 1;$
- $n_{\mathcal{N}_P}(2) + n_{\mathcal{N}_P}(3) = \mu_{P^{\mathcal{M}}}(v, w) = n_{\mathcal{M}}(2) + n_{\mathcal{M}}(3) + 1.$

Then we conclude that there is $s \in \{0, 1\}$ and $t \in \{2, 3\}$, such that $n_{\mathcal{N}_P}(s) > n_{\mathcal{M}}(s)$ and $n_{\mathcal{N}_P}(t) > n_{\mathcal{M}}(t)$, which is impossible. □

Lemma 2.3 *Let \mathcal{M} be a multiset of k perfect matchings of P and let $\mu = \mu(P^{\mathcal{M}})$. Then, $\lambda(P^{\mathcal{M}}) = \min\{k + 3, 2k + 6 - 2\mu\}$.*

Proof Note that P is a 3-graph, and $P^{\mathcal{M}}$ is a $(k+3)$ -graph since every perfect matching of P intersects each edge-cut that separates two vertex sets of odd cardinality. Let X be a non-empty proper subset of $V(P^{\mathcal{M}})$ minimizing $|\partial_{P^{\mathcal{M}}}(X)|$. It implies that $P^{\mathcal{M}}[X]$ is connected. If $|X|$ is odd, then $|\partial_{P^{\mathcal{M}}}(X)| \geq k + 3$ since $P^{\mathcal{M}}$ is a $(k + 3)$ -graph. If $|X|$ is even, then it suffices to consider the cases $|X| \in \{2, 4\}$. Since P has girth 5, either $P[X]$ is a path on two or four vertices, or it is isomorphic to $K_{1,3}$. Then $\partial_{P^{\mathcal{M}}}(X)$ contains at least $k + 3 - \mu$ edges for each vertex of degree 1 in $P[X]$, and so we get that $|\partial_{P^{\mathcal{M}}}(X)| \geq 2(k + 3 - \mu)$. Consequently, $\lambda(P^{\mathcal{M}}) \geq \min\{k + 3, 2k + 6 - 2\mu\}$. Finally, observe that $|\partial_{P^{\mathcal{M}}}(\{u, v\})| = 2k + 6 - 2\mu$ for every two vertices u, v with $\mu_{P^{\mathcal{M}}}(u, v) = \mu$. Thus, the statement follows. □

Definition 2.4 Let G and H be two graphs with $u, v \in V(G)$ and $x, y \in V(H)$ such that $\mu_G(u, v) \geq t$ and $\mu_H(x, y) \geq r - t$. Then, $(G, u, v) \oplus_t (H, x, y)$ is the graph obtained from G and H by deleting exactly t edges joining u and v in G and $r - t$ edges joining x and y in H , identifying u and x to a new vertex w_{ux} , and identifying v and y to a new vertex w_{vy} , see Fig. 2.

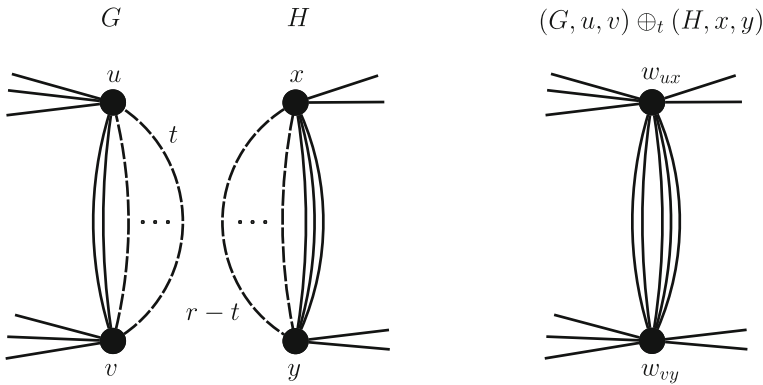


Fig. 2 The operation of Definition 2.4

Lemma 2.5 *Let G and H be two r -graphs with $u, v \in V(G)$ and $x, y \in V(H)$ such that $\mu_G(u, v) \geq t$ and $\mu_H(x, y) \geq r - t$. Then, $G' = (G, u, v) \oplus_t (H, x, y)$ is an r -graph with*

