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ORIGINAL PAPER

Separating Polynomial χ-Boundedness from χ-Boundedness

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Abstract

Extending the idea from the recent paper by Carbonero, Hompe, Moore, and Spirkl, for every function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ with $f(1) = 1$ and $f(n) \geqslant \binom{3n+1}{3}$, we construct a hereditary class of graphs $\mathcal{G}$ such that the maximum chromatic number of a graph in $\mathcal{G}$ with clique number n is equal to $f(n)$ for every $n \in \mathbb{N}$. In particular, we prove that there exist hereditary classes of graphs that are χ-bounded but not polynomially χ-bounded.

Keywords Graph colouring · clique number · $\mathcal{X}$-bounded · polynomially $\mathcal{X}$-bounded

Mathematics Subject Classification 05C15

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1 Introduction

Given a class of graphs $\mathcal{C}$ its *χ-bounding function* is the function $\chi_{\mathcal{C}}\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined as

$$\chi_{\mathcal{C}}(n) = \sup\{\chi(G)\colon G \in \mathcal{C} \text{ and } \omega(G) = n\},$$

where $\chi(G)$ and $\omega(G)$ denote, respectively, the chromatic number and the clique number of G. A class of graphs $\mathcal{C}$ is *χ-bounded* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leqslant f(\omega(G))$ for every graph $G \in \mathcal{C}$, or equivalently if $\chi_{\mathcal{C}}(n)$ is finite for every $n \in \mathbb{N}$. A class $\mathcal{C}$ is *polynomially χ-bounded* if such a function f can be chosen to be a polynomial. A class $\mathcal{C}$ is *hereditary* if it is closed under taking induced subgraphs.

A well-known and fundamental open problem, due to Esperet [6], has been to decide whether every hereditary class of graphs which is χ-bounded is polynomially χ-bounded. We provide a negative answer to this question. More generally, we prove that χ-bounding functions may be arbitrary, so long as they are bounded from below by a certain cubic function.

Theorem 1 *Let $f\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ be such that $f(1) = 1$ and $f(n) \geqslant \binom{3n+1}{3}$ for every $n \geqslant 2$. Then there exists a hereditary class of graphs $\mathcal{G}$ such that $\chi_{\mathcal{G}}(n) = f(n)$ for every $n \in \mathbb{N}$.*

On the other hand, χ-bounding functions are not entirely arbitrary. For instance, Scott and Seymour [11] proved that every hereditary class of graphs $\mathcal{C}$ with $\chi_{\mathcal{C}}(2) = 2$ satisfies $\chi_{\mathcal{C}}(n) \leqslant 2^{2^{n+1}}$.

The proof of Theorem 1 is heavily based on the idea used by Carbonero, Hompe, Moore, and Spirkl [2] in their very recent solution to another well-known problem attributed to Esperet [12]. They proved that for every $k \in \mathbb{N}$, there is a graph G with $\omega(G) = 3$ and $\chi(G) \geqslant k$ such that every triangle-free induced subgraph of G has chromatic number at most 4. Their proof, in turn, relies on an idea by Kierstead and Trotter [8], who proved in 1992 that the class of oriented graphs excluding a directed path on four vertices as an induced subgraph is not χ-bounded. We further generalise the aforesaid result of Carbonero, Hompe, Moore, and Spirkl [2] to higher clique numbers. Specifically, we prove the following general bound, which we use to derive Theorem 1.

Theorem 2 *For every pair of integers n and k with $k \geqslant n \geqslant 2$, there exists a graph G with clique number n and chromatic number k such that every induced subgraph of G with clique number $m < n$ has chromatic number at most $\binom{3m+1}{3}$.*

In case that n is a prime number, we prove a better bound, which matches the bound of 4 from [2] when $n = 3$.

Theorem 3 *For every pair of integers p and k with p a prime and $k \geqslant p$, there exists a graph G with clique number p and chromatic number k such that every induced subgraph of G with clique number $m < p$ has chromatic number at most $\binom{m+2}{3}$.*

In the first version of this paper [1], we proved a weaker version of Theorem 3 with $\binom{m+2}{3}$ replaced by m^{m^2}. Despite the worse bound obtained, that alternative proof may still be of interest. The mere qualitative statement that for every prime p, there are graphs with clique number p and arbitrarily large chromatic number whose induced subgraphs with clique number less than p have bounded chromatic number suffices to imply the negative answer to Esperet's question.

