Formulation and experimental validation of space-fractional Timoshenko beam model with functionally graded materials effects

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Abstract

In this study, the static bending behaviour of a size-dependent thick beam is considered including FGM (Functionally Graded Materials) effects. The presented theory is a further development and extension of the space-fractional (non-local) Euler–Bernoulli beam model (s-FEBB) to space-fractional Timoshenko beam (s-FTB) one by proper taking into account shear deformation. Furthermore, a detailed parametric study on the influence of length scale and order of fractional continua for different boundary conditions demonstrates, how the non-locality affects the static bending response of the s-FTB model. The differences in results between s-FTB and s-FEBB models are shown as well to indicate when shear deformations need to be considered. Finally, material parameter identification and validation based on the bending of SU-8 polymer microbeams confirm the effectiveness of the presented model.

Keywords Timoshenko beam · Non-local model · Fractional calculus · Microbeam bending

Abbreviations

FGM	Functionally graded materials
s-FEBB	Space-fractional Euler–Bernoulli beam model
s-FTB	Space-fractional Timoshenko beam
SE	Scale effect
RVE	Representative volume element
BD	Body dimensions
LS	Length scale
FDO	Fractional differential operator
CTB	Classical Timoshenko beam

1 Introduction

The first works on scale effect (SE), which manifest dependence on the answer of the system (e.g. deformation of a material body - the crux of this paper) in relation to its dimensions, date to the investigations of Leonardo da Vinci

at the very beginning of XVI century [1,2]. Since that time both experimental techniques [3–7] and theoretical concepts [8–13] for SE analysis were considerably developed and moreover SE phenomena were utilised successfully in the modern industry [14–18]. The main message is that from the standpoint of mechanics, which constitutes the central point of presented considerations, one can say that each structure, due to the complexity of material over different scales of observation, reveals SE. On the other hand, it is clear that the strength of SE phenomena is different depending on the analysed case and proportional to the ratio of the representative volume element (RVE) of a specific material to body dimensions (BD).

From the theoretical side, when constructing a mathematical model for the description of mechanical phenomena, one should choose certain mathematical objects proper to the experimental observation scale [19–22]. Herein, we operate on meso/macro level, therefore for SE modelling a phenomenological approach is used, thus in consequence RVE to BD ratio is mapped utilising so-called length scale (LS) parameter (it is clear that LS meaning is different depending on certain theory [15,23–26]). To be precise, the developed s-FTB theory is defined in the framework of space-Fractional Continuum Mechanics (s-FCM) [27,28], where LS is introduced through the fractional differential operator (FDO) and furthermore, an additional parameter which controls SE is

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introduced, namely order of FDO. Finally, because of the complex nature of mechanical properties through the beam thickness FGM concept is also used [29–32].

This paper is a continuation of previous studies focused on the development of non-local beam models [27,28]. The currently formulated space-Fractional Timoshenko beam model, compared to the space-Fractional Euler–Bernoulli beam model, takes into account the shear effect to extend modelling range for thick beams. The definition of rotation has been therefore extended by additional rotation resulting from non-local shear deformations. Hence, the appropriate governing equations and enriched numerical algorithm are provided. The discussion also includes a study of the influence of non-locality parameters for different boundary conditions on the static beam response to bending, a comparison study between s-FTB and s-FEBB models, and identification and validation of s-FTB model parameters for the experimental data of the SU-8 polymer microcantilever bending test (including FGM effects).

The paper is structured as follows. Section 2 deals with the s-FTB definition. Section 3 is devoted to the numerical scheme and parametric study. Section 4 provides experimental validation and finally Sect. 5 concludes the paper.

2 Theory

The space-Fractional Timoshenko beam (s-FTB) theory is an extension of the space-Fractional Euler–Bernoulli beam (s-FEBB) theory [28] to include shear deformation and make it suitable for thick beams as mentioned in the introduction. As in the previous study, the non-locality is taken into account, based on the fractional elasticity concept [33,34], by the following definition of small strain

$$\varepsilon_{ij}^{\diamond} = \frac{1}{2} \ell_f^{\alpha-1} \left(\frac{\alpha}{x_j} D u_i(\mathbf{x}) + \frac{\alpha}{x_i} D u_j(\mathbf{x}) \right), \quad (1)$$