$$\lambda(G') = \min\{\lambda(G), \lambda(H)\}.$$

Proof By the construction in Definition 2.4, $G' = (G, u, v) \oplus_t (H, x, y)$ is r -regular. For every $Y \subseteq V(G) \setminus \{v\}$, we have $|\partial_G(Y)| = |\partial_{G'}((Y \setminus \{u\}) \cup \{w_{ux}\})|$ if $u \in Y$ and $|\partial_G(Y)| = |\partial_{G'}(Y)|$ otherwise. Similarly, for every $Y' \subseteq V(H) \setminus \{y\}$, we have $|\partial_H(Y')| = |\partial_{G'}((Y' \setminus \{x\}) \cup \{w_{ux}\})|$ if $x \in Y'$ and $|\partial_H(Y')| = |\partial_{G'}(Y')|$ otherwise. As a consequence, there is an $X \subseteq V(G')$ with $|\partial_{G'}(X)| = \min\{\lambda(G), \lambda(H)\}$. Thus, it suffices to prove that, for each non-empty proper subset $S \subset V(G')$, we have $|\partial_{G'}(S)| \geq \min\{\lambda(G), \lambda(H)\}$ if $|S|$ is even, and $|\partial_{G'}(S)| \geq r$ if $|S|$ is odd. Since, for all $Y \subseteq V(G')$, $\partial_{G'}(Y) = \partial_{G'}(V(G') \setminus Y)$, we just need to consider the two cases when $|S \cap \{w_{ux}, w_{vy}\}| = 0$ and $|S \cap \{w_{ux}, w_{vy}\}| = 1$.

First, assume $|S \cap \{w_{ux}, w_{vy}\}| = 0$. It is clear that $|\partial_{G'}(S)| = |\partial_G(S \cap V(G))| + |\partial_H(S \cap V(H))|$. Note that one of $|S \cap V(G)|$ and $|S \cap V(H)|$ is odd if $|S|$ is odd. So $|\partial_{G'}(S)| \geq \min\{\lambda(G), \lambda(H)\}$ if $|S|$ is even, and $|\partial_{G'}(S)| \geq r$ if $|S|$ is odd.

Next, we assume $|S \cap \{w_{ux}, w_{vy}\}| = 1$. Without loss of generality, say $w_{ux} \in S$. Note that $|\partial_{G'}(S)| = |\partial_G((S \setminus \{w_{ux}\}) \cup \{u\}) \cap V(G)| - t + |\partial_H((S \setminus \{w_{ux}\}) \cup \{x\}) \cap V(H)| - (r - t)$. If one of $|(S \setminus \{w_{ux}\}) \cup \{u\}) \cap V(G)|$ and $|(S \setminus \{w_{ux}\}) \cup \{x\}) \cap V(H)|$ is odd, then the other has the same parity as $|S|$. This implies $|\partial_{G'}(S)| \geq \min\{\lambda(G), \lambda(H)\}$ if $|S|$ is even, and $|\partial_{G'}(S)| \geq r$ if $|S|$ is odd. Thus, the remaining case is that both $|(S \setminus \{w_{ux}\}) \cup \{u\}) \cap V(G)|$ and $|(S \setminus \{w_{ux}\}) \cup \{x\}) \cap V(H)|$ are even, and $|S|$ is odd. Since $|(S \setminus \{w_{ux}\}) \cap V(G)|$ is odd in this case and G is an r -graph, we obtain $|\partial_G((S \setminus \{w_{ux}\}) \cup \{u\}) \cap V(G)| \geq 2\mu_G(u, v) \geq 2t$. Similarly, $|\partial_H((S \setminus \{w_{ux}\}) \cup \{x\}) \cap V(H)| \geq 2\mu_H(x, y) \geq 2(r - t)$. So $|\partial_{G'}(S)| \geq 2t - t + 2(r - t) - (r - t) = r$. This completes the proof. \square

