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ORIGINAL PAPER

On the Maximum of the Sum of the Sizes of Non-trivial Cross-Intersecting Families

P. Frankl[1]

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Abstract
Let $n \ge 2k \ge 4$ be integers, $\binom{[n]}{k}$ the collection of k-subsets of $[n] = \{1, \ldots, n\}$. Two families $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are said to be *cross-intersecting* if $F \cap G \ne \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. A family is called non-trivial if the intersection of all its members is empty. The best possible bound $|\mathcal{F}| + |\mathcal{G}| \le \binom{n}{k} - 2\binom{n-k}{k} + \binom{n-2k}{k} + 2$ is established under the assumption that $\mathcal{F}$ and $\mathcal{G}$ are non-trivial and cross-intersecting. For the proof a strengthened version of the so-called *shifting technique* is introduced. The most general result is Theorem 4.1.

Keywords Subsets · Intersection · Maximal size

1 Introduction

Let n, k be integers, $n \ge 2k \ge 4$. Let $[n] = \{1, \ldots, n\}$ be the standard n-element set, $\binom{[n]}{k}$ the collection of its k-subsets. Subsets of $\binom{[n]}{k}$ are called *k-graphs* or *k-uniform* families. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *intersecting* if $F \cap F' \ne \emptyset$ for all $F, F' \in \mathcal{F}$. Similarly, if $F \cap G \ne \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$ then $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are called *cross-intersecting*.

For a family $\mathcal{F}$ set $\cap\mathcal{F} = \cap\{F : F \in \mathcal{F}\}$. If $\cap\mathcal{F} = \emptyset$ then $\mathcal{F}$ is called *non-trivial* and if $\cap\mathcal{F} \ne \emptyset$ then it is called a *star*. For $i \in [n]$ let $\mathcal{S}_i = \left\{S \in \binom{[n]}{k} : i \in S\right\}$ be the *full star*, k is understood from the context. Note that $|\mathcal{S}_i| = \binom{n-1}{k-1}$.

Let us recall the Erdős–Ko–Rado Theorem, one of the central results in extremal set theory.

✉ P. Frankl
frankl.peter@renyi.hu

1 Rényi Institute, Budapest, Hungary

Theorem 1.1 *([6]) Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$*,* $n \geq 2k \geq 4$ *and* $\mathcal{F}$ *is intersecting. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{1.1}$$

The full star shows that (1.1) is best possible.

The Hilton–Milner Theorem shows in a strong way that for $n > 2k$ only full stars achieve equality in (1.1).

Theorem 1.2 *([16]) Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$*,* $n > 2k \geq 4$*,* $\mathcal{F}$ *is intersecting and non-trivial. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1.2}$$

For $k \neq 3$ the only family providing equality in (1.2) is the Hilton–Milner Family, $\mathcal{H}(n,k) = \left\{H \in \binom{[n]}{k} : 1 \in H, [2, k+1] \cap H \neq \emptyset\right\} \cup \{[2, k+1]\}$. For $k = 3$ the *triangle family* $\mathcal{T}(n,k) = \left\{T \in \binom{[n]}{k} : |T \cap [3]| \geq 2\right\}$ is the only other family attaining the bound (1.2).

By now there are dozens of papers proving and reproving results related to these basic theorems ([1, 2, 4, 5, 8, 10–13, 17–20], etc.).

The author might lack modesty, but he pretends to have found a closely related natural question that has not been investigated before.

Prblem 1.3 *Let* $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ *be non-trivial cross-intersecting families. Determine or estimate*

$$h(n,k) := \max\left\{|\mathcal{F}| + |\mathcal{G}|\right\}. \tag{1.3}$$

The construction that we propose is really simple. Let $\mathcal{A} = \left\{A_1, A_2\right\}$ where $A_1, A_2 \in \binom{[n]}{k}$ are disjoint. Set

$$\mathcal{B} = \left\{B \in \binom{[n]}{k} : B \cap A_i \neq \emptyset,\ i = 1, 2\right\}.$$

Clearly, $|\mathcal{B}| = \binom{n}{k} - 2\binom{n-k}{k} + \binom{n-2k}{k}$. Note that for k fixed and $n \to \infty$,

$$|\mathcal{B}| = k^2\binom{n-2}{k-2} + O\left(\binom{n-3}{k-3}\right).$$

Theorem 1.4 *Let* $n > 2k \geq 4$ *be integers. Then*

$$h(n,k) = 2 + \binom{n}{k} - 2\binom{n-k}{k} + \binom{n-2k}{k}, \tag{1.4}$$

moreover for $k \geq 3$ up to automorphism the above example is unique.

Let us recall the *shifting partial order* that can be traced back to Erdős, Ko and Rado [6]. For two k-sets $A = \{x_1, \ldots, x_k\}$ and $B = \{y_1, \ldots, y_k\}$ where $x_1 < x_2 < \ldots < x_k$, $y_1 < \ldots < y_k$ we say that A *precedes* B and denote it by $A \prec B$ if $x_i \leq y_i$ for all $1 \leq i \leq k$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *initial* or *shifted* if $A \prec B$ and $B \in \mathcal{F}$ always imply $A \in \mathcal{F}$.

Note that the full star $\mathcal{S}_1$ and the Hilton–Milner Family as well as $\mathcal{T}(n,k)$ and many other important families are initial.

Erdős, Ko and Rado invented the shifting operator (cf. definition below) that maintains the size of a family along the intersection or cross-intersection properties. It might destroy non-triviality, but there are certain ways to circumvent this difficulty (cf. [11]).

In great contrast the families $\mathcal{A}$ and $\mathcal{B}$ defined above are *not* initial. In fact the answer for initial families is completely different. Define $\mathcal{P} = \binom{[k+1]}{k}$, $\mathcal{R} = \left\{R \in \binom{[n]}{k} : |R \cap [k+1]| \geq 2\right\}$. It is easy to check that $\mathcal{P}$ and $\mathcal{R}$ are non-trivial and cross-intersecting.

Theorem 1.5 *Suppose that $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are non-trivial, cross-intersecting initial families, then for $n \geq 2k \geq 4$,*

$$|\mathcal{F}| + |\mathcal{G}| \leq k + 1 + \sum_{2 \leq i \leq k} \binom{k+1}{i}\binom{n-k-1}{k-i}. \tag{1.5}$$

Note that for fixed k and $n \to \infty$ the RHS is $\binom{k+1}{2}\binom{n-2}{k-2} + O\left(\binom{n-3}{k-3}\right)$, i.e., it is asymptotic to $\frac{k+1}{2k} h(n,k)$.

For the proof of Theorem 1.4 we need the following old result.

Theorem 1.6 *([13]) Let $k \geq \ell > 0$ be integers. Suppose that $n \geq k + \ell$, $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{\ell}$ and the families $\mathcal{F}, \mathcal{G}$ are non-empty and cross-intersecting. Then*

$$|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{k} - \binom{n-\ell}{k} + 1. \tag{1.6}$$

Moreover unless $n = k + \ell$ or $k = \ell = 2$ the equality is strict for $|\mathcal{G}| > 1$.

Since in [13] uniqueness was not proven, we provide a full proof of (1.6). This proof uses no computation what the reader might find nice.

Let us recall some standard notations. For subsets A, B define $\mathcal{F}(A) = \{F \backslash A : A \subset F \in \mathcal{F}\}$, $\mathcal{F}(\overline{B}) = \{F \in \mathcal{F} : F \cap B = \emptyset\}$ and if $A \cap B = \emptyset$ then set $\mathcal{F}(A, \overline{B}) = \{F \backslash A : A \subset F \in \mathcal{F}, F \cap B = \emptyset\}$. If $A = \{i\}$, $B = \{j\}$ then we use the shorthand notations $\mathcal{F}(i)$, $\mathcal{F}(\overline{j})$, $\mathcal{F}(i, \overline{j})$, etc. Let us mention that $\mathcal{F}(A, \overline{B})$ is a family on the ground set $[n] \setminus (A \cup B)$. The notation $\mathcal{F}(A, A \cup B)$ is also quite common for disjoint sets A, B. Note that $\mathcal{F}(A, A \cup B) = \mathcal{F}(A, \overline{B})$ in this case.

Let us close this section by an inequality that we need in Sect. 3. Somewhat surprisingly the proof is via linear independence.

Proposition 1.7 *Let* m, p, q, a, b *be integers* $0 \le p < q \le m - p$, $0 \le a \le b < m$. *Then*

$$\binom{m}{p} - \sum_{0 \le i < a} \binom{b}{i}\binom{m-b}{p-i} \le \binom{m}{q} - \sum_{0 \le i < a} \binom{b}{i}\binom{m-b}{q-i}. \qquad (1.7)$$

Proof Let us consider the inclusion matrix $T = T(p, q, m)$ defined in the following way. Let $P_1, P_2, \ldots, P_{\binom{m}{p}}$ be a list of all p-subsets of $[m]$ and $Q_1, \ldots, Q_{\binom{m}{q}}$ a list of all q-subsets of $[m]$. The (i, j)-th entry $t(i, j)$ of T is:

$$t(i, j) = \begin{cases} 0 & \text{if } P_i \not\subset Q_j, \\ 1 & \text{if } P_i \subset Q_j. \end{cases}$$

It is well-known (cf. e.g. [3]) that T has full rank $\binom{n}{p}$, that is, the rows are linearly independent over the rationals.