where u_i are the components of the displacement vector, $\mathbf{x}$ is a spatial variable, whereas the term $\frac{\alpha}{x_j} D(\cdot)$ denotes the Riesz–Caputo fractional derivative [35],

$$\frac{\alpha}{x_j} D u_i(\mathbf{x}) = \frac{1}{2} \frac{\Gamma(2-\alpha)}{\Gamma(2)} \left({}^C_{x_j-\ell_f} D_{x_j}^{\alpha} u_i(\mathbf{x}) + (-1)^n {}^C_{x_j} D_{x_j+\ell_f}^{\alpha} u_i(\mathbf{x}) \right), \quad (2)$$

with the left-side and right-side Caputo derivatives

$${}^C_{x_j-\ell_f} D_{x_j}^{\alpha} u_i(\mathbf{x}) = \frac{1}{\Gamma(n-\alpha)} \int_{x_j-\ell_f}^{x_j} \frac{u_i^{(n)}(\tau)}{(x_j-\tau)^{\alpha-n+1}} d\tau, \quad (3)$$

$${}^C_{x_j} D_{x_j+\ell_f}^{\alpha} u_i(\mathbf{x}) = \frac{-1}{\Gamma(n-\alpha)} \int_{x_j}^{x_j+\ell_f} \frac{u_i^{(n)}(\tau)}{(x_j-\tau)^{\alpha-n+1}} d\tau, \quad (4)$$

where Γ is an Euler gamma function, $n = [\alpha] + 1$, and $[\cdot]$ denotes the integer part of a real number, $\alpha \in (0, 1]$ is the order of FDO, and ℓ_f is LS i.e. the surrounding affecting the considered material point. The concept of variable length scale [36] $\ell_f = \ell_f(\mathbf{x})$, as function decreasing at the boundaries, has been kept. These two parameters α and ℓ_f are regarded as associated with microstructure [37] and responsible for SE mapping.

In the presented study one considers the static bending behavior in the x_1x_3 -plane, therefore, keeping the assumption that the cross-section is infinitely rigid in its own plane and remains plane after deformation. In consequence, the displacement field takes the form

$$u_1(x_1, x_2, x_3) = x_3 \Phi_2, \quad u_2(x_1, x_2, x_3) = 0, \quad u_3(x_1, x_3, x_3) = \bar{u}_3, \quad (5)$$

where $\bar{u}_3 = \bar{u}_3(x_1)$ is the rigid body translation of the cross-section in 3-rd axis direction of the coordinate system and $\Phi_2 = \Phi_2(x_1)$ is the rigid body rotation (positive keeping the right-hand rule). Rotation Φ_2 depends on the Riesz–Caputo fractional derivative with the proportionality factor $\ell_f^{\alpha-1}$ and, by comparison to the already developed s-FEBB [28], is extended by an additional rotation due to the fractional shear deformation $\dot{\gamma}_{13}^{\diamond}$,

$$\Phi_2 = -\ell_f^{\alpha-1} \frac{\alpha}{x_1} D \bar{u}_3 + \dot{\gamma}_{13}^{\diamond}. \quad (6)$$

Herein, it should be emphasised that on one hand side the assumption Eq. (6) reflects the influence of microstructure on beam cross section rotation, but simultaneously acts as a consistency condition. Namely, it allows to obtain proper relation between the component of the fractional Cauchy strain $\varepsilon_{13}^{\diamond}$ and the fractional shear deformation $\dot{\gamma}_{13}^{\diamond}$. The last statement causes that in limit case when s-FTB reduces to s-FEBB $\varepsilon_{13}^{\diamond} = 0$ which is fundamental for s-FEBB [28].

Next, using Eqs (1) and (6) the nonzero fractional Cauchy strains are

$$\begin{aligned} \varepsilon_{11}^{\diamond} &= x_3 \ell_f^{\alpha-1} \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1} \frac{\alpha}{x_1} D \bar{u}_3 \right) + \frac{\alpha}{x_1} \dot{\gamma}_{13}^{\diamond} \right], \\ \varepsilon_{13}^{\diamond} &= \varepsilon_{31}^{\diamond} = \frac{1}{2} \left[\left(-\ell_f^{\alpha-1} \frac{\alpha}{x_1} D \bar{u}_3 + \dot{\gamma}_{13}^{\diamond} \right) \frac{\alpha}{x_3} D x_3 + \frac{\alpha}{x_1} D \bar{u}_3 \right] \ell_f^{\alpha-1} = \frac{1}{2} \dot{\gamma}_{13}^{\diamond}. \end{aligned} \quad (7)$$