After [1] appeared, Girão et al. [7] proved another generalisation of the aforesaid qualitative version of Theorem 3. Namely, they proved that for every graph F with at least one edge, there are graphs of arbitrarily large chromatic number and the same clique number as F in which every F-free induced subgraph has chromatic number at most some constant c_F depending only on F. They also showed the analogous statement where clique number is replaced by odd girth.

See [12] and [9] for recent surveys on χ-boundedness and polynomial χ-boundedness.

2 Proof

First, we show that Theorem 2 implies Theorem 1.

Proof of Theorem 1 Assuming Theorem 2 Fix a function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ such that $f(1) = 1$ and $f(n) \geqslant \binom{3n+1}{3}$ for every $n \geqslant 2$. By Theorem 2, for every pair of integers n and k with $k \geqslant n \geqslant 2$, there exists a graph $H_{n,k}$ with clique number n and chromatic number k such that every induced subgraph of $H_{n,k}$ with clique number $m < n$ is $\binom{3m+1}{3}$-colourable.

We now consider two cases. If $f(n)$ is finite, we put $\mathcal{H}_n = \{H_{n,f(n)}\}$. Otherwise $f(n) = \infty$, and we put $\mathcal{H}_n = \{H_{n,k} : k \geqslant n\}$. Finally, we let $\mathcal{H} = \bigcup_{n=2}^{\infty} \mathcal{H}_n$, and we let $\mathcal{G}$ be the hereditary closure of $\mathcal{H}$, i.e., the family of all induced subgraphs of the graphs in $\mathcal{H}$.

We now argue that $\chi_{\mathcal{G}}(n) = f(n)$ for all $n \in \mathbb{N}$. The claim holds trivially for $n = 1$, so assume $n \geqslant 2$. If $f(n) = \infty$, then the sequence of graphs $\{H_{n,k} : k \geqslant n\} \subseteq \mathcal{G}$ all have clique number equal to n and have unbounded chromatic number, thus showing that $\chi_{\mathcal{G}}(n) = \infty$, as claimed. Otherwise, $f(n)$ is finite. The graph $H_{n,f(n)} \in \mathcal{G}$ shows that $\chi_{\mathcal{G}}(n) \geqslant f(n)$. For the reverse inequality, let $G \in \mathcal{G}$ be such that $\omega(G) = n$. Then there exist integers k and n^* with $k \geqslant n^* \geqslant n$ such that G is an induced subgraph of $H_{n^*,k} \in \mathcal{H}$. The unique graph of $\mathcal{H}$ with clique number n is $H_{n,f(n)}$. So if $n^* = n$, then $\chi(G) \leqslant \chi(H_{n,f(n)}) = f(n)$, and if $n^* > n$, then $\chi(G) \leqslant \binom{3n+1}{3}$. Combining these inequalities, we conclude that

$$f(n) \leqslant \chi_{\mathcal{G}}(n) \leqslant \max\{\tbinom{3n+1}{3},\ f(n)\} = f(n),$$

and the theorem follows. □

The rest of the paper is devoted to proving Theorem 2. We begin with the following lemma.

Lemma 4 *For every positive integer k, there is a graph G_k and an acyclic orientation of its edges with the following properties:*

(1) $\chi(G_k) = k$;
(2) *for every pair of vertices u and v, there is at most one directed path from u to v in G_k;*
(3) *there is a directed path in G_k on k vertices;*
(4) *there is a k-colouring ϕ of G_k such that $\phi(u) \neq \phi(v)$ for any two distinct vertices u and v such that there is a directed path from u to v in G_k.*

Various well-known constructions of triangle-free graphs with arbitrarily high chromatic number, such as Zykov's [13] and Tutte's [3, 4], satisfy the condition of Lemma 4 once the edges are oriented in a way that follows naturally from the construction. See [2] and [8] for an explicit construction of the graphs G_k with the appropriate acyclic orientations, based on Zykov's construction. It is only implicit that the acyclic orientations of the graphs in [2] and [8] satisfy all of the properties in the conclusion of Lemma 4, so for the sake completeness we provide a proof based on Tutte's construction.