Define $\mathcal{P} = \left\{P \in \binom{[m]}{p} : |P \cap [b]| \ge a\right\}$. Then the corresponding rows of T form a $|\mathcal{P}|$ by $\binom{n}{q}$ submatrix T_0 with linearly independent rows. Note that for $Q \in \binom{m}{q}$ with $|Q \cap [b]| < a$, the column corresponding to Q in T_0 consists entirely of 0. Hence omitting these columns will not alter the row-independence. As the number of rows and columns of the reduced matrix equals the left and right side of (1.7), the inequality follows. □

2 Shifting and Some More Tools

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and integers $1 \le i \ne j \le n$ one defines the shifting (operator) S_{ij} by

$$S_{ij}(\mathcal{F}) = \left\{S_{ij}(F) : F \in \mathcal{F}\right\}$$

where

$$S_{ij}(F) = \begin{cases} F' := (F \setminus \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F \text{ and } F' \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Note that $\left|S_{ij}(\mathcal{F})\right| = |\mathcal{F}|$ and $S_{ij}(\mathcal{F}) \subset \binom{[n]}{k}$. It is well known (cf. [9]) that if $\mathcal{F}, \mathcal{G}$ are cross-intersecting, then $S_{ij}(\mathcal{F})$ and $S_{ij}(\mathcal{G})$ are also.

In the present paper we are mostly dealing with non-trivial families. However it might happen that $\mathcal{F}$ is non-trivial but $S_{ij}(\mathcal{F})$ is a star. The next statement is easy to prove.

Fact 2.1 (cf. e.g. [14]) Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is non-trivial but $S_{ij}(\mathcal{F})$ is a star. Then $S_{ij}(\mathcal{F}) \subset \mathcal{S}_i$ and (i) $\mathcal{F}(i) \cap \mathcal{F}(j) = \emptyset$, (ii) $F \cap \{i, j\} \ne \emptyset$ for all $F \in \mathcal{F}$.

The cross-intersecting families $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are called *saturated* if they cease to be cross-intersecting upon the addition of any new k-set $H \in \binom{[n]}{k}$.

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and a positive integer ℓ define

$$\mathcal{T}^{(\ell)}(\mathcal{F}) = \left\{ H \in \binom{[n]}{\ell} : H \cap F \neq \emptyset \, \forall F \in \mathcal{F} \right\}.$$

With this notation $\mathcal{F}$ and $\mathcal{G}$ are saturated iff $\mathcal{F} = \mathcal{T}^{(k)}(\mathcal{G})$ and $\mathcal{G} = \mathcal{T}^{(k)}(\mathcal{F})$.

Claim 2.2 Suppose that $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are cross-intersecting and saturated and $F \cap \{i, j\} \neq \emptyset$ for all $F \in \mathcal{F}$ (equivalently $\{i, j\} \in \mathcal{T}^{(2)}(\mathcal{F})$). Then $\mathcal{S}_{ij} = \left\{ S \in \binom{[n]}{k} : \{i, j\} \subset S \right\} \subset \mathcal{G}$. □

Claim 2.3 Let $1 \leq i \neq j \leq n$, $\mathcal{F} \subset \binom{[n]}{k}$. Then

$$\left| \mathcal{T}^{(2)}(\mathcal{F}) \right| \leq \left| \mathcal{T}^{(2)} \left(S_{ij}(\mathcal{F}) \right) \right|.$$

□

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and an integer ℓ, $0 \leq \ell \leq k$ define the ℓ-*shadow* $\sigma_\ell(\mathcal{F}) := \left\{ H \in \binom{[n]}{\ell} : \exists F \in \mathcal{F}, H \subset F \right\}$.

Let us recall the shadow theorem of Sperner [21].

$$|\sigma_\ell(\mathcal{F})| \Big/ \binom{n}{\ell} \geq |\mathcal{F}| \Big/ \binom{n}{k}. \tag{2.1}$$

This has an important consequence for us.

Lemma 2.4 *Let m, a, b be positive integers, $m \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[m]}{a}$ and $\mathcal{B} \subset \binom{[m]}{b}$ are cross-intersecting. Then*

$$\frac{|\mathcal{A}|}{\binom{m}{a}} + \frac{|\mathcal{B}|}{\binom{m}{b}} \leq 1. \tag{2.2}$$

Proof Define the family of complements $\mathcal{B}^c = \{[m] \backslash B : B \in \mathcal{B}\}$. By the cross-intersecting property $\mathcal{A} \cup \mathcal{B}^c$ is an antichain. Consequently $\sigma_a(\mathcal{B}^c) \cap \mathcal{A} = \emptyset$. In view of (2.1) $\frac{|\mathcal{B}|}{\binom{n}{b}} \binom{n}{a} + |\mathcal{A}| \leq \binom{n}{a}$ which is equivalent to (2.2). □

Let us prove a corollary of (2.1) for initial families.

Lemma 2.5 *Let $\mathcal{F} \subset \binom{[n]}{k}$ be initial and q a positive integer, $q < n$. Let $P \subset R \subset [q]$ with $|R| \leq k$. Then*

$$|\mathcal{F}(P, [q])| \Big/ \binom{n-q}{k-|P|} \leq |\mathcal{F}(R, [q])| \Big/ \binom{n-q}{k-|R|}. \tag{2.3}$$

Proof Set $d = |R \setminus P|$. Note that $R \subset [q]$ implies $R \backslash P \prec S$ for every $S \in \binom{[q+1,n]}{|R \backslash P|}$. Take a set $F \in \mathcal{F}$ satisfying $F \cap [q] = P$ and a d-element subset S of $F \backslash [q]$. Then $(F \cup (R \setminus P)) \setminus S \prec F$ implies that $F \backslash ([q] \cup S)$ is in $\mathcal{F}(R, [q])$. Now (2.3) follows from (2.1). □

3 The Proof of Theorem 1.5

Let us define $H_i = [k+1] \backslash \{i\}$ for $1 \le i \le k+1$. Let $H \subset \binom{[2,n]}{k}$. Then $H_{k+1} \prec H_k \prec \ldots \prec H_1 \prec H$. Let $\mathcal{H} = \mathcal{F}$ or $\mathcal{G}$. Non-triviality implies $\mathcal{H} \cap \binom{[2,n]}{k} \neq \emptyset$. By initiality $\binom{[k+1]}{k} = \{H_1, \ldots, H_{k+1}\} \subset \mathcal{H}$ follows. By cross-intersection $|H \cap [k+1]| \ge 2$ for all $H \in \mathcal{F} \cup \mathcal{G}$. For $P \subset [k+1]$, $2 \le |P| \le k$ define $\mathcal{F}(P) = \mathcal{F}\left(P, \overline{[k+1] \backslash P}\right)$ and $\alpha(P) = |\mathcal{F}(P)| \Big/ \binom{n-k-1}{k-|P|}$. Define $\beta(P)$ analogously using $\mathcal{G}$ instead of $\mathcal{F}$.

In view of Lemma 2.5 for $P \subset R \subsetneq [k+1]$, $\alpha(P) \le \alpha(R)$.

Claim 3.1 If $P \cap Q = \emptyset$ then

$$\alpha(P) + \beta(Q) \le 1. \tag{3.1}$$

Proof If $Q = [k+1] \backslash P$ then $\mathcal{F}(P)$ and $\mathcal{G}(Q)$ are cross-intersecting and $(k - |P|) + k - |Q| = k - 1 \le n - (k+1)$ implies that we can apply Lemma 2.4. Then (3.1) follows from (2.2). If $Q \subsetneq [k+1] \setminus P$ then we can deduce (3.1) using monotonicity: $\beta(Q) \le \beta([k+1] \setminus P)$. □

Let us use (3.1) to prove the main lemma.

