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ORIGINAL PAPER

A Book Proof of the Middle Levels Theorem

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Abstract

We give a short constructive proof for the existence of a Hamilton cycle in the subgraph of the $(2n+1)$-dimensional hypercube induced by all vertices with exactly n or $n+1$ many 1s.

Keywords Hamilton cycle · Hypercube · Middle levels conjecture

Mathematics Subject Classification 05C38

The *n-dimensional hypercube* Q_n is the graph that has as vertices all bitstrings of length n, and an edge between any two bitstrings that differ in a single bit. The *weight* of a vertex x of Q_n is the number of 1s in x. The *kth level* of Q_n is the set of vertices with weight k.

Theorem 1 *For all $n \geq 1$, the subgraph of Q_{2n+1} induced by levels n and $n+1$ has a Hamilton cycle.*

Theorem 1 solves the well-known *middle levels conjecture*, and it was first proved in [2] (see this paper for a history of the problem). A shorter proof was presented in [1] (12 pages). Here, we present a proof from 'the book'.

Proof We write D_n for all Dyck words of length $2n$, i.e., bitstrings of length $2n$ with weight n in which every prefix contains at least as many 1s as 0s. We also define $D := \bigcup_{n\geq 0} D_n$. Any $x \in D_n$ can be decomposed uniquely as $x = 1u0v$ with $u, v \in D$. Furthermore, Dyck words of length $2n$ can be identified by ordered rooted trees with n edges as follows; see Fig. 1: Traverse the tree with depth-first search and

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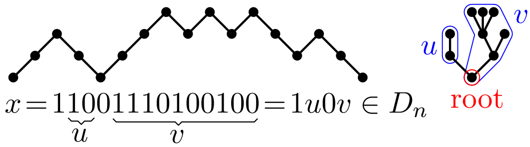

Fig. 1 A Dyck word (left) and the corresponding ordered rooted tree (right)

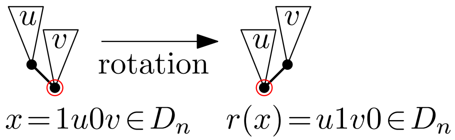

Fig. 2 Tree rotation

write a 1-bit for every step away from the root and a 0-bit for every step towards the root. For any bitstring x, we write $\sigma^s(x)$ for the cyclic right rotation of x by s steps. We write A_n and B_n for the vertices of Q_{2n+1} in level n or $n+1$, respectively, and we define $M_n := Q_{2n+1}[A_n \cup B_n]$. For any $x \in D_n$, $b \in \{0, 1\}$ and $s \in \{0, \ldots, 2n\}$ we define $\langle x, b, s\rangle := \sigma^s(xb)$. Note that we have $A_n = \{\langle x, 0, s\rangle \mid x \in D_n \wedge 0 \leq s \leq 2n\}$ and $B_n = \{\langle x, 1, s\rangle \mid x \in D_n \wedge 0 \leq s \leq 2n\}$. Thus, we think of every vertex of M_n as a triple $\langle x, b, s\rangle$, i.e., an ordered rooted tree x with n edges referred to as the *nut*, a bit $b \in \{0, 1\}$, and an integer $s \in \{0, \ldots, 2n\}$ referred to as the *shift*.

The first step is to construct a cycle factor in the graph M_n. For this we define a mapping $f : A_n \cup B_n \to A_n \cup B_n$ as follows. Given an ordered rooted tree $x = 1u0v \in D_n$ with $u, v \in D$, a *tree rotation* yields the tree $r(x) := u1v0 \in D_n$; see Fig. 2. We define $f(\langle x, 0, s\rangle) := \langle r(x), 1, s+1\rangle$ and $f(\langle x, 1, s\rangle) := \langle x, 0, s\rangle$. It is easy to see that f is a bijection. Indeed, the inverse mapping is $f^{-1}(\langle x, 0, s\rangle) = \langle x, 1, s\rangle$ and $f^{-1}(\langle x, 1, s\rangle) = \langle r^{-1}(x), 0, s-1\rangle$. Furthermore, f changes only a single bit. To see this observe that for $x = 1u0v$ with $u, v \in D$ the bitstrings $\langle x, 0, s\rangle = \sigma^s(1u0v0)$ and $f(\langle x, 0, s\rangle) = \langle r(x), 1, s+1\rangle = \sigma^{s+1}(u1v01) = \sigma^s(1u1v0)$ differ only in the bit between the substrings u and v. We also note that $f^2(\langle x, 0, s\rangle) = \langle r(x), 0, s+1\rangle \neq \langle x, 0, s\rangle$. Consequently, for any vertex y of M_n, the sequence $C(y) := \big(y, f(y), f^2(y), \ldots\big)$ is a cycle, and $F_n := \{C(y) \mid y \in A_n \cup B_n\}$ is a cycle factor in M_n.

As $f^2(\langle x, 0, s\rangle) = \langle r(x), 0, s+1\rangle$, moving two steps forward along a cycle of F_n applies a tree rotation to the nut, and increases the shift by $+1$. As the ordered rooted tree $x \in D_n$ has n edges, we have $x = r^{2n}(x)$. Consequently, the minimum integer $t > 0$ such that $x = r^t(x)$ must divide $2n$. It follows that $\gcd(t, 2n+1) = 1$, hence all shifts of the nut x are contained in the cycle $C(\langle x, 0, 0\rangle)$, i.e., $\langle x, 0, s\rangle \in C(\langle x, 0, 0\rangle)$ for all $s \in \{0, \ldots, 2n\}$. Therefore, the cycles of F_n are in bijection with equivalence classes of ordered rooted trees with n edges under tree rotation, also known as *plane trees*. In particular, the number of cycles of F_n is the number of plane trees with n edges (OEIS A002995).