It is so because in Eq. (7) $\ell_f^{\alpha-1} \frac{\alpha}{x_3} D x_3 = 1$, then $\varepsilon_{13}^{\diamond} = \frac{1}{2} \dot{\gamma}_{13}^{\diamond}$. The corresponding stresses are

$$\begin{aligned} \sigma_{11} &= x_3 \ell_f^{\alpha-1} \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1} \frac{\alpha}{x_1} D \bar{u}_3 \right) + \frac{\alpha}{x_1} \dot{\gamma}_{13}^{\diamond} \right] E(x_2, x_3), \\ \sigma_{13} &= G \varepsilon_{13}^{\diamond}, \end{aligned} \quad (8)$$

where $E(x_2, x_3)$ is Young's modulus, $G = \frac{E(x_2, x_3)}{2(1+\nu)}$ is Kirchhoff modulus and ν is Poisson's ratio. Based

on the above results, the bending moment M_2 and the shear force V_3 can be expressed as

$$M_2 = M_2(x_1) = \int_A x_3 \sigma_{11} \, dA \quad \text{and} \quad V_3 = V_3(x_1) = \int_A \sigma_{13} \, dA. \quad (9)$$

and, by introducing Eq. (8), as

$$M_2 = \int_A x_3 \sigma_{11} \, dA = \ell_f^{\alpha-1} \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \bar{u}_3 \right) + \bar{D}_{x_1}^{\alpha} \gamma_{13} \right] (EI)^*, \quad (10)$$

$$V_3 = \int_A \sigma_{13} \, dA = k(GA)^* \bar{\gamma}_{13}, \quad (11)$$

where k is the shear correction factor, $(EI)^* = \int_A E(x_2, x_3) x_3^2 \, dA$ and $(GA)^* = \int_A G(x_2, x_3) \, dA$. The relationship among V_3 , M_2 and distributed load $p_3 = p_3(x_1)$ (for the derivation of following formulas from the principle of virtual work see Appendix and [28]) is

$$V_3 = \frac{\alpha}{x_1} \left(M_2 \ell_f^{\alpha-1} \right), \quad (12)$$

$$p_3 = -\frac{\alpha}{x_1} \left(V_3 \ell_f^{\alpha-1} \right). \quad (13)$$

Moreover, the shear force V_3 , by using Eqs. (10) and (12), can be also expressed as

$$V_3 = \frac{\alpha}{x_1} \left\{ \ell_f^{2\alpha-2} \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \bar{u}_3 \right) + \bar{D}_{x_1}^{\alpha} \gamma_{13} \right] \right\} (EI)^*. \quad (14)$$

Finally, substituting Eq. (14) in Eq. (13), and comparing Eq. (14) and Eq. (11) we obtain the fractional equations governing the bending of s-FTB

$$\begin{cases} \frac{\alpha}{x_1} \left\{ \ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \left[\ell_f^{2\alpha-2} \bar{D}_{x_1}^{\alpha} \left(\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \bar{u}_3 \right) \right] \right\} (EI)^* = p_3 + \frac{\alpha}{x_1} \left[\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \left(\ell_f^{2\alpha-2} \bar{D}_{x_1}^{\alpha} \gamma_{13} \right) \right] (EI)^*, \\ \frac{\alpha}{x_1} \left[\ell_f^{2\alpha-2} \bar{D}_{x_1}^{\alpha} \left(\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \bar{u}_3 \right) \right] EI - \frac{\alpha}{x_1} \left(\ell_f^{2\alpha-2} \bar{D}_{x_1}^{\alpha} \gamma_{13} \right) (EI)^* + k(GA)^* \bar{\gamma}_{13} = 0. \end{cases} \quad (15)$$

It is worth noting that when the shear deformation is neglected, i.e. $\bar{\gamma}_{13} \approx 0$, the rotation Eq. (6) takes the form

$$\Phi_2 = -\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \bar{u}_3, \quad (16)$$

and consequently, Eq. (15) reduces to equation governing the bending of a s-FEBB model [28]

$$\frac{\alpha}{x_1} \left\{ \ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \left[\ell_f^{2\alpha-2} \bar{D}_{x_1}^{\alpha} \left(\ell_f^{\alpha-1} \bar{D}_{x_1}^{\alpha} \bar{u}_3 \right) \right] \right\} (EI)^* = p_3. \quad (17)$$