Lemma 2.6 *Let r, t be two integers with $2 \leq t < r$, let G be an r -graph and let $u, v \in V(G)$ such that $\mu_G(u, v) \geq t$. Let \mathcal{M} be a multiset of $r - 3$ perfect matchings of P , let*

$x, y \in V(P^{\mathcal{M}})$ such that $\mu_{P^{\mathcal{M}}}(x, y) \geq r - t$ and let $G' = (G, u, v) \oplus_t (P^{\mathcal{M}}, x, y)$. If G' has a k -PDPM \mathcal{N}' , then G has a k -PDPM \mathcal{N} such that

- (i) \mathcal{N} avoids at least one edge connecting u and v ,
- (ii) for every $e \in E(G'[V(G) \setminus \{u, v\}])$, if \mathcal{N}' avoids e , then \mathcal{N} avoids e .

Proof Assume that \mathcal{N}' is a k -PDPM of G' . Every perfect matching of G' contains either zero or exactly two edges of $\partial_{G'}(V(P^{\mathcal{M}}) \setminus \{x, y\})$, since $|V(P^{\mathcal{M}}) \setminus \{x, y\}|$ is even. The same holds for $V(G) \setminus \{u, v\}$, since $|V(G) \setminus \{u, v\}|$ is also even. Hence, every perfect matching of G' can be transformed into a perfect matching of G and of $P^{\mathcal{M}}$ by adding either uv or xy . In particular, \mathcal{N}' can be transformed into a k -PDPM \mathcal{N} of G , which satisfies (ii). Suppose that \mathcal{N} contains all edges connecting u and v , which implies that \mathcal{N}' contains all edges of $\partial_{G'}(V(G))$. As a consequence, $P^{\mathcal{M}}$ has a k -PDPM that contains all edges of $\partial_{P^{\mathcal{M}}}(\{x, y\})$. This means that $P^{\mathcal{M}}$ has a k -PDPM containing all edges incident with y and not with x , a contradiction to Lemma 2.2. \square

3 An Upper Bound for $m(t, r)$ Depending on t

Recall that $m(t, r) \leq m(t', r)$ whenever $t \leq t'$. For an r -graph G with a subset $X \subseteq V(G)$, we observe that $|\partial_G(X)| = r \cdot |X| - 2|E(G[X])|$ is even if $|X|$ is even. Therefore, the edge-connectivity of an r -graph is either r or an even number. By Rizzi [5], $m(4, r) = 1$ for every $r \geq 4$. Furthermore, $r - 2$ is a trivial upper bound for $m(2l, r)$ since $m(4, 5) = 1$ and for each $r \neq 5$ there are r -edge-connected r -graphs that are class 2 [2, 4]. We will improve this bound as follows.

Theorem 3.1 For every $l \geq 3$ and $r \geq 2l$, $m(2l, r) \leq 3l - 6$.

As mentioned above, we know $m(2l, r) \leq r - 2$ for every $r \geq 3$. It implies that Theorem 3.1 trivially holds for the case $2l \leq r \leq 3l - 4$. Thus, it suffices to prove Theorem 3.1 for the case $r \geq 3l - 3$.

We will construct $2l$ -edge-connected r -graphs inductively starting with a $2l$ -edge-connected $(3l - 4)$ -graph without a $(3l - 5)$ -PDPM if $l \geq 4$ and a 6-edge-connected 6-graph without a 4-PDPM if $l = 3$.

For this we describe the induction step in the next section and then in the following sections we give the base graphs for the two cases. Finally we deduce the statement of Theorem 3.1.

Induction Step from r to $r + 1$

Lemma 3.2 Let r, l, k be integers such that $r \geq 3l - 4$, $l \geq 2$ and $2 \leq k \leq r$. If there is an r -graph G such that

- $\lambda(G) \geq 2l$,
- G has a perfect matching M such that $\mu_G(u, v) \geq l - 1$ for every $uv \in M$,
- G has no k -PDPM,

then there is an $(r + 1)$ -graph G' such that

- $\lambda(G') \geq 2l$,
- G' has a perfect matching M' such that $\mu_{G'}(u, v) \geq l - 1$ for every $uv \in M'$,
- G' has no k -PDPM.

