ERRATUM

HIGH ORDER RANDOM WALKS: BEYOND SPECTRAL GAP

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This is an erratum to [1, Theorem 1.5 (3), Theorem 5.10, Corollary 5.11].

We recall that for a two-sided λ -local spectral expander, we defined constants:

$$\varepsilon_k = \begin{cases} \lambda & k = 0 \\ 2k(1 + 2k\sqrt{k})\varepsilon_{k-1} + (k+1)\lambda & 0 < k \end{cases}$$

(see [1, Theorem 5.6]). With this definition, the corrected version of [1, Theorem 5.10] is:

Theorem 1. *Let X is a two-sided λ -local spectral expander, and let $\{\varepsilon_k : 0 \leq k \leq n-1\}$ be the constants defined above. Assume that $\varepsilon_k \leq \frac{1}{2(1+2(k+1)\sqrt{k+1})}$ for all $0 \leq k \leq n-2$ and $\varepsilon_{n-1} < \frac{1}{2\sqrt{n}}$. For every $0 \leq k \leq n-1$ and $0 \leq j_0 \leq k$, if $\varphi \in U_k^{j_0}$, then*

$$\|\varphi - P_{W_k^{j_0}} \varphi\| \leq \sqrt{k+2}\varepsilon_k \|\varphi\|.$$

Moreover, for every $\phi \in C_0^k(X)$,

$$\begin{aligned} \|P_{W_k^j} \phi\|^2 &\leq \|P_{U_k^j} \phi\|^2 + \left(\sqrt{k+2}\varepsilon_k + \frac{\sqrt{k+1}\varepsilon_k}{1 - \sqrt{k+1}\varepsilon_k} \right) \|\phi\|^2, \\ \|P_{W_k^j} \phi\|^2 &\geq \|P_{U_k^j} \phi\|^2 - \left(\sqrt{k+2}\varepsilon_k + \frac{\sqrt{k+1}\varepsilon_k}{1 - \sqrt{k+1}\varepsilon_k} \right) \|\phi\|^2. \end{aligned}$$

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Proof. The proof of the inequality

$$(1) \quad \|\varphi - P_{W_k^{j_0}} \varphi\| \leq \sqrt{k+2}\varepsilon_k \|\varphi\|, \forall \varphi \in U_k^{j_0}$$

given in [1] is correct and thus we are left to prove the second and third inequalities. Fix $0 \leq j \leq k$. To avoid cumbersome notation, we denote $U = U_k^j, W = W_k^j$. Note that by (1), it holds that

$$\|(I - P_W)P_U\| \leq \sqrt{k+2}\varepsilon_k,$$

and thus

$$(2) \quad \|P_U(I - P_W)\| \leq \sqrt{k+2}\varepsilon_k.$$

By [1, Theorem 5.9], it holds that

$$\|(I - P_U)P_W\phi\| \leq \frac{\sqrt{k+1}\varepsilon_k}{1 - \sqrt{k+1}\varepsilon_k} \|P_W\phi\|,$$

and thus

$$(3) \quad \|P_W(I - P_U)\| = \|(I - P_U)P_W\| \leq \frac{\sqrt{k+1}\varepsilon_k}{1 - \sqrt{k+1}\varepsilon_k}.$$

We note that

$$\begin{aligned} \|P_W\phi\|^2 &= \langle P_W\phi, \phi \rangle = \\ &\langle P_W\phi, P_U\phi \rangle + \langle P_W\phi, (I - P_U)\phi \rangle = \\ \|P_U\phi\|^2 &+ \langle (P_W - P_U)\phi, P_U\phi \rangle + \langle (I - P_U)P_W\phi, \phi \rangle = \\ \|P_U\phi\|^2 &+ \langle P_U(P_W - I)\phi, \phi \rangle + \langle (I - P_U)P_W\phi, \phi \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|P_W\phi\|^2 &\leq \|P_U\phi\|^2 + |\langle P_U(I - P_W)\phi, \phi \rangle| + |\langle (I - P_U)P_W\phi, \phi \rangle| \\ &\leq \|P_U\phi\|^2 + (\|P_U(I - P_W)\| + \|(I - P_U)P_W\|) \|\phi\|^2 \\ &\stackrel{(2),(3)}{\leq} \|P_U\phi\|^2 + \left(\sqrt{k+2}\varepsilon_k + \frac{\sqrt{k+1}\varepsilon_k}{1 - \sqrt{k+1}\varepsilon_k} \right) \|\phi\|^2, \end{aligned}$$

as needed. Similarly,

$$\|P_W\phi\|^2 \geq \|P_U\phi\|^2 - \left(\sqrt{k+2}\varepsilon_k + \frac{\sqrt{k+1}\varepsilon_k}{1 - \sqrt{k+1}\varepsilon_k} \right) \|\phi\|^2. \quad \blacksquare$$

Next, we prove a corrected version of Corollary [1, Corollary 5.11]:

Corollary 2. *Let X is a two-sided λ -local spectral expander, and let $\{\varepsilon_k : 0 \leq k \leq n-1\}$ be the constants defined above. Assume that $\varepsilon_k \leq \frac{1}{2(1+2(k+1)\sqrt{k+1})}$ for all $0 \leq k \leq n-2$ and $\varepsilon_{n-1} < \frac{1}{2\sqrt{n}}$. For $0 \leq k \leq n-1$ and $i \in \mathbb{N}$, define*

$$b_k(i, \lambda) = \sum_{j=0}^k \left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} \left(\frac{\sqrt{k+1} \varepsilon_k}{1 - \sqrt{k+1} \varepsilon_k} \right).$$