Lemma 3.2 *Let* $2 \le i \le \frac{k}{2}$ *and let* $P, Q \in \binom{[k+1]}{i}$ *be disjoint. Then*

$$|\mathcal{F}(P)| + |\mathcal{G}(Q)| + |\mathcal{G}([k+1] \setminus P)| + |\mathcal{F}([k+1] \setminus Q)| \le \binom{n-k-1}{k-i} + \binom{n-k-1}{i-1}. \tag{3.2}$$

Proof From (3.1) we derive the following three inequalities

$$|\mathcal{F}(P)| + |\mathcal{G}(Q)| \le \binom{n-k-1}{k-i}, \tag{3.3}$$

$$\alpha(P) + \beta([k+1] \setminus P) \le 1, \tag{3.4}$$

$$\beta(Q) + \alpha([k+1] \setminus Q) \le 1. \tag{3.5}$$

Note that $2i \le k$ implies $i - 1 < k - i$ and for $n \ge 2k$, $(k-i) + (i-1) = k - 1 \le n - k - 1$. Thus $\binom{n-k-i}{i-1} \le \binom{n-k-1}{k-i}$. Set $\gamma = \binom{n-k-1}{i-1} \Big/ \binom{n-k-1}{k-i}$. Multiplying by $\binom{n-k-1}{i-1}$ the sum of (3.4) and (3.5) yields

$$|\mathcal{G}([k+1] \setminus P)| + |\mathcal{F}([k+1] \setminus Q)| + \gamma\big(|\mathcal{F}(P)| + |\mathcal{G}(Q)|\big) \le 2\binom{n-k-1}{i-1}.$$

Adding $(1-\gamma)$ times (3.3) yields (3.2). $\square$

To deduce Theorem 1.5 from Lemma 3.2 is easy. Let us define for $0 \le i \le k$, $f_i = \big|\{F \in \mathcal{F} : |F \cap [k+1]| = i\}\big|$. Define g_i analogously. Obviously,

$$|\mathcal{F}| = f_0 + \ldots + f_k = f_2 + \ldots + f_{k-1} + \binom{k+1}{k},$$
$$|\mathcal{G}| = g_2 + \ldots + g_{k-1} + \binom{k+1}{k}.$$

For a fixed i, $2 \le i \le \frac{k}{2}$, averaging (3.2) over all choices of disjoint i-sets P and Q yields

$$f_i + g_i + f_{k+1-i} + g_{k+1-i} \le \binom{k+1}{i}\binom{n-k-1}{k-i} + \binom{k+1}{k+1-i}\binom{n-k-1}{i-1}.$$

In case of $k+1 = 2q$ is even summing (3.3) over all ordered complementary pairs (P, Q) we infer

$$f_q + g_q \le \binom{k+1}{q}\binom{n-k-1}{k-q}.$$

Summing these inequalities we obtain

$$|\mathcal{F}| + |\mathcal{G}| \le \sum_{2 \le j \le k-1} \binom{k+1}{j}\binom{n-k-1}{k-j} + 2\binom{k+1}{k} \quad \text{proving (1.5).}$$

In case of equality, equality must hold all the way, in particular in (3.3) for all $2 \le i \le \frac{k+1}{2}$ and all pairs of disjoint i-sets P and Q. Using uniqueness in case of Sperner's shadow theorem (3.1), we infer that one of $\mathcal{F}(P)$ and $\mathcal{G}(Q)$ must be empty and the other the full set $\binom{[k+2,n]}{k-i}$. Using equality in Lemma 2.5 we eventually arrive at the conclusion that either $\mathcal{F} = \binom{[k+1]}{k}$ or $\mathcal{G} = \binom{[k+1]}{k}$ and the other is $\left\{H \in \binom{[n]}{k} : |H \cap [k+1]| \ge 2\right\}$. $\square$

Let us prove the analogous result for families of distinct uniformities as well.

First we define two pairs of CI families, $n \ge k + \ell$, $k > \ell \ge 1$.

Example 3.3 $\mathcal{G}_1 = \binom{[\ell+1]}{\ell}$, $\mathcal{F}_1 = \left\{F \in \binom{[n]}{k} : |F \cap [\ell+1]| \ge 2\right\}$.

Example 3.4 $\mathcal{G}_2 = \left\{G \in \binom{[n]}{\ell} : 1 \in G\right\} \cup \{[2, \ell+1]\}$, $\mathcal{F}_2 = \left\{F \in \binom{[n]}{k} : 1 \in F, F \cap [2, \ell+1] \ne \emptyset\right\}$.

It should be clear that both pairs form initial CI families, $\mathcal{G}_1 \subset \mathcal{G}_2$, $\mathcal{F}_2 \subset \mathcal{F}_1$. Let us define $g_i(n, k, \ell) = |\mathcal{G}_i| + |\mathcal{F}_i|$, $i = 1, 2$. It is easy to check that $g_i(k+\ell, k, \ell) = \binom{k+\ell}{\ell} = \binom{k+\ell}{k}$ and $|\mathcal{F}| + |\mathcal{G}| \le \binom{k+\ell}{\ell}$ for all CI pairs $\mathcal{F} \subset \binom{[k+\ell]}{k}$, $\mathcal{G} \subset \binom{[k+\ell]}{\ell}$. This takes care of the $n = k + \ell$ case of the following.

Theorem 3.5 *Suppose that $n \geq k+\ell$, $k > \ell \geq 1$ and the initial families $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{\ell}$ form a CI pair. Then* (a) *and* (b) *hold.*
(a) *If both $\mathcal{F}$ and $\mathcal{G}$ are non-trivial then*

$$|\mathcal{F}| + |\mathcal{G}| \leq g_1(n,k,\ell) = \binom{n}{k} - \binom{n-\ell-1}{k} - (\ell+1)\binom{n-\ell-1}{k-1} + (\ell+1). \tag{3.6}$$

(b) *$\mathcal{F}$ is trivial but non-empty and $\mathcal{G}$ is non-trivial, then*

$$|\mathcal{F}| + |\mathcal{G}| \leq g_2(n,k,\ell) = \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1} + \binom{n-1}{\ell-1} + 1. \tag{3.7}$$

Let us note that in the nearly trivial case $\ell = 1$, $g_i(n,k,1) = \binom{n-2}{k-2} + 2$, $i = 1, 2$. Strictly speaking case (a) does not exist: if $\mathcal{G} \subset \binom{[n]}{1}$ is initial and non-empty then $\{1\} \in \mathcal{G}$. Hence $1 \in F$ for all $F \in \mathcal{F}$. In any case the inequalities (3.6) and (3.7) are straightforward to verify for $\ell = 1$.

As we noted above for $n = k+\ell$, $g_i(n,k,\ell) = \binom{n}{k} = \binom{n}{\ell}$. Then $|\mathcal{F}| + |\mathcal{G}| \leq \binom{k+\ell}{\ell}$ follows from the CI property: if $F \in \mathcal{F}$ then $([k+\ell] \setminus F) \notin \mathcal{G}$.

Thus in the sequel we may assume that $k \geq \ell \geq 2$, $n > k+\ell$. The only reason that we discussed the above cases is that we use them during the induction argument.

Assume that we have proved (3.6) and (3.7) for $n-1$ and let us establish it for n. Let us first consider (b), that is, $1 \in F$ for all $F \in \mathcal{F}$. Then

$$|\mathcal{G}| + |\mathcal{F}| = |\mathcal{G}(1)| + |\mathcal{G}(\bar{1})| + |\mathcal{F}(1)|, \quad \mathcal{F}(1) \neq \emptyset \neq \mathcal{G}(\bar{1}).$$

As $\mathcal{G}(\bar{1})$, $\mathcal{F}(1)$ form a CI pair on $[2, n]$ and $\ell \leq k-1$, (1.6) yields

$$|\mathcal{G}(\bar{1})| + |\mathcal{F}(1)| \leq \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1} + 1.$$

Adding to this the obvious $|\mathcal{G}(1)| \leq \binom{n-1}{\ell-1}$, (3.7) follows.

Now let us turn to case (a). By non-triviality all four families $\mathcal{G}(1)$, $\mathcal{F}(\bar{1})$; $\mathcal{G}(\bar{1})$, $\mathcal{F}(1)$ are non-empty. Using non-triviality and initiality $[2, \ell+1] \in \mathcal{G}$ whence $\binom{[\ell+1]}{\ell} \subset \mathcal{G}$. Thus $\binom{[2,\ell+1]}{\ell-1} \subset \mathcal{G}(1)$, proving that $\mathcal{G}(1)$ is non-trivial. In the same way, $\mathcal{F}(1)$ is non-trivial as well. Since $\mathcal{G}_1(\bar{1}) = \{[2, \ell+1]\}$, we can't hope for a proof of $\mathcal{G}(\bar{1})$ being non-trivial. We simply use $\mathcal{G}(\bar{1}) \neq \emptyset$ and (1.6):

$$|\mathcal{G}(\bar{1})| + |\mathcal{F}(1)| \leq \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1} + 1. \tag{3.8}$$

Now we distinguish two cases.
(i) $\mathcal{F}(\bar{1})$ is non-trivial.

Using the induction hypothesis yields

$$|\mathcal{F}(\bar{1})| + |\mathcal{G}(1)| \le \binom{n-1}{k} - \binom{n-\ell-1}{k} - \ell\binom{n-\ell-1}{k-1} + \ell. \tag{3.9}$$

Adding (3.9) to (3.8) yields exactly (3.6).
(ii) $\mathcal{F}(\bar{1})$ is trivial.