Proof Assume that the order of G is $2s$ and let $M = \{x_1y_1, \dots, x_sy_s\}$. In order to construct G' we define a graph $P_{(r+1,l)}$ by

$$P_{(r+1,l)} = P + \left\lceil \frac{r-l}{2} \right\rceil M_0 + \left\lfloor \frac{r-l}{2} \right\rfloor M_1 + (l-2)M_2.$$

Since G is $2l$ -edge-connected, we have $r \geq 2l$. Thus, $P_{(r+1,l)}$ is well defined. For every $i \in \{1, \dots, s\}$, take a copy $P^i_{(r+1,l)}$ of $P_{(r+1,l)}$. In each copy, the vertices and perfect matchings are labelled accordingly by using an upper index, i.e. the vertex of $P^i_{(r+1,l)}$ corresponding to u_1 in $P_{(r+1,l)}$ is labeled as u^i_1 . Define graphs H^0, \dots, H^s inductively as follows:

$$H^0 := G + M,$$

$$H^i := (H^{i-1}, x_i, y_i) \oplus_l (P^i_{(r+1,l)}, u^i_1, v^i_1) \text{ for every } i \in \{1, \dots, s\}.$$

Note that H^0 and $P_{(r+1,l)}$ are both $(r+1)$ -graphs. Furthermore, $\mu_{H^0}(x_i, y_i) \geq l$ for every $i \in \{1, \dots, s\}$ by the choice of M . Recall that $u_1v_1 \in E(P)$ is the unique edge in $M_0 \cap M_1$. Thus, $\mu_{P_{(r+1,l)}}(u_1, v_1) = (r+1) - l$ by the definition of $P_{(r+1,l)}$. As a consequence, H^0, \dots, H^s are well defined. Set

$$G' := H^s \quad \text{and} \quad M' := \bigcup_{i=1}^s M^i_2.$$

An example is given in Fig. 3. We claim that G' and M' have the desired properties.

The perfect matching M_2 does not contain the edge u_1v_1 . Thus, M' is well defined. Furthermore, M' is a perfect matching of G' since M is a perfect matching of G . By the definition of $P_{(r+1,l)}$, we have $\mu_{G'}(u, v) \geq l - 1$ for every $uv \in M'$. Hence, M' has the desired properties.

The graph H^0 is a $2l$ -edge-connected $(r+1)$ -graph, since G is a $2l$ -edge-connected r -graph. Furthermore, $\left\lceil \frac{r-l}{2} \right\rceil \geq l-2$ since $r \geq 3l-4$. Thus, $r-l+1$ is the maximum number of parallel edges of $P_{(r+1,l)}$ and hence, $\lambda(P_{(r+1,l)}) = 2l$ by Lemma 2.3. Therefore, for each $i \in \{1, \dots, s\}$, H^i is a $2l$ -edge-connected $(r+1)$ -graph by Lemma 2.5, and so is G' .

Now, suppose that H^s has a k -PDPM \mathcal{N}^s . By applying Lemma 2.6 with $t = l$ to the $(r+1)$ -graph H^s and \mathcal{N}^s we obtain a k -PDPM \mathcal{N}^{s-1} of H^{s-1} , which avoids $x^s y^s$ by property (i). Apply Lemma 2.6 to H^{s-1} and \mathcal{N}^{s-1} to obtain a k -PDPM \mathcal{N}^{s-2} of H^{s-2} , which avoids $x^{s-1} y^{s-1}$ by property (i) and $x^s y^s$ by property (ii). By inductively repeating this process, we obtain a k -PDPM of H^0 that avoids every edge of M . This is not possible, since G has no k -PDPM. Therefore, G' has no k -PDPM, which completes the proof. \square