Proof of Lemma 4 We proceed by induction on k. The base case $k = 1$ follows by taking a single-vertex graph as G_1. For the induction step, assume G_{k-1} is an acyclically oriented graph satisfying conditions (1)–(4) for $k - 1$. To construct G_k, begin with a stable set S with $|S| = (k-1)(|V(G_{k-1})| - 1) + 1$, and for every subset X of S with $|X| = |V(G_{k-1})|$, add an isomorphic copy G_X of G_{k-1} (with the same orientation as in G_{k-1}) and an arbitrary perfect matching between the vertices in X and the vertices of G_X, oriented from X to G_X. This clearly preserves acyclicity of the orientation. Since the vertices in S have in-degree zero, either every directed path is contained in some copy G_X of G_{k-1}, or the starting vertex u is contained in S and every other vertex is contained in some copy G_X of G_{k-1}. As every vertex in S has at most one edge to each copy G_X of G_{k-1}, the induction hypothesis implies that condition (2) is preserved. Any directed path on $k - 1$ vertices in G_X extends to a directed path on k vertices in G_k by adding a vertex from S, so (3) holds. Any colouring of the copies G_X of G_{k-1} with a common palette of $k - 1$ colours extends to a k-colouring of G_k by using a single new colour on S, which shows that $\chi(G_k) \leqslant \chi(G_{k-1}) + 1$ and condition (4) is preserved. Finally, suppose there exists a $(k - 1)$-colouring of G_k. Then, since $|S| > (k-1)(|V(G_{k-1})| - 1)$, there is a monochromatic set $X \subset S$ with $|X| = |V(G_{k-1})|$. Since X and G_X are connected by a perfect matching, at most $k-2$ colours are used on G_X, which contradicts the fact that $\chi(G_X) = \chi(G_{k-1}) = k - 1$. Hence $\chi(G_k) = k$, as claimed in (1). □

For the rest of the argument, we fix an arbitrary sequence $(G_k)_{k\in\mathbb{N}}$ of graphs given by Lemma 4. Now, for every pair of positive integers k and p, where p is a prime number, we construct a graph $G_{k,p}$ by adding edges to G_k as follows.

Let $\leqslant$ be the directed reachability order of the vertices of G_k, that is, $u \leqslant v$ if and only if there is a (unique) directed path from u to v in G_k. Since the orientation of G_k given by Lemma 4 is acyclic, $\leqslant$ is indeed a partial order. For every pair of vertices u and v in G_k such that $u \leqslant v$, let $d(u, v)$ be the length of the unique directed path from u to v in G_k (i.e., the number of edges in that path). The graph $G_{k,p}$ has the same vertex set as G_k and has the set $\{uv : u < v \text{ and } d(u, v) \not\equiv 0 \pmod{p}\}$ as the edge set. We consider each such edge uv as oriented from u to v. Since the original (oriented) edges uv of G_k satisfy $u < v$ and $d(u, v) = 1$, the graph $G_{k,p}$ contains G_k as a subgraph. Furthermore, every edge of $G_{k,p}$ connects vertices with different colours in a k-colouring ϕ of G_k claimed in Lemma 4. Therefore, $\chi(G_{k,p}) = k$. Furthermore, $G_{k,p}$ is acyclic since G_k is acyclic.