Moreover, for $\alpha = 1$, the rotations Eqs. (6) and (16) take the classical form for Timoshenko and Euler–Bernoulli beam theories, respectively

$$\Phi_2 = -\frac{d\bar{u}_3}{dx_1} + \gamma_{13} \quad \text{and} \quad \Phi_2 = -\frac{d\bar{u}_3}{dx_1}, \quad (18)$$

and the governing equations Eqs. (15)₁ and (17) also reduce to the classical local descriptions

$$\frac{d^4 \bar{u}_3}{dx_1^4} (EI)^* = p_3 + \frac{d^3 \gamma_{13}}{dx_1^3} (EI)^* \quad \text{and} \quad \frac{d^4 \bar{u}_3}{dx_1^4} (EI)^* = p_3, \quad (19)$$

Timoshenko and Euler–Bernoulli beam theories, respectively.

3 Numerical study

3.1 Discretization

Equation (15) has been solved utilising the numerical method [38]. The beam was discretized in n intervals of length h (see Fig. 1). The trapezoidal rule [38–40] was applied to approximate Caputo fractional derivatives $\bar{D}_{x_1}^{\alpha}(\cdot)_i$ at the node

x_1^i by the sum of first-order derivatives at nodes $x_1^{i-m} \div x_1^{i+m}$ (see the Fig. 1) with appropriate weight coefficients marked as B , C^a and C^b , according to the Eq. (20)

$$\frac{\alpha}{x_1} D(\cdot)_i = h^{1-\alpha} A \left[B(\cdot)'_{i-m} + \sum_{j^a=i-m+1}^{i-1} C^a(\cdot)'_{j^a} + 2(\cdot)'_i + \sum_{j^b=i+1}^{i+m-1} C^b(\cdot)'_{j^b} + B(\cdot)'_{i+m} \right], \quad (20)$$

where

$$\begin{aligned} m = m_i = (\ell_f)_i/h \geq 2, \quad A = \frac{\Gamma(2-\alpha)}{2\Gamma(2)\Gamma(3-\alpha)}, \quad B = (m-1)^{2-\alpha} - (m+\alpha-2)m^{1-\alpha}, \\ C^a = (i-j^a+1)^{2-\alpha} - 2(i-j^a)^{2-\alpha} + (i-j^a-1)^{2-\alpha}, \\ C^b = (j^b-i+1)^{2-\alpha} - 2(j^b-i)^{2-\alpha} + (j^b-i-1)^{2-\alpha}, \end{aligned} \quad (21)$$

and $(\ell_f)_i = \ell_f(x_1^i)$.

Based on the Eq. (20), the numerical representation of the distributed load Eq. (15), shear force Eq. (14), bending moment Eq. (10) and rotation Eq. (6) at node x_1^i are given by

$$\begin{aligned} \mathcal{D}_i(EI)^* &= p_3(x_1^i) + \mathcal{C}_i(EI)^*, \quad \mathcal{C}_i(EI)^* - \mathcal{B}_i(EI)^* + k(GA)^* \hat{\gamma}_{13}(x_1^i) = 0, \\ V_3(x_1^i) &= (-\mathcal{C}_i + \mathcal{B}_i)(EI)^*, \quad \text{or} \quad V_3(x_1^i) = k(GA)^* \hat{\gamma}_{13}(x_1^i), \\ M_2(x_1^i) &= (\ell_f^{\alpha-1})_i (-\mathcal{B}_i + \mathcal{A}_i)(EI)^*, \quad \Phi_2(x_1^i) = -(\ell_f^{\alpha-1})_i \mathcal{A}_i + \hat{\gamma}_{13}(x_1^i), \end{aligned} \quad (22)$$

where

$$\begin{aligned} \mathcal{D}_i &= \frac{\alpha}{x_1} D \left\{ (\ell_f^{\alpha-1})_i \frac{\alpha}{x_1} D \left[(\ell_f^{2\alpha-2})_i \frac{\alpha}{x_1} D \left((\ell_f^{\alpha-1})_i \frac{\alpha}{x_1} D \bar{u}_3(x_1^i) \right) \right] \right\} = \frac{\alpha}{x_1} D \left[(\ell_f^{\alpha-1} C)_i \right] = h^{1-\alpha} A \left[B(\ell_f^{\alpha-1} C)'_{i-m} \right. \\ &\quad \left. + \sum_{j^a=i-m+1}^{i-1} C^a(\ell_f^{\alpha-1} C)'_{j^a} + 2(\ell_f^{\alpha-1} C)'_i + \sum_{j^b=i+1}^{i+m-1} C^b(\ell_f^{\alpha-1} C)'_{j^b} + B(\ell_f^{\alpha-1} C)'_{i+m} \right], \end{aligned} \quad (23)$$