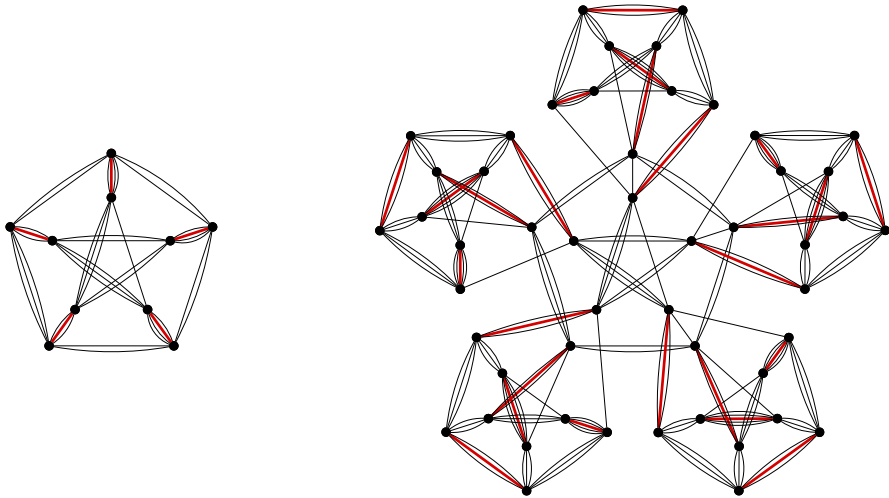


Fig. 3 The graph $G = P + 2M_0 + M_1 + M_2 + M_3$ (left) and the graph G' (right) constructed from G in the proof of Lemma 3.2. The edges of M and M' respectively are drawn in bold red lines

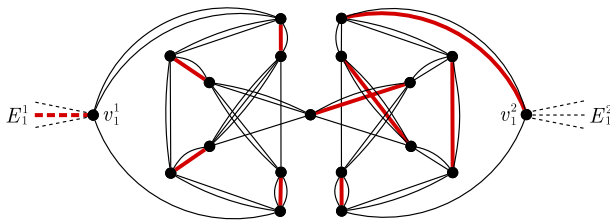


Fig. 4 The graph Q_1 (solid lines) and the edge sets E_1^1, E_1^2 (dashed lines). The bold red edges are used to construct M^6 in the proof of Theorem 3.1

We note that the condition $r \geq 3l - 4$ is necessary in Lemma 3.2 since $\lambda(P_{(r+1,l)}) < 2l$ if $r < 3l - 4$. In view of Lemma 3.2, we need to construct suitable base graphs for all $l \geq 3$, which will be done now.

Base Graph if $l = 3$.

Let $l = 3$ and let P_1^1 and P_1^2 be two copies of the graph $P + M_0 + M_1 + M_2$. For $i \in \{1, 2\}$, remove from P_1^i all parallel edges connecting $u_1^i v_1^i$, call this new graph P^i . Let Q_1 be the graph constructed by identifying the vertices u_1^1 and u_1^2 of P^1 and P^2 respectively. If a graph G contains Q_1 as a subgraph, then let $E_1^i = E_G(\{v_1^i\}, V(G) \setminus V(Q_1))$ for every $i \in \{1, 2\}$, see Fig. 4.

Recall the following property of Q_1 , proved in [2].

Lemma 3.3 [2] *Let G be a graph that contains Q_1 as an induced subgraph. Let $\{N_1, \dots, N_4\}$ be a set of pairwise disjoint perfect matchings of G and let $N = \bigcup_{i=1}^4 N_i$. If $\partial(V(Q_1)) = E_1^1 \cup E_1^2$, then $|E_1^1 \cap N| = |E_1^2 \cap N| = 2$.*