As $\mathcal{F}$ is non-trivial, $\mathcal{F}(\bar{1})$ is non-empty. Also $k > \ell$ implies $k \ge (\ell - 1) + 2$. Applying (3.7) to the CI-pair $\mathcal{F}(\bar{1})$, $\mathcal{G}(1)$ yields

$$|\mathcal{F}(\bar{1})| + |\mathcal{G}(1)| \le \binom{n-2}{k-1} - \binom{n-\ell-1}{k-1} + \binom{n-2}{\ell-2} + 1. \tag{3.10}$$

To obtain (3.6) it is sufficient to prove that the RHS of (3.10) does not exceed the RHS of (3.9).

$$\binom{n-2}{k-1} - \binom{n-\ell-1}{k-1} + \binom{n-2}{\ell-2} + 1 \\ \le \binom{n-1}{k} - \binom{n-\ell-1}{k} - \ell\binom{n-\ell-1}{k-1} + \ell.$$

Equivalently,

$$\binom{n-2}{\ell-2} - (\ell-2) - 1 \le \binom{n-2}{k} - \binom{(n-2)-(\ell-2)-1}{k} \\ -((\ell-2)+1)\binom{(n-2)-(\ell-2)-1}{k-1}. \tag{3.11}$$

To derive (3.11) from (1.7) simply set $m = n-2$, $q = k$, $p = \ell-2$, $a = 2$, $b = \ell-2$.

To conclude the proof we only have to check the inequalities $0 \le \ell-2 < k \le n-\ell$ where the last one is valid by $n \ge k+\ell$. □

4 The Proof of Theorem 1.4

Let us fix $k \ge \ell \ge 1$, $n \ge k+\ell$ and define $\mathcal{G}_0 = \{U, V\}$ where U and V are two disjoint ℓ-subsets of $[n]$. Set $\mathcal{F}_0 = \left\{F \in \binom{[n]}{k} : F \cap U \neq \emptyset, F \cap V \neq \emptyset\right\}$.

Obviously $\mathcal{F}_0$ and $\mathcal{G}_0$ are cross-intersecting and

$$|\mathcal{F}_0| + |\mathcal{G}_0| = \binom{n}{k} - 2\binom{n-\ell}{k} + \binom{n-2\ell}{k} + 2 =: h(n,k,\ell). \tag{4.1}$$

Moreover, for $\ell \ge 2$ both $\mathcal{F}_0$ and $\mathcal{G}_0$ are non-trivial.

Let $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{\ell}$ be cross-intersecting. In the case $n = k+\ell$, $h(n,k,\ell) = \binom{k+\ell}{k}$ and the bound $|\mathcal{F}| + |\mathcal{G}| \le \binom{n}{k+\ell}$ is very easy to prove. Just note that $F \in \mathcal{F}$

implies $[n]\backslash F \notin \mathcal{G}$. If $\ell = 1$ and $|\mathcal{G}| = r > 1$ then $|\mathcal{F}| \le \binom{n-r}{k-r}$ is obvious. Now $\binom{n-r}{k-r}+r \le \binom{n-2}{k-2}+2 = h(n, k, 1)$. I.e., in the case $\ell = 1$, (4.2) provides the maximum of $|\mathcal{F}| + |\mathcal{G}|$ for cross-intersecting families subject to $|\mathcal{G}| \ge 2$ without requiring that $\mathcal{F}$ is non-trivial. Our main result shows that (4.2) provides the maximum in general.

Theorem 4.1 *Let $k \ge \ell \ge 2$, $n > k + \ell$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{G} \subset \binom{[n]}{\ell}$ are non-trivial and cross-intersecting. Then*

$$|\mathcal{F}| + |\mathcal{G}| \le h(n, k, \ell). \tag{4.2}$$

Moreover, unless $k = \ell = 2$, up to symmetry $\mathcal{F}_0, \mathcal{G}_0$ are the only families achieving equality in (4.2).

Let us note that in the case $k = \ell = 2$ there are two more essentially different constructions: $\mathcal{F}_1 = \mathcal{G}_1 = \binom{[3]}{2}$ and $\mathcal{F}_2 = \{(1, 2), (2, 3), (3, 4)\}$, $\mathcal{G}_2 = \{(1, 3), (2, 3), (2, 4)\}$. For fixed k, ℓ and $n \to \infty$, $h(n, k, \ell) = (\ell^2 - o(1))\binom{n-2}{k-2}$ while $g_1(n, k, \ell)$ from Theorem 3.5 satisfies $g_1(n, k, \ell) = \left(\binom{\ell+1}{2} - o(1)\right)\binom{n-2}{k-2}$.

For large n and $k \ge 3$ these show that $h(n, k, \ell) > g_1(n, k, \ell)$. To prove it for all $n > k + \ell$, $k \ge 3$ is not evident.

Proposition 4.2 *Let $n > k + \ell$, $k \ge 3$, $\ell \ge 2$. Then*

$$g_1(n, k, \ell) < h(n, k, \ell). \tag{4.3}$$

Note that Theorem 3.5 and (4.3) imply Theorem 4.1 for shifted (initial) families. Nevertheless shifting (cf. Sect. 2) plays a crucial role in the proof of Theorem 4.1.

Recall that (i, j) and more generally $(x_1, \ldots, x_r)$ denote a set where we know that $i < j$, $x_1 < \ldots < x_r$.

Let us now prove (4.3) in a more general form that will be useful later.

Proposition 4.3 *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[2\ell]}{\ell}$ are non-trivial, cross-intersecting and saturated. Let $n > k + \ell$, $k \ge \ell \ge 2$, $k \ge 3$. Then*

$$|\mathcal{F}| + |\mathcal{G}| \le h(n, k, \ell) \text{ and the inequality is strict unless } |\mathcal{G}| = 2. \tag{4.4}$$

Proof Let us define the family $\mathcal{T}_r$ of r-transversals of $\mathcal{G}$ by

$$\mathcal{T}_r = \left\{T \in \binom{[2\ell]}{r} : T \cap G \ne \emptyset \text{ for all } G \in \mathcal{G}\right\}, \quad 1 \le r \le k.$$

Set $t_r = |\mathcal{T}_r|$. Note that $t_r = \binom{2\ell}{r}$ for $\ell < r \le k$. The non-triviality of $\mathcal{G}$ implies $t_1 = 0$. Also,

$$t_\ell = \binom{2\ell}{\ell} - |\mathcal{G}|. \tag{4.5}$$

Lemma 4.4 *For* $2 \le r < \ell$,

$$t_r \le \binom{2\ell}{r} - 2\binom{\ell}{r}. \tag{4.6}$$

Let us prove (4.4) assuming (4.6). Note that saturatedness, i.e., the maximality of $|\mathcal{F}| + |\mathcal{G}|$ implies

$$|\mathcal{F}| = \sum_{2 \le r \le k} t_r \binom{n-2\ell}{k-r}$$

and using (4.5)

$$|\mathcal{F}| + |\mathcal{G}| = |\mathcal{G}| + \sum_{\substack{2 \le r < k \\ r \ne \ell}} t_r \binom{n-2\ell}{k-r} + \left(\binom{2\ell}{\ell} - |\mathcal{G}|\right)\binom{n-2\ell}{k-\ell}.$$

Setting $g = |\mathcal{G}| - 2$ and noting that for $\mathcal{G}_0 = \{[\ell], [\ell+1, 2\ell]\}$ equality holds in (4.6),

$$|\mathcal{F}| + |\mathcal{G}| \le h(n, k, \ell) - g\left(\binom{n-2\ell}{k-\ell} - 1\right) \le h(n, k, \ell).$$

□

To prove (4.6) we show

$$\binom{2\ell}{r} - t_r \ge 2\binom{\ell}{r}, \tag{4.7}$$

i.e., there are at least $2\binom{\ell}{r}$ subsets $R \in \binom{[2\ell]}{r}$ that are not transversals (covers) of $\mathcal{G}$. To this end pick an arbitrary $x \in [\ell]$ and a $G \in \mathcal{G}$ with $x \notin G$. Then all $\binom{\ell-1}{r-1}$ subsets $R \in \binom{[2\ell]\setminus G}{r}$ satisfying $x \in R$ are non-covers. Summing this over the 2ℓ choices for x we infer that there are at least $2\ell\binom{\ell-1}{r-1}/r = 2\binom{\ell}{r}$ non-covers, as desired. This concludes the proof of (4.6).

To prove (4.2) we introduce a special way of simultaneously transforming the two families. We call it *shifting ad extremis*. Let us say that two families $\mathcal{F}$ and $\mathcal{G}$ are *CI* if they are *cross-intersecting*. We proved in [9] that this property is maintained by shifting, that is $S_{ij}(\mathcal{F})$ and $S_{ij}(\mathcal{G})$ are CI as well. Consequently we can keep on applying S_{ij} simultaneously for arbitrary pairs $1 \le i < j \le n$. However it might happen that $S_{ij}(\mathcal{F})$ or $S_{ij}(\mathcal{G})$ ceases to be non-trivial.