Next, we examine cliques in $G_{k,p}$ (and its induced subgraphs). Since $G_{k,p}$ is acyclic, every clique of $G_{k,p}$ induces a transitive tournament. Given a clique C of an acyclic oriented graph, we let $t(C)$ be the unique in-degree zero vertex of the transitive tournament induced by C. We call $t(C)$ the *tail* of C. Given a clique C of $G_{k,p}$, we let $r(C)$ be the subset of $\mathbb{Z}_p$ such that $r(C) \equiv \{d(t(C), v) : v \in C\} \pmod{p}$. We call $r(C)$ the *residue* of the clique C. Note that 0 is always contained in $r(C)$ since $t(C) \in C$. Furthermore, $|C| = |r(C)|$, otherwise there would exist two distinct vertices $u, v \in C$ such that $d(t(C), u) \equiv d(t(C), v) \pmod{p}$, and so $d(u, v) \equiv 0 \pmod{p}$, which would contradict the fact that u and v are adjacent. This observation allows us to determine the clique number of $G_{k,p}$.

Lemma 5 *For every positive integer k and every prime $p \leqslant k$, the graph $G_{k,p}$ has clique number p.*

Proof Since G_k contains a directed path on k vertices and $p \leqslant k$, the graph $G_{k,p}$ contains a clique of size p. Conversely, if C is a clique in $G_{k,p}$, then $|C| = |r(C)| \leqslant |\mathbb{Z}_p| = p$. □

A *rotation* of a subset X of $\mathbb{Z}_p$ is a subset of $\mathbb{Z}_p$ of the form $X + a = \{x + a : x \in X\}$ for any $a \in \mathbb{Z}_p$. A subset of $\mathbb{Z}_p$ is *rooted* if it contains 0. The rotation $X + a$ of a rooted subset X of $\mathbb{Z}_p$ is rooted if and only if $-a \in X$. Let $\sim_p$ be the equivalence relation on the rooted subsets of $\mathbb{Z}_p$ such that $X \sim_p Y$ whenever Y is a rotation of X. Let $[X]_p$ denote the equivalence class of X in $\sim_p$. For every proper rooted subset X of $\mathbb{Z}_p$ (such that $X \neq \mathbb{Z}_p$), since p is a prime, all rotations $X + a$ of X with $a \in \mathbb{Z}_p$ are distinct. (Indeed, if $X + a = X$, then $\sum_{x \in X} x \equiv \sum_{x \in X}(x + a) \equiv \sum_{x \in X} x + a \cdot |X| \pmod{p}$, so $a \cdot |X| \equiv 0 \pmod{p}$, which yields $a \equiv 0 \pmod{p}$.) In particular, we have $|[X]_p| = |X|$. Order every equivalence class arbitrarily, and for every proper rooted subset X of $\mathbb{Z}_p$, let $c(X) \in \{1, \ldots, |X|\}$ denote the position of X in this ordering.

Lemma 6 *For every positive integer k, every prime p, and every induced subgraph G of $G_{k,p}$ with clique number $m < p$, we have $\chi(G) \leqslant \binom{m+2}{3}$.*

Proof We will colour the vertices of G by triples of integers (a, b, c) with $m \geqslant a \geqslant b \geqslant c \geqslant 1$. Since there are $\binom{m+2}{3}$ choices for such a triple, this will be a $\binom{m+2}{3}$-colouring of G.

For each vertex v of G, let $a(v)$ be the maximum size of a clique in G with tail v. Thus $m \geqslant a(v) \geqslant 1$. Let $B(v)$ be the intersection of the residues of all cliques of size $a(v)$ with tail v in G. Since 0 belongs to the residue of every clique, we have $0 \in B(v)$. Let $b(v) = |B(v)|$, so that $a(v) \geqslant b(v) \geqslant 1$. Let $c(v) = c(B(v))$, so that $b(v) \geqslant c(v) \geqslant 1$, as $|[B(v)]_p| = |B(v)| = b(v)$. Finally, let $\psi(v) = (a(v), b(v), c(v))$. We have

$m \geqslant a(v) \geqslant b(v) \geqslant c(v) \geqslant 1$ for every v, so it remains to show that ψ is a proper colouring of G.