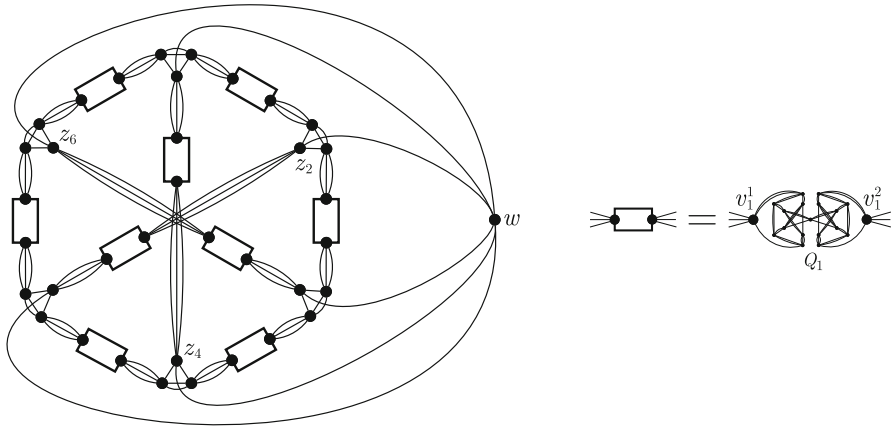


Fig. 5 The graph G_1

In order to construct the required base graph G^6 , we need the graph G_1 shown in Fig. 5, where the boxes denote copies of the graph Q_1 .

The graph G_1 was constructed in [2] to provide a negative answer to a question of Thomassen [7]. The following theorem describes its properties.

Theorem 3.4 [2] G_1 is a 6-edge-connected 6-graph without a 4-PDPM.

Every perfect matching of G_1 contains an edge in $\partial_{G_1}(w)$, which is simple. Thus, in order to use Lemma 3.2 we need to slightly modify G_1 . For any $v \in V(G_1)$, we define a 3-expansion to be the operation that splits v into two vertices v' and v'' (edges formerly incident with v will be incident with exactly one of v' and v'') and adds three parallel edges between them.

Let G^6 be the graph (depicted in Fig. 6) obtained from G_1 by applying a 3-expansion to the vertices z_2, z_4, z_6 and w . Let w' and w'' be the new vertices in which w has been split. It is straightforward that G^6 is still a 6-edge-connected 6-graph.

Proposition 3.5 The graph G^6 has no 4-PDPM.

Proof In this proof vertex labelings of G^6 are considered with reference to Fig. 6. Assume by contradiction that G^6 has a 4-PDPM $\mathcal{M} = \{N_1, \dots, N_4\}$. Then, there is $j \in \{1, \dots, 4\}$ such that $\partial_{G^6}(\{w', w''\}) \cap N_j \neq \emptyset$. Let $e \in \partial_{G^6}(\{w', w''\}) \cap N_j$. We can assume without loss of generality that e is incident with z'_2 . Let $X = \{x_2, y_2, z'_2\} \subseteq V(G^6)$. Then, from Lemma 3.3, we infer that $|\partial_{G^6}(X) \cap N|$ is odd, where $N = \cup_{i=1}^4 N_i$. On the other hand, since X is an odd set, we have that for every $i \in \{1, \dots, 4\}$, $|X \cap N_i|$ is an odd number. Thus, $|X \cap N| = \sum_{i=1}^4 |X \cap N_i|$ must be an even number, a contradiction. \square

Base Graphs if $l \geq 4$

Let $l \geq 4$ and consider the following graph

$$G^{3l-4} = P + (l - 2)M_0 + (l - 3)M_1 + (l - 3)M_2 + M_3.$$

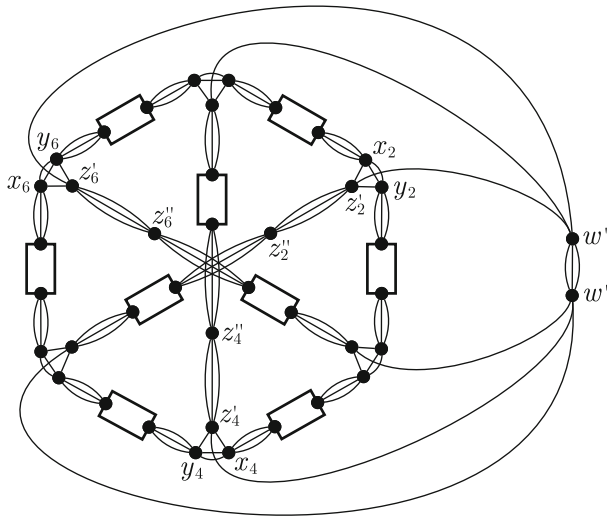


Fig. 6 The graph G^6

The graph G^8 is shown in the left-hand side of Fig. 3. By definition, G^{3l-4} is a $(3l-4)$ -graph, which is $2l$ -edge-connected by Lemma 2.3. It is well known, see [1], that G^{3l-4} is of class 2 and hence has no $(3l-5)$ -PDPM.