Throughout the whole proof we keep assuming that $|\mathcal{F}| + |\mathcal{G}|$ is maximal. Consequently, whenever $\mathcal{G}(\bar{i}, \bar{j}) = \emptyset$ its counterpart $\mathcal{F}(i, j)$ must be *full*, i.e., $\mathcal{F}(i, j) = \left\{F \in \binom{[n]}{k} : (i, j) \subset F\right\}$. Similarly, if $\mathcal{F}(\bar{i}, \bar{j}) = \emptyset$ then $\mathcal{G}(i, j)$ is full.

The important thing about the shifting process is that if $S_{ij}(\mathcal{F})$ or $S_{ij}(\mathcal{G})$ is trivial (a star) then we renounce at S_{ij} and choose an arbitrary different pair (i', j') and perform $S_{i'j'}$. We should stress three things. First, by abuse of notation, we keep denoting the *current* families by $\mathcal{F}$ and $\mathcal{G}$. Second, since $i < j$, the sum $\sum_{F\in\mathcal{F}}\sum_{x\in F} x + \sum_{G\in\mathcal{G}}\sum_{y\in G} y$ keeps decreasing. Third, we return to previously failed pairs $1 \le i < j \le n$, because it might happen that at a later stage in the process S_{ij} does not destroy non-triviality any longer. The important thing is that shifting ad extremis eventually produces two non-trivial families $\mathcal{F}$ and $\mathcal{G}$ that are CI and for each $1 \le i < j \le n$ one of (a), (b) and (c) holds.

(a) $S_{ij}(\mathcal{F}) = \mathcal{F}$ and $S_{ij}(\mathcal{G}) = \mathcal{G}$.
(b) $S_{ij}(\mathcal{F})$ is a star.
(c) $S_{ij}(\mathcal{G})$ is a star.

Then we say that $(\mathcal{F}, \mathcal{G})$ is *shifted ad extremis*. We are going to see below that this is a very strong property.

For $\mathcal{H} = \mathcal{F}$ or $\mathcal{G}$ let us define an ordinary graph $\mathcal{T}_2(\mathcal{H})$ where (i, j) is an edge of $\mathcal{T}_2(\mathcal{H})$ if $\mathcal{H}(\bar{i}, \bar{j}) = \emptyset$. By the maximality of $|\mathcal{F}| + |\mathcal{G}|$, for $\mathcal{K} = \{\mathcal{F}, \mathcal{G}\}\setminus\{\mathcal{H}\}$, $\mathcal{K}(i, j)$ is full.

Fact 4.5 $\mathcal{T}_2(\mathcal{F})$ and $\mathcal{T}_2(\mathcal{G})$ are cross-intersecting.

Proof Suppose the contrary and fix $(i, j) \in \mathcal{T}_2(\mathcal{F})$ and $(a, b) \in \mathcal{T}_2(\mathcal{G})$ that are disjoint. By $n \ge k + \ell$ we can find $G \in \binom{[n]}{\ell}$ and $H \in \binom{[n]}{k}$ satisfying $G \cap \{i, j, a, b\} = (i, j)$, $H \cap \{i, j, a, b\} = (a, b)$ and $H \cap G = \emptyset$.

By the maximality of $|\mathcal{G}| + |\mathcal{F}|$, $G \in \mathcal{G}$ and $H \in \mathcal{F}$, a contradiction. □

We should note that if (b) holds for (i, j) then $(i, j) \in \mathcal{T}_2(\mathcal{F})$, but not necessarily vice versa.

If (a) holds for all $1 \le i < j \le n$ then $\mathcal{F}$ and $\mathcal{G}$ are initial and $|\mathcal{F}|+|\mathcal{G}| \le g_1(n, k, \ell)$ follows from Theorem 3.5. Consequently we may assume that (b) or (c) holds for at least one pair (i, j).

Let us first show that we may assume without loss of generality that $S_{ij}(\mathcal{G})$ is a star for some $1 \le i < j \le n$. If $k = \ell$ then if necessary we could interchange $\mathcal{F}$ and $\mathcal{G}$. Let $k > \ell$ and assume that $\widetilde{\mathcal{F}} := S_{ij}(\mathcal{F})$ is a star. Set $\widetilde{\mathcal{G}} = S_{ij}(\mathcal{G})$. Note that $\left|\widetilde{\mathcal{F}}\right| = \left|\widetilde{\mathcal{F}}(i)\right|$ and the two families $\widetilde{\mathcal{F}}(i)$, $\widetilde{\mathcal{G}}(\bar{i})$ are non-empty CI. Since $k - 1 \ge \ell$, Theorem 1.6 yields

$$\left|\widetilde{\mathcal{F}}(i)\right| + \left|\widetilde{\mathcal{G}}(\bar{i})\right| \le \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1} + 1.$$

Using $|\mathcal{F}| + |\mathcal{G}| = \left|\widetilde{\mathcal{F}}\right| + \left|\widetilde{\mathcal{G}}\right|$ and $\left|\widetilde{\mathcal{G}}\right| = \left|\widetilde{\mathcal{G}}(i)\right| + \left|\widetilde{\mathcal{G}}(\bar{i})\right|$ we infer

$$|\mathcal{F}| + |\mathcal{G}| \le \binom{n-1}{k-1} + \binom{n-1}{\ell-1} - \binom{n-\ell-1}{k-1} + 1 < \binom{n}{k} - 2\binom{n-\ell}{k} + 2$$

for $n > k + \ell$, where the last inequality is (5.8) from [15].

Since $\mathcal{G}(\bar{i}, \bar{j}) = \emptyset$ but $\mathcal{G}(\bar{i})$ and $\mathcal{G}(\bar{j})$ are non-empty,we can choose $K, H \in \mathcal{G}$ with $K \cap (i, j) = \{i\}$, $H \cap \{i, j\} = \{j\}$. Let us fix these K and H maximizing $|K \cap H|$. In view of Fact 2.1, $|K \cap H| \le \ell - 2$.

Lemma 4.6 *Suppose that $x \neq i$, $y \neq j$ but $x \in K \setminus H$, $y \in H \setminus K$. Then $\{x, y\}$ is in* (c).

Proof To simplify notation assume $x < y$ for the proof. The case $y < x$ can be treated in the same way. Since $S_{xy}(H) =: H' = (H \setminus \{y\}) \cup \{x\}$ and $|H' \cap K| = |H \cap K| \cup \{x\}$, $H' \notin \mathcal{G}$. Consequently, (x, y) cannot be of type (a). In view of $(i, j) \cap (x, y) = \emptyset$, it is not of type (b) either. □

In view of the fullness of $\mathcal{F}(x, y)$, $\mathcal{F}(i, \bar{j})$ and $\mathcal{F}(\bar{i}, j)$ are both non-empty.

At this point let us clarify our plan for proving (4.2). There is a Plan A and a Plan B. Plan A is to find a pair (i, j) of type (c) with the additional property that $\mathcal{F}(\bar{i}, \bar{j})$ is non-trivial. Then use induction.

Accordingly Plan B relates to the case that no such (i, j) exists. In this case we show that $\mathcal{F}$ and $\mathcal{G}$ are of a rather restricted type, e.g., they are shifted on $[n] \setminus \{i, j, x, y\}$, $\{i, j, x, y\}$ is a transversal for $\mathcal{F}$ etc. Eventually we use these structural properties to prove (4.2).

Case A. There are $1 \leq i < j \leq n$ so that (i, j) is of type (c) and $\mathcal{F}(\bar{i}, \bar{j})$ is non-trivial.

We assume that (4.2) has been proved for all pairs $k' \geq \ell' \geq 2$ with $k' + \ell' < k + \ell$ and apply induction. As noted before, for $\ell = 1$ and $k \geq 2$, (4.2) holds whenever $\mathcal{F}, \mathcal{G}$ are CI and $|\mathcal{G}| \geq 2$. This can serve as a base for the induction.

To prove (4.2) first note that $\mathcal{G}(\bar{i}, \bar{j}) = \emptyset$ implies

$$|\mathcal{F}(i, j)| + |\mathcal{G}(\bar{i}, \bar{j})| = \binom{n-2}{k-2}. \tag{4.8}$$

Note also that $\mathcal{G}(\bar{i}, \bar{j}) = \emptyset$ and non-triviality imply that $\mathcal{G}(\bar{i}, j)$ and $\mathcal{G}(i, \bar{j})$ are non-empty.