Suppose for the sake of contradiction that some two vertices u and v of G with $\psi(u) = \psi(v)$ are connected by an edge of G oriented from u to v. Let $d \in \mathbb{Z}_p$ be such that $d(u, v) \equiv d \pmod p$. Since u and v are adjacent in G, we have $d \neq 0$. Observe that if C is a clique in G with residue X and tail v, then prepending u to C and possibly removing the unique vertex w in C with $d(v, w) \equiv -d \pmod p$ (if it exists) gives us a clique with residue $(X + d) \cup \{0\}$ and tail u. Therefore, since $a(u) = a(v)$, the residue of every clique of size $a(v)$ with tail v must contain $-d$. Thus $-d \in B(v)$, and if X is the residue of a clique of size $a(v)$ with tail v, then $X + d$ is the residue of a clique of the same size with tail u. Hence $B(u) \subseteq B(v) + d$, and since $b(u) = b(v)$, we further conclude that $B(u) = B(v) + d$. Since 0 belongs to the residue of every clique, both $B(u)$ and $B(v)$ are rooted and $B(u) \sim_p B(v)$. Thus $B(u) = B(v)$, as $c(u) = c(v)$. However, since $b(u) = b(v) \leqslant m < p$ and $d \neq 0$, we have $B(u) = B(v) + d \neq B(v)$, which is a contradiction. This shows that ψ is a proper colouring of G, as desired. □

By combining Lemmas 5 and 6, we have so far proven Theorem 3. Next, we extend the construction to non-primes in order to prove Theorem 2.

For every triple of positive integers k, n, p with p prime and $p > 2n$, we construct a subgraph $G_{k,n,p}$ of $G_{k,p}$ by including the edge uv if and only if $d(u, v) \equiv x \pmod p$ where $x \in \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$. We will now determine the clique number and the chromatic number of $G_{k,n,p}$.

Lemma 7 *Let k, n, and p be positive integers with p prime, $p > 2n$, and $k \geqslant n$. Then $G_{k,n,p}$ has clique number n and chromatic number k.*

Proof We have $\chi(G_k) = \chi(G_{k,p}) = k$. Since G_k is a subgraph of $G_{k,n,p}$ and $G_{k,n,p}$ is a subgraph of $G_{k,p}$, it follows that $\chi(G_{k,n,p}) = k$. Next, we determine the clique number of $G_{k,n,p}$.

Let $I = \{n, n+1, \ldots, p-n\} \subset \mathbb{Z}_p$. Note that uv is an edge of $G_{k,n,p}$ if and only if $u < v$ and $d(u, v) \notin \{0\} \cup I \pmod p$. Since G_k contains a directed path on k vertices and $n \leqslant k$, the graph $G_{k,n,p}$ has clique number at least n. It remains to show that $G_{k,n,p}$ has clique number at most n.

Let C be a clique in $G_{k,n,p}$, and let $v = t(C)$. If there are vertices $x, y \in C$ with $d(v, y) \in I + d(v, x) \pmod p$, then either $y < x$ and $d(y, x) = d(v, x) - d(v, y) = -(d(v, y) - d(v, x)) \in -I = I \pmod p$, or $x < y$ and $d(x, y) = d(v, y) - d(v, x) \in I \pmod p$. In either case, xy is not an edge of $G_{k,n,p}$, contradicting the assumption that C is a clique. Thus the set $r(C)$ is disjoint from the set $\bigcup_{x \in C}(I + d(v, x)) = I + r(C)$, which implies that $r(C)$ contains at most one of i and $p - n + i$ for each $i \in \{1, 2, \ldots, n-1\}$. Since $r(C) \subset \{0, 1, \ldots, n-1, p-n+1, p-n+2, \ldots, p-1\}$, we conclude that $|C| = |r(C)| \leqslant n$. □

Next we examine the maximum size of a clique in an induced subgraph of $G_{k,n,p}$ that is induced by the vertices of a clique in $G_{k,p}$. This will allow us to compare the chromatic number of induced subgraphs of $G_{k,p}$ and $G_{k,n,p}$ that have the same vertex set.