The second step is to glue the cycles of the factor F_n to a single Hamilton cycle. We call an ordered rooted tree $x \in D_n$ *pullable* if $x = 110u0v$ for $u, v \in D$, and we define $p(x) := 101u0v \in D_n$. We refer to $p(x)$ as the tree obtained from x by a *pull* operation. In words, the leftmost leaf of x is in distance 2 from the root, and the edge leading to this leaf is removed and reattached as the new leftmost

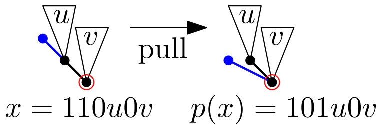

Fig. 3 Pull operation

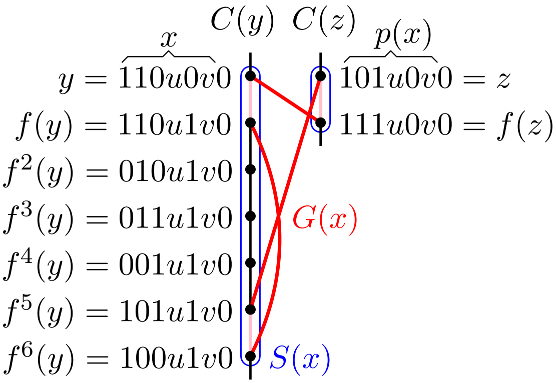

Fig. 4 Gluing 6-cycle $G(x)$

child of the root in $p(x)$; see Fig. 3. For any pullable tree $x = 110u0v \in D_n$ with $u, v \in D$, we define $y := \langle x, 0, 0\rangle = x0$ and $z := \langle p(x), 0, 0\rangle = p(x)0$, and we consider the 6-cycle $G(x) := (y, f(y), f^6(y), f^5(y), z, f(z)) = (110u0v0, 110u1v0, 100u1v0, 101u1v0, 101u0v0, 111u0v0)$, which has the edges $(y, f(y))$ and $(f^6(y), f^5(y))$ in common with the cycle $C(y)$, and the edge $(z, f(z))$ in common with the cycle $C(z)$; see Fig. 4. Consequently, if $C(y)$ and $C(z)$ are two distinct cycles, then the symmetric difference between the edge sets of $C(y)$, $C(z)$ and $G(x)$ is a single cycle on the same set of vertices, i.e., $G(x)$ glues the cycles $C(y)$ and $C(z)$ together.

We define $S(x) := \{f^i(y) \mid i = 0, \ldots, 6\} \cup \{z, f(z)\}$, and we claim that for any two pullable trees $x \neq x'$, we have $S(x) \cap S(x') = \emptyset$, i.e., the cycles $C(x)$ and $C(x')$ are (vertex-)disjoint. To see this, consider the shifts of the vertices in $S(x)$ and $S(x')$, which are 0, 1, 1, 2, 2, 3, 3, 0, 1. It follows that if $S(x) \cap S(x') \neq \emptyset$, then we have $x = x'$, $p(x) = x'$, or $x = p(x')$. These cases are ruled out by the assumption $x \neq x'$, the fact that $p(x) = 10\cdots$ and $x' = 11\cdots$ differ in the second bit, and that $x = 11\cdots$ and $p(x') = 10\cdots$ differ in the second bit, respectively.

To complete the proof, it remains to show that the cycles of the factor F_n can be glued to a single cycle via gluing cycles $G(x)$ for a suitable set of pullable trees $x \in D_n$. As argued before, none of the gluing operations interfere with each other. Using the interpretation of the cycles of F_n as equivalence classes of ordered rooted trees under tree rotation, it suffices to prove that every cycle can be glued to the cycle that corresponds to the star with n edges. As each gluing cycle corresponds to a pull operation, this amounts to proving that any ordered rooted tree $x \in D_n$ can be transformed to the star $(10)^n$ via a sequence of tree rotations and/or pulls.

Indeed, this is achieved as follows: We fix a vertex c of x to become the center of the star (this vertex never changes), and we repeatedly perform the following three steps; see Fig. 5: (i) rotate x to a tree x' such that c is root and the leftmost leaf of x'

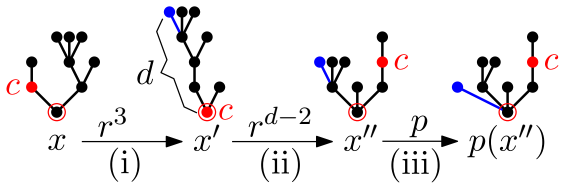

Fig. 5 Illustration of steps (i)–(iii) that make a tree more star-like

is in distance $d > 1$ from c; (ii) apply $d - 2$ rotations to x' to obtain a tree x'' whose leftmost leaf has distance 2 from the root; (iii) perform a pull. As step (iii) decreases the sum of distances of all vertices from c, we reach the star after finitely many steps.

This completes the proof of the theorem. □

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