Applying Theorem 1.6 to the non-empty CI families $\big(\mathcal{F}(i, \bar{j}), \mathcal{G}(\bar{i}, j)\big)$ and $\big(\mathcal{F}(\bar{i}, j), \mathcal{G}(i, \bar{j})\big)$ yields

$$\big|\mathcal{F}(i, \bar{j})\big| + \big|\mathcal{G}(\bar{i}, j)\big| \leq \binom{n-2}{k-1} - \binom{n-\ell-1}{k-1} + 1, \tag{4.9}$$

$$\big|\mathcal{F}(\bar{i}, j)\big| + \big|\mathcal{G}(i, \bar{j})\big| \leq \binom{n-2}{k-1} - \binom{n-\ell-1}{k-1} + 1. \tag{4.10}$$

Moreover, the inequalities are strict unless $\big|\mathcal{G}(\bar{i}, j)\big| = 1$, $\big|\mathcal{G}(i, \bar{j})\big| = 1$, respectively. Finally, we need to prove

$$\big|\mathcal{F}(\bar{i}, \bar{j})\big| + |\mathcal{G}(i, j)| \leq \binom{n-2}{k} - 2\binom{n-\ell-1}{k} + \binom{n-2\ell}{k}. \tag{4.11}$$

To prove (4.11) using the induction hypothesis we construct a non-trivial family $\widetilde{\mathcal{G}} \subset \binom{[n] \setminus (i,j)}{\ell-1}$ satisfying $\big|\widetilde{\mathcal{G}}\big| \geq |\mathcal{G}(i, j)| + 2$.

First note that the non-triviality of $\mathcal{G}$ and $\mathcal{G}(\bar{i}, \bar{j}) = \emptyset$ and (i, j) being of type (c) imply that $\mathcal{G}(\bar{i}, j)$ and $\mathcal{G}(i, \bar{j})$ are disjoint and $\mathcal{G}(\bar{i}, j) \cup \mathcal{G}(i, \bar{j}) \cup \mathcal{G}(i, j)$ is non-trivial on $[n] \setminus (i, j)$. In the case $\mathcal{G}(i, j) = \emptyset$, we set $\widetilde{\mathcal{G}} = \mathcal{G}(\bar{i}, j) \cup \mathcal{G}(i, \bar{j})$ and apply the induction hypothesis to the pair $\left(\mathcal{F}(\bar{i}, \bar{j}), \widetilde{\mathcal{G}}\right)$. Noting $\left|\widetilde{\mathcal{G}}\right| \geq 2$, (4.11) follows. If $\mathcal{G}(i, j) \neq \emptyset$ let $\widehat{\mathcal{G}}$ be its *shade*,

$$\widehat{\mathcal{G}} = \left\{ \widehat{G} \in \binom{[n] \setminus (i, j)}{\ell - 1} : \exists G \in \mathcal{G}(i, j), G \subset \widehat{G} \right\}.$$

By Sperner's Shadow Theorem,

$$\left|\widehat{\mathcal{G}}\right| \geq |\mathcal{G}(i, j)| \frac{n - \ell}{\ell - 1}. \tag{4.12}$$

We define $\widetilde{\mathcal{G}} = \widehat{\mathcal{G}} \cup \mathcal{G}(\bar{i}, j) \cup \mathcal{G}(i, \bar{j})$. Obviously $\widetilde{\mathcal{G}} \subset \binom{[n] \setminus (i, j)}{\ell - 1}$ and the pair $\left(\mathcal{F}(\bar{i}, \bar{j}), \widetilde{\mathcal{G}}\right)$ is CI. The final piece to prove is

$$\left|\widetilde{\mathcal{G}}\right| \geq |\mathcal{G}(i, j)| + 2. \tag{4.13}$$

Since for $n > k + \ell$ every $(\ell - 2)$-subset of $[n] \setminus (i, j)$ is contained in $(n-2)-(\ell-2) > k \geq \ell$ subsets of size $\ell - 1$, $\left|\widehat{\mathcal{G}}\right| \geq \ell + 1$. Thus in proving (4.13) we may assume $|\mathcal{G}(i, j)| \geq \ell$. From (4.12) we infer

$$\begin{aligned} \left|\widetilde{\mathcal{G}}\right| &\geq |\mathcal{G}(i, j)| + |\mathcal{G}(i, j)| \frac{n - 2\ell + 1}{\ell - 1} \\ &\geq |\mathcal{G}(i, j)| + \frac{\ell}{\ell - 1}(n - 2\ell + 1) > |\mathcal{G}(i, j)| + 1 \end{aligned}$$

proving (4.13). □

Remark The above proof shows that (4.2) holds in the case $n = k + \ell + 1$ even if $\mathcal{F}(\bar{i}, \bar{j})$ is trivial. Indeed, the RHS of (4.11) is $\binom{k+\ell-1}{k} - 2$. Above we constructed $\widetilde{\mathcal{G}} \subset \binom{[n]}{\ell - 1}$ so that $\mathcal{F}(\bar{i}, \bar{j})$ and $\widetilde{\mathcal{G}}$ are CI on the $(k + \ell - 1)$-set $[n] \setminus (i, j)$. Thus $|\mathcal{F}(\bar{i})| + \left|\widetilde{\mathcal{G}}\right| \leq \binom{k+\ell-1}{k}$. Hence (4.13) implies (4.11).

From now on we assume $n \geq k + \ell + 2$.

CaseB. For all pairs (i, j) of type (c), $\mathcal{F}(\bar{i}, \bar{j})$ is a star.

Let $z(i, j)$ denote an element common to all members of $\mathcal{F}(\bar{i}, \bar{j})$. Let us set $Z = \{i, j, x, y\}$ from Lemma 4.6. We claim that (a, b) is of type (a) for all $(a, b) \subset [n] \setminus Z$.

First note that (a, b) is not of type (b) because of Fact 4.5. Should it be of type (c), we infer $(a, b) \in \mathcal{T}_2(\mathcal{G})$ whence all k-sets $F \subset [n]$ satisfying $F \cap \{i, j, a, b\} = (a, b)$ are in $\mathcal{F}(\bar{i}, \bar{j})$. Since the same holds for all k-sets with $F \cap \{i, j, x, y\} = \{x, y\}$, $\mathcal{F}(\bar{i}, \bar{j})$ is non-trivial, a contradiction. Note that this argument shows $z(i, j) \in \{x, y\} \subset Z$ as well.

Note that via Lemma 4.6, $|K \cap H| = \ell - 2$ follows as well. Set $W = [n] \setminus Z = (w_1, \ldots, w_{n-4})$. Define $\widetilde{K} = \{i, x, w_1, \ldots, w_{\ell-2}\}$, $\widetilde{H} = \{j, y, w_1, \ldots, w_{\ell-2}\}$. Since all pairs in W are of type (a), $K, H \in \mathcal{G}$ imply $\widetilde{K}, \widetilde{H} \in \mathcal{G}$. WLOG assume $K = \widetilde{K}$, $H = \widetilde{H}$.

Fact 4.7 If $|L \cap Z| \le 2$ for some $L \in \mathcal{G}$ then $L \cap Z = \{i, x\}$ or $\{j, y\}$.

Proof From $L \cap (i, j) \ne \emptyset$ and $L \cap (x, y) \ne \emptyset$, $|L \cap Z| \ge 2$ follows. We have to show that $L \cap Z$ is neither $\{i, y\}$ nor $\{j, x\}$. By symmetry assume $L \cap Z = \{i, y\}$. By shiftedness on W, $\widetilde{L} = \{i, y, w_1, \dots, w_{\ell-2}\} \in \mathcal{G}$. However this contradicts $\mathcal{G}(\overline{x}) \cap \mathcal{G}(\overline{y}) = \emptyset$. □

Corollary 4.8 $\mathcal{G}(\overline{x}, \overline{j}) = \emptyset = \mathcal{G}(\overline{i}, \overline{y})$.

Proof If $G \in \mathcal{G}(\overline{x}, \overline{j})$ then $|G \cap Z| \ge 2$ implies $G \cap Z = \{i, y\}$, a contradiction. □

Corollary 4.9 $\{x, j\}$ *and* $\{i, y\}$ *are of type* (c).

Proof By Corollary 4.8 and the maximality of $|\mathcal{F}| + |\mathcal{G}|$, both $\mathcal{F}(x, j)$ and $\mathcal{F}(i, y)$ are full. This implies that $\{x, j\}$ and $\{i, y\}$ could not be of type (b). Should, say, $\{x, j\}$ be of type (a), we infer according to whether $x < j$ or $j < x$ that $\{x, y, w_1, \dots, w_{\ell-2}\} \in \mathcal{G}$ or $\{i, j, w_1, \dots, w_{\ell-2}\} \in \mathcal{G}$. Both contradict Fact 4.7. □

Fact 4.10 Z is a transversal of $\mathcal{F}$.

Proof Since $\mathcal{F}(x, y)$ is full $z(i, j) \in \{x, y\}$.

However, if $F \in \mathcal{F}$ satisfies $F \cap Z = \emptyset$ then $F \in \mathcal{F}(\bar{i}, \overline{j})$ but $z(i, j) \notin F$, a contradiction. □

Now we are in a position to prove an important lemma.