Lemma 8 *Let k, n, and p be positive integers with p prime, $p > 2n$, and $k \geqslant n$. Then for every clique C of $G_{k,p}$, the induced subgraph $G_{k,n,p}[C]$ of $G_{k,n,p}$ contains a clique of size at least $\frac{n}{p}|C|$.*

Proof Let C be a clique in $G_{k,p}$. For each $i \in \mathbb{Z}_p$, let $J_i = \{i, i+1, \ldots, i+n-1\} \subset \mathbb{Z}_p$. Since each $i \in \mathbb{Z}_p$ is contained in exactly n of the p sets $J_0, \ldots, J_{p-1}$, by the pigeonhole principle, there exists $i \in \mathbb{Z}_p$ such that $|r(C) \cap J_i| \geqslant \frac{n}{p}|r(C)|$. Let $C_i = \{v \in C : d(t(C), v) \in J_i \pmod p\}$. It follows that $|C_i| = |r(C) \cap J_i| \geqslant \frac{n}{p}|r(C)| = \frac{n}{p}|C|$. It remains to show that C_i is a clique in $G_{k,n,p}$.

Let x and y be distinct vertices in C_i. Since $x, y \in C$, they are adjacent in $G_{k,p}$, so $d(t(C), x) \not\equiv d(t(C), y) \pmod p$, and we can assume without loss of generality that $x < y$. It follows that $d(x, y) = d(t(C), y) - d(t(C), x) \in \{\pm 1, \pm 2, \ldots, \pm(n-1)\} \pmod p$, as $d(t(C), x),\ d(t(C), y) \in J_i \pmod p$. Hence x and y are adjacent in $G_{k,n,p}$. We conclude that C_i is indeed a clique in $G_{k,n,p}$. □

Lemma 9 *Let k, n, and p be positive integers with p prime, $p > 2n$, and $k \geqslant n$, and let G be an induced subgraph of $G_{k,n,p}$ with $m = \omega(G) < n$. Then $\chi(G) \leqslant \binom{\lfloor mp/n \rfloor + 2}{3}$.*

Proof Let $G' = G_{k,p}[V(G)]$. Lemma 8 yields $\omega(G') \leqslant \lfloor mp/n \rfloor$. The fact that G is a subgraph of G' and Lemma 6 yield $\chi(G) \leqslant \chi(G') \leqslant \binom{\lfloor mp/n \rfloor + 2}{3}$. □

Theorem 2 now follows from Lemma 7, Lemma 9, and the following theorem of Schur [10] on the gaps between prime numbers.

Theorem 10 *For every integer $n \geqslant 2$, there is a prime p such that $2n < p < 3n$.*

3 Concluding Remarks

To better understand χ-bounding functions, it is of course of interest to improve the bound of $\binom{3m+1}{3}$ in Theorem 2 (and equivalently this same lower bound function for f in Theorem 1).

A slight tweak to the last step of the proof improves this bound slightly to $\binom{2m}{3} + o(m^3)$. To do this, instead of using Theorem 10, we can use the fact that for any $\epsilon > 0$, there exists a n_ϵ such that for every $n \geqslant n_\epsilon$, there is always a prime p with $2n < p < (2+\epsilon)n$. This follows from the prime number theorem that the number of primes at most n is asymptotically equal to $n/\ln n$. For a more recent and explicit result on the gaps between primes, see [5].

One may hope that another way to further improve this bound would be to improve the bound of $\binom{m+2}{3}$ in Lemma 6. However, in our construction, Lemma 6 is in some sense best possible. For every prime p, we have been able to construct a graph G'_k (with k large enough) that satisfies the conclusion of Lemma 4, and such that for every positive integer $m < p$, the graph $G'_{k,p}$ (as constructed from G'_k) contains an induced subgraph with clique number m and chromatic number $\binom{m+2}{3}$. So any improvements would require an entirely new construction.

In the other direction, the only result restricting χ-bounding functions is that of Scott and Seymour [11] stating that if a hereditary class of graphs $\mathcal{C}$ satisfies $\chi_{\mathcal{C}}(2) \leqslant 2$, then $\mathcal{C}$ is χ-bounded. We conjecture the following generalisation.

Conjecture 11 *For every integer $k \geqslant 2$, if $\mathcal{C}$ is a hereditary class of graphs such that $\chi_{\mathcal{C}}(n) \leqslant k$ for every positive integer $n \leqslant k$, then the class $\mathcal{C}$ is χ-bounded.*

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