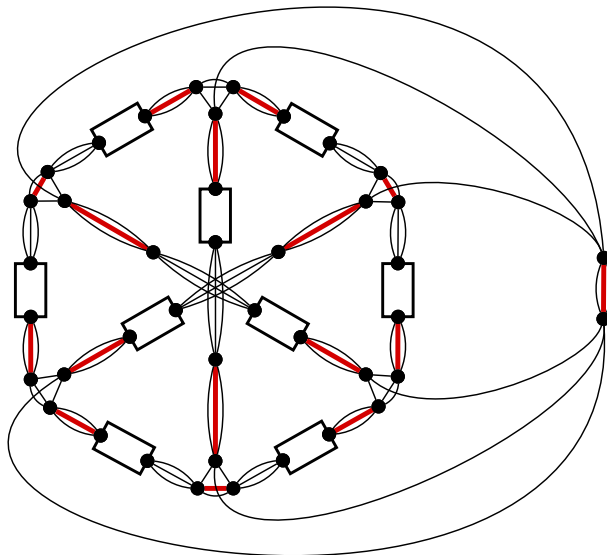


Fig. 7 The chosen edges of G^6 needed to construct M^6

Proof of Theorem 3.1

We prove the statement by induction on r . When $l \geq 4$ we choose G^{3l-4} as base graph (defined above) and we consider the perfect matching M_0 of G^{3l-4} .

Recall that G^{3l-4} is a $2l$ -edge-connected $(3l-4)$ -graph with no $(3l-5)$ -PDPM. Furthermore, for all $uv \in M_0$, $\mu_{G^{3l-4}}(u, v) \geq l-1$. Hence the base case is settled. Then, the inductive step follows by Lemma 3.2 and the statement is proved.

When $l = 3$, we again argue by induction on r . We choose G^6 as base graph. We have already proved that it is a 6-edge-connected 6-graph without a 4-PDPM. Hence, $m(6, 6) \leq 3$.

Let M^6 be the perfect matching of G^6 defined as follows. Consider the matching consisting of the bold red edges depicted in Fig. 7. Extend this matching to a perfect matching of G^6 by choosing, for every copy of Q_1 , the bold red edges depicted in Fig. 4. Note that the chosen set of edges is indeed a perfect matching and each edge of such perfect matching has at least one other parallel edge. This means that the condition on the multiplicities of Lemma 3.2 is satisfied, i.e. for every edge $uv \in M^6$, $\mu_{G^6}(u, v) \geq 2 = l-1$. Therefore the base step is settled. Again, by Lemma 3.2, the inductive step follows. Then Theorem 3.1 is proved. \square

By asking for lower bounds on the parameter $m(t, r)$, one can prove the existence of sets of perfect matchings having specific intersection properties in regular graphs. For example, it can be proved that for $l \geq 5$, if $m(2l, 3l) \geq 2l-1$, then every bridgeless cubic graph admits a perfect matching cover of cardinality $2l-1$. As another example, it can be proved that, for $l \geq 3$, if $m(2l, 3l) \geq l$, then every bridgeless cubic graph has l perfect matchings with empty intersection. Both these proofs rely on the properties of the Petersen graph described in Lemma 2.2.

We though believe that these lower bounds are quite strong conditions. We believe the following statement to be true.

Conjecture 3.6 For all $l \geq 2$ and $r \geq 2l$, $m(2l, r) \leq l-1$.

Note that when $l = 2$, Conjecture 3.6 is true by Rizzi [5].

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