Lemma 4.11 *Set* $V = Z \cup (w_1, \dots, w_{k+\ell-4})$, $|V| = k + \ell$. *Then for all* $F \in \mathcal{F}$ *and* $G \in \mathcal{G}$,

$$F \cap G \cap V \ne \emptyset.$$

Proof Arguing for contradiction choose $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \cap G \cap V = \emptyset$ and among such sets $|F \cap G|$ is as small as possible. Choose $v \in F \cap G$. Then $F \cap Z \ne \emptyset$ implies $|F \cap (w_1, \dots, w_{k+\ell-4})| \le k-2$ and by $|G \cap Z| \ge 2$, $|G \cap (w_1, \dots, w_{k+\ell-4})| \le \ell-3$.

Consequently we can pick $w \in (w_1, \dots, w_{k+\ell-4})$ with $w \notin F \cup G$. By shiftedness $F' := (F \backslash \{v\}) \cup \{w\}$ is in $\mathcal{F}$. However, $F' \cap G \cap V = \emptyset$ and $|F' \cap G| < |F \cap G|$, a contradiction. □

Seeing $\ell-3$ in the proof the careful reader might wonder, what about the case $\ell = 2$? Actually, that case is readily settled by Fact 4.7. If $\ell = 2$ it implies $\mathcal{G} = \{\{i, x\}, \{j, y\}\}$ and (4.2) follows together with uniqueness.

Let us use Lemma 4.11 to prove (4.2) in an important special case.

Proposition 4.12 *Suppose that* $\mathcal{G}(n) = \emptyset$ *and* $k > \ell$. *Then* (4.2) *holds. Moreover, inequality is strict unless* $|\mathcal{G}| = 2$.

Proof Note that the fullness of $\mathcal{F}(i, j)$ and $\mathcal{F}(x, y)$ imply that $\mathcal{F}(n)$ is non-trivial. In view of Lemma 4.11, $\mathcal{F}(n)$ and $\mathcal{G}$ are CI. The same is true for $\mathcal{F}(\overline{n})$ and $\mathcal{G}$. Thus

$$|\mathcal{F}(\overline{n})| + |\mathcal{G}| - 2 \leq \binom{n-1}{k} - 2\binom{n-\ell-1}{k} + \binom{n-2\ell-1}{k},$$
$$|\mathcal{F}(n)| + |\mathcal{G}| - 2 \leq \binom{n-1}{k-1} - 2\binom{n-\ell-1}{k-1} + \binom{n-2\ell-1}{k-1}.$$

Adding these yields $|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{k} - 2\binom{n-\ell}{k} + \binom{n-2\ell}{k} + 2 - (|\mathcal{G}| - 2)$. □

As useful as Proposition 4.12 might look, it is not needed in the rest of the proof. We believe that it might be useful in other situations. On the other hand one can use Lemma 4.11 to settle the case $k = \ell$ as well.

Claim 4.13 If $\mathcal{G}(n) = \emptyset$ then $G \subset V$ for all $G \in \mathcal{G}$.

Proof Suppose that $u \in G \backslash V$ for some $G \in \mathcal{G}$. Then Lemma 4.11 implies $G \backslash \{u\} \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Consequently the same holds for $(G \setminus \{u\}) \cup \{n\} =: G'$. Now the maximality of $|\mathcal{G}| + |\mathcal{F}|$ implies $G' \in \mathcal{G}$. Hence $\mathcal{G}(n) \neq \emptyset$. □

If $k = \ell$ and $\mathcal{G}(n) = \emptyset$ then Claim 4.13 and Lemma 4.4 imply (4.2) together with uniqueness.

Recall that for $a \in \{i, x\}$, $b \in \{j, y\}$, (a, b) is of type (c) and $\mathcal{F}(\overline{a}, \overline{b})$ is trivial. Let $z = z(a, b) \in F$ for all $F \in \mathcal{F}(\overline{a}, \overline{b})$. Recall that $z \in \{x, y\} \subset Z$. This implies that $\{a, b, z\} \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. That is, for each of the four choices of (a, b) there is a T, $(a, b) \subset T \in \binom{Z}{3}$ where T is a transversal of $\mathcal{F}$. Using this for $(a, b) = (i, j)$ and $\{x, y\}$ we infer that at least two of the four sets in $\binom{Z}{3}$ are transversals of $\mathcal{F}$. Could it be three or four?

Proposition 4.14 *Exactly two members, $T, T' \in \binom{Z}{3}$ are transversals of $\mathcal{F}$. Moreover, either $\{T, T'\} = \big\{Z\backslash\{x\}, Z\backslash\{i\}\big\}$ or $\{T, T'\} = \big\{Z\backslash\{y\}, Z\backslash\{j\}\big\}$.*

Proof If $T \in \binom{Z}{3}$ is a transversal of $\mathcal{F}$ then saturatedness implies that all ℓ-sets containing T are in $\mathcal{G}$. In particular, $G_T := T \cup \{w_1, \ldots, w_{\ell-3}\}$. For a transversal $T \in \binom{Z}{3}$ define $z(T)$ by $T = Z\backslash\{z(T)\}$. If for two transversals T, T', $\{z(T), z(T')\} = \{i, j\}$ or $\{x, y\}$ then $G(T)$ and $G(T')$ contradict $\mathcal{G}(i) \cap \mathcal{G}(j) \neq \emptyset$ or $\mathcal{G}(x) \cap \mathcal{G}(y) \neq \emptyset$, respectively. This implies that we could not have three or more such T. Finally, if $\{z(T), z(T')\} = \{i, y\}$ or $\{j, x\}$ then there is no transversal of size 3 containing $\{i, y\}$ or $\{j, x\}$, respectively. This concludes the proof. □

Let us suppose that $T = \{i, x, j\}$ and $T' = \{i, x, y\}$.

Claim 4.15 There exist $F, F' \in \mathcal{F}$ such that $F \cap Z = \{i\}$ and $F' \cap Z = \{x\}$.

Proof By Proposition 4.14, there is some $F \in \mathcal{F}$ satisfying $F \cap \{x, j, y\} = \emptyset$. By Fact 4.10, $F \cap Z = \{i\}$. Considering the non-transversal $\{i, j, y\}$ we obtain $F' \in \mathcal{F}$ with $F' \cap Z = \{x\}$. □

Since $T \cap H = \{j\}$, $\mathcal{G}(j)$ is non-trivial. Let us show:

Claim 4.16 $\mathcal{F}(\overline{j})$ is non-trivial.

Proof Since $\mathcal{F}(i, x)$ is full, the only candidates for membership in $\cap\mathcal{F}(\overline{j})$ are i and x. However, F, F' from Claim 4.15 are in $\mathcal{F}(\overline{j})$ and $F \cap F' \cap Z = \emptyset$. □

Applying the induction hypothesis to $\mathcal{F}(\overline{j})$ and $\mathcal{G}(j)$ yields

$$|\mathcal{F}(\overline{j})| + |\mathcal{G}(j)| \leq \binom{n-1}{k} - 2\binom{n-\ell}{k} + \binom{n-2\ell+1}{k} + 2. \tag{4.14}$$

Here we distinguish two cases, namely $k > \ell$ and $k = \ell$. If $k > \ell$, i.e., $k - 1 \geq \ell$ then we apply (1.6) to the non-empty CI pair $\mathcal{F}(j)$ and $\mathcal{G}(\overline{j})$:

$$|\mathcal{F}(j)| + |\mathcal{G}(\overline{j})| \leq \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1} + 1. \tag{4.15}$$

The sum of (4.14) and (4.15) is:

$$|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{k} - 2\binom{n-\ell}{k} + \binom{n-2\ell}{k} + 2 - \left(\binom{n-\ell-1}{k-1} - \binom{n-2\ell}{k-1} - 1\right).$$

That is, to prove (4.2) we need to show,

$$\binom{n-\ell-1}{k-1} - \binom{n-2\ell}{k-1} = \binom{n-\ell-2}{k-2} + \binom{n-\ell-3}{k-2} + \ldots \binom{n-2\ell}{k-2} > 1.$$

This is true for $n \geq k + \ell + 1$, $k \geq 3$ by $\binom{n-\ell-2}{k-2} \geq \binom{k-1}{k-2} = k - 1 > 1$.

The second case is $k = \ell$.