Then for every $0 \leq k \leq n-1$, every $i \in \mathbb{N}$ and every $\phi \in C_0^k(X)$,

$$\|(M_k^+)^i \phi\| \leq \sqrt{\sum_{j=0}^k \left(\left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} + b_k(i, \lambda) \right) \|P_{U_k^j} \phi\|^2},$$

and

$$\|(M_k^+)^i \phi\| \geq \sqrt{\sum_{j=0}^k \left(\frac{k+1-j}{k+2} - \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} (1 - b_k(i, \lambda)) \|P_{U_k^j} \phi\|^2}.$$

Proof. The proofs of the two inequalities are similar and we will prove only the first one and leave the second one to the reader. Fix $0 \leq k \leq n-1$. Every $\phi \in C_0^k$ has two orthogonal decompositions:

$$\phi = \sum_{j=0}^k P_{W_k^j} \phi,$$

and

$$\phi = \sum_{j=0}^k P_{U_k^j} \phi.$$

By definition, W_k^j is an invariant subspace spanned by eigenvectors with eigenvalues in $\left[\frac{k+1-j}{k+2} - \frac{\sqrt{k+1}}{k+2} \varepsilon_k, \frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right]$. Thus $M_k^+ P_{W_k^j} = P_{W_k^j} M_k^+ P_{W_k^j}$ and $\|M_k^+ P_{W_k^j}\| \leq \frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k$. Combining these facts yields

$$\begin{aligned} \|(M_k^+)^i \phi\|^2 &= \sum_{j=0}^k \left\| P_{W_k^j} (M_k^+)^i \phi \right\|^2 = \sum_{j=0}^k \left\| (P_{W_k^j} M_k^+)^i P_{W_k^j} \phi \right\|^2 \\ &\leq \sum_{j=0}^k \left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} \|P_{W_k^j} \phi\|^2. \end{aligned}$$

By Theorem 1,

$$\|P_{W_k^j}\phi\|^2 \leq \|P_{U_k^j}\phi\|^2 + \left(\sqrt{k+2}\varepsilon_k + \frac{\sqrt{k+1}\varepsilon_k}{1-\sqrt{k+1}\varepsilon_k}\right)\|\phi\|^2,$$

and it follows that

$$\begin{aligned} & \| (M_k^+)^i \phi \|^2 \\ & \leq \sum_{j=0}^k \left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} \left(\|P_{U_k^j}\phi\|^2 + \left(\frac{\sqrt{k+1}\varepsilon_k}{1-\sqrt{k+1}\varepsilon_k} \right) \|\phi\|^2 \right) \\ & = \sum_{j=0}^k \left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} \|P_{U_k^j}\phi\|^2 \\ & \quad + \sum_{j=0}^k \left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} \left(\frac{\sqrt{k+1}\varepsilon_k}{1-\sqrt{k+1}\varepsilon_k} \right) \|\phi\|^2 \\ & = \sum_{j=0}^k \left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} \|P_{U_k^j}\phi\|^2 + b_k(i, \lambda) \|\phi\|^2. \end{aligned}$$

Using the fact that

$$\|\phi\|^2 = \sum_{j=0}^k \|P_{U_k^j}\phi\|^2,$$

completes the proof. ■

Remark 3 (Mistakes in the original proof and false inequalities).

The mistake in the original proof of [1, Theorem 5.10] was very silly: We showed that for every $j' \neq j$ it followed that $\|P_{W_k^{j'}}P_{U_k^j}\| \leq \sqrt{k+2}\varepsilon_k$ (which is correct), but falsely deduced from it that $\|P_{W_k^j}P_{U_k^{j'}}\phi\| \leq \sqrt{k+2}\varepsilon_k\|P_{U_k^j}\phi\|$. The mistake came from a silly indexation mistake – we confused $P_{W_k^j}P_{U_k^{j'}}$ with $P_{W_k^{j'}}P_{U_k^j}$. We then got the following **false** inequalities: for every ϕ

$$\|P_{W_k^j}\phi\| \leq (1 + (k+1)\sqrt{k+2}\varepsilon_k)\|P_{U_k^j}\phi\|,$$

and

$$\|P_{W_k^j}\phi\| \geq (1 - (k+1)\sqrt{k+2}\varepsilon_k)\|P_{U_k^j}\phi\|.$$

We did not notice it when submitting the paper, but if these inequalities were correct they would imply that $U_k^j = W_k^j$. Indeed, let $\phi \perp U_k^j$, then by

the first inequality it follows that $\phi \perp W_k^j$. Similarly, if $\phi \perp W_k^j$, then by the second inequality (for ε_k sufficiently small) it follows that $\phi \perp U_k^j$. Thus, $(W_k^j)^\perp = (U_k^j)^\perp$ and $U_k^j = W_k^j$. However, we do not expect $U_k^j = W_k^j$ to hold in general, since U_k^j are only approximations of the eigenspaces and not the actual eigenspaces.

Remark 4. Note that for every $j > 0$,

$$\lim_{i \rightarrow \infty} \frac{b_k(\lambda, i)}{\left(\frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i}} = \infty.$$

This means that when performing many steps of the random walk, the decomposition to U_k^i spaces does not give a good approximation of the decomposition to eigenspaces. In other words, the fact that the spaces U_k^i and W_k^i are not identical (but only very close to one another) comes into play when performing many steps. This behavior did not appear in the incorrect version of the Corollary in [1].

Theorem [1, Theorem 1.5 (3)] should also be corrected to the statement in Corollary 2.

References

- [1] T. KAUFMAN and I. OPPENHEIM: High order random walks: beyond spectral gap, *Combinatorica* **40** (2020), 245–281.

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