Since $\{i, x, y\}$ is a transversal of $\mathcal{F}$, $|\mathcal{F}(j)| = |\mathcal{F}_1| + |\mathcal{F}_2|$ where

$$\mathcal{F}_1 = \left\{F \in \mathcal{F} : j \in F, F \cap \{i, x\} \neq \emptyset\right\} \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F}(\{y, j\}, Z).$$

For $G \in \mathcal{G}(\overline{j})$, Fact 4.7 implies $\{i, x\} \subset G$. Thus

$$|\mathcal{G}(\overline{j})| = \left|\mathcal{G}(\{i, x, y\}, Z)\right| + \left|\mathcal{G}(\{i, x\}, Z)\right|.$$

Obviously,

$$|\mathcal{F}_1| \leq \binom{n-1}{k-1} - \binom{n-3}{k-1},$$
$$\left|\mathcal{G}(\{i, x, y\}, Z)\right| \leq \binom{n-4}{k-3}.$$

The important observation is that $\mathcal{F}(\{y, j\}, Z)$ and $\mathcal{G}(\{i, x\}, Z)$ are cross-intersecting $(k-2)$-graphs on $[n] \setminus Z$. By (2.2), $\left|\mathcal{F}(\{y, j\}, Z)\right| + \left|\mathcal{G}(\{i, x\}, Z)\right| \leq \binom{n-4}{k-2}$. Thus we infer

$$|\mathcal{F}(j)| + |\mathcal{G}(\bar{j})| \leq \binom{n-1}{k-1} - \binom{n-3}{k-1} + \binom{n-3}{k-2}. \tag{4.16}$$

Adding this to (4.14) yields

$$|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{k} - 2\binom{n-k}{k} + \binom{n-2k}{k} + 2 - \binom{n-2k}{k-1} - \left(\binom{n-3}{k-1} - \binom{n-3}{k-2}\right),$$

proving (4.2) with strict inequality for $n > 2k$. □

5 The New Proof of Theorem 1.6

Since simultaneous shifting maintains cross-intersection we may assume that $\mathcal{F}$ and $\mathcal{G}$ are initial families. For $G \in \mathcal{G}$ define the quantity $p(G)$ as the maximal integer p with the property

$$\left|G \cap [2p + k - \ell]\right| \geq p. \tag{5.1}$$

Note that (5.1) is always satisfied for $p = 0$. This implies $0 \leq p \leq \ell$.

In the case $k = \ell$, should $p(G) = 0$ hold for some G, then $(3, 5, 7, \ldots, 2k+1) \prec G$ follows. Indeed, otherwise if $G = (x_1, \ldots, x_k)$ then for some $1 \leq p \leq k$, $x_p \leq 2p$ and thereby $|G \cap [2p]| \geq p$ would hold.

Thus $p(G) = 0$ and shiftedness imply $(3, 5, \ldots, 2k+1) \in \mathcal{G}$. We claim that $p(F) > 0$ for all $F \in \mathcal{F}$. In the opposite case we infer that $(3, 5, \ldots, 2k+1) \in \mathcal{F}$. However, $\mathcal{F}$ is initial, yielding $(2, 4, \ldots, 2k) \in \mathcal{F}$ which contradicts cross-intersection.

In the $k = \ell$ case, if necessary we interchange $\mathcal{F}$ and $\mathcal{G}$. Then we can suppose that $p(G) > 0$ for all $G \in \mathcal{G}$.

Let us now define the map $\varphi : \mathcal{G} \to \binom{[n]}{k}$ by $\varphi(G) = G\Delta[2p(G) + k - \ell]$ (Δ denotes symmetric difference).

Lemma 5.1 (i) $|\varphi(G)| = k$,
(ii) *φ is an injection with $\varphi(G) \notin \mathcal{F}$,*
(iii) *$\varphi(G) \cap [\ell] \neq \emptyset$ for $G \neq [\ell]$.*

Proof The maximal choice of p in (5.1) implies $\left|G \cap [2p(G) + k - \ell]\right| = p(G)$. Thus $\left|G\Delta[2p(G) + k - \ell]\right| = |G| + k - \ell = k$, proving (i).

Let us show $\varphi(G) \neq \varphi(G')$ for $G \neq G' \in \mathcal{G}$. If $p(G) = p(G')$ then this is evident. Suppose that $p(G) > p(G')$. Then $\varphi(G) \cap [2p(G) + k - \ell] = p(G) + k - \ell$. However $\left|G' \cap [2p(G) + (k - \ell)]\right| < p(G)$ implies $\left|\varphi(G') \cap [2p(G) + k - \ell]\right| = p(G') + k - \ell + \left|G' \cap [2p(G') + k - \ell + 1, 2p(G) + k - \ell]\right| < p(G) + k - \ell$. This shows that φ is injective.

To prove $\varphi(G) \notin \mathcal{F}$ first note that the maximal choice of $p = p(G)$ implies that for $G = (x_1, \dots, x_\ell)$, $x_{p+i} > 2p + k - \ell + 2i$, $p < i \le \ell$. Using shiftedness for $\mathcal{G}$ we infer

$$(G \cap [2p + k - \ell] \cup \{k - \ell + 2(p+1), k - \ell + 2(p+2), \dots, k - \ell + 2\ell\}) \in \mathcal{G}.$$

If $\varphi(G) \in \mathcal{F}$ then the shiftedness of $\mathcal{F}$ implies in the same way

$$\big([2p + k - \ell] \setminus G \cup \{k - \ell + 2p + 1, k - \ell + 2p + 3, \dots, k - \ell + 2\ell - 1\}\big) \in \mathcal{F}.$$

However these two sets are disjoint, a contradiction.

To prove (iii) is easy. If $G \neq [\ell]$ then let x be the minimal element of $[\ell] \setminus G$. Now $[x-1] \subset G$ implies $p(G) \ge x - 1$ and $2p(G) + k - \ell \ge x$ because $p(G) + k - \ell \ge 1$ either by $k > \ell$ or by $k = \ell$ and $p(G) \ge 1$. Hence $x \in \varphi(G)$. □

Let us deduce Theorem 1.6 from the lemma. By shiftedness $[\ell] \in \mathcal{G}$. Set $\mathcal{H} = \left\{H \in \binom{[n]}{k} : H \cap [\ell] \neq \emptyset\right\}$. By cross-intersection $\mathcal{F} \subset \mathcal{H}$. By the lemma $\varphi(\mathcal{G} \setminus \{[\ell]\}) \cap \mathcal{F} = \emptyset$ and $\varphi(\mathcal{G}) \subset \mathcal{H}$ as well. Consequently

$$|\mathcal{F}| + |\mathcal{G}| - 1 \le |\mathcal{H}| = \binom{n}{k} - \binom{n-\ell}{k} \text{ proving(1.6).} \tag{5.2}$$

Let us show that the inequality is strict if $|\mathcal{G}| > 1$ unless $k = \ell = 2$. By shiftedness $[\ell+1] \setminus \{\ell\} =: G_\ell \in \mathcal{G}$. Obviously, $p(G_\ell) = \ell$. Thus $\varphi(G_\ell) = (\ell, \ell+2, \ell+3, \dots, k+\ell) =: H_0 \notin \mathcal{F}$ by Lemma 5.1 (ii).

Proof Define $H_1 = (\ell, \ell+2) \cup [\ell+4, k+\ell+1]$. Note that $k \ge 3$ implies $[\ell+4, k+\ell+1] \neq \emptyset$. Since $H_0 \prec H_1$, $H_1 \notin \mathcal{F}$. Should equality hold in (1.6), that is, in (5.2), there is some $G_1 \in \mathcal{G}$ with $\varphi(G_1) = H_1$.

Now $H_1 = G_1 \Delta \big[2p(G_1) + k - \ell\big]$ implies $H_1 \Delta \big[2p(G_1) + k - \ell\big] = G_1$.

Using $p(G_1) + k - \ell > 0$, $1 \in G_1$ follows. This implies $p(G_1) \ge 1$ and $2p(G_1) + k - \ell \ge 2$. Using also $[\ell - 1] \cap H_1 = \emptyset$ we can prove successively $p(G_1) > \ell - 1$ and therefore $p(G_1) = \ell$. However $H_1 \Delta [k + \ell] = [\ell - 1] \cup \{\ell+1, \ell+3\}$ is an $(\ell+1)$-set contradicting $H_1 = \varphi(G_1)$. This concludes the proof. □

We should mention that in the case $k = \ell = 2$, $H_1 = (2, 4)$ and $k + \ell = 4$. Thus $H_1 = \varphi(G_1)$ with $G_1 = (1, 3)$. As a matter of fact, setting $\mathcal{F} = \mathcal{G} = \{(1, i) : 2 \le i \le n\}$ gives equality in (1.6).

6 Concluding Remarks

In the present paper we considered the problem of determining the maximum of $|\mathcal{F}| + |\mathcal{G}|$ for families $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ where $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting and non-trivial.

Recently, in a joint paper with Jian Wang [14] we proved the following result concerning the product $|\mathcal{F}||\mathcal{G}|$.

Theorem 6.1 *([14]) Let $n \geq 4k$, $k \geq 8$, and suppose that $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are cross-intersecting and non-trivial. Then*

$$|\mathcal{F}||\mathcal{G}| \leq \left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right)^2. \tag{6.1}$$

Note that in the above range (6.1) implies (1.2), that is, the Hilton–Milner Theorem.

Note that (6.1) can be proved easily for $k = 2$, $n \geq 4$. However, we do not know whether it holds for all (n, k) satisfying $n \geq 2k \geq 4$.

We hope that the inequality (4.2) will find application in a wide range of extremal problems. As a matter of fact, in [15] it played an important role in the complete solution concerning the maximum of $|\mathcal{F}|$, $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ is intersecting and has covering number at least 3.

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