$$\begin{aligned} \mathcal{C}_i &= \frac{\alpha}{x_1} D \left[(\ell_f^{2\alpha-2})_i \frac{\alpha}{x_1} D \left((\ell_f^{\alpha-1})_i \frac{\alpha}{x_1} D \bar{u}_3(x_1^i) \right) \right] = \frac{\alpha}{x_1} D \left[(\ell_f^{2\alpha-2} B)_i \right] = h^{1-\alpha} A \left[B(\ell_f^{2\alpha-2} B)'_{i-m} \right. \\ &\quad \left. + \sum_{j^a=i-m+1}^{i-1} C^a(\ell_f^{2\alpha-2} B)'_{j^a} + 2(\ell_f^{2\alpha-2} B)'_i + \sum_{j^b=i+1}^{i+m-1} C^b(\ell_f^{2\alpha-2} B)'_{j^b} + B(\ell_f^{2\alpha-2} B)'_{i+m} \right], \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{B}_i &= \frac{\alpha}{x_1} D \left[(\ell_f^{\alpha-1})_i \frac{\alpha}{x_1} D \bar{u}_3(x_1^i) \right] = \frac{\alpha}{x_1} D \left[(\ell_f^{\alpha-1} \mathcal{A})_i \right] = h^{1-\alpha} A \left[B(\ell_f^{\alpha-1} \mathcal{A})'_{i-m} \right. \\ &\quad \left. + \sum_{j^a=i-m+1}^{i-1} C^a(\ell_f^{\alpha-1} \mathcal{A})'_{j^a} + 2(\ell_f^{\alpha-1} \mathcal{A})'_i + \sum_{j^b=i+1}^{i+m-1} C^b(\ell_f^{\alpha-1} \mathcal{A})'_{j^b} + B(\ell_f^{\alpha-1} \mathcal{A})'_{i+m} \right], \end{aligned} \quad (25)$$

$$\mathcal{A}_i = \frac{\alpha}{x_1} D \bar{u}_3(x_1^i) = h^{1-\alpha} A \left[B \bar{u}'_{i-m} + \sum_{j^a=i-m+1}^{i-1} C^a \bar{u}'_{j^a} + 2\bar{u}'_i + \sum_{j^b=i+1}^{i+m-1} C^b \bar{u}'_{j^b} + B \bar{u}'_{i+m} \right] \quad (26)$$

$$\begin{aligned} \mathcal{C}_i &= \frac{\alpha}{x_1} D \left[(\ell_f^{\alpha-1})_i \frac{\alpha}{x_1} D \left((\ell_f^{2\alpha-2})_i \frac{\alpha}{x_1} D \hat{\gamma}_{13}(x_1^i) \right) \right] = \frac{\alpha}{x_1} D \left[(\ell_f^{\alpha-1} \mathcal{B})_i \right] = h^{1-\alpha} A \left[B(\ell_f^{\alpha-1} \mathcal{B})'_{i-m} \right. \\ &\quad \left. + \sum_{j^a=i-m+1}^{i-1} C^a(\ell_f^{\alpha-1} \mathcal{B})'_{j^a} + 2(\ell_f^{\alpha-1} \mathcal{B})'_i + \sum_{j^b=i+1}^{i+m-1} C^b(\ell_f^{\alpha-1} \mathcal{B})'_{j^b} + B(\ell_f^{\alpha-1} \mathcal{B})'_{i+m} \right], \end{aligned} \quad (27)$$

Fig. 1 Discretization of the analysed beam of length L —homogeneous grid: real nodes $x_1^0 \div x_1^n$; fictitious nodes $x_1^{-8} \div x_1^{-1}$ and $x_1^{n+1} \div x_1^{n+8}$

$$\mathcal{B}_i = \frac{\alpha}{x_1} \left[(\ell_f^{2\alpha-2})_i \frac{\alpha}{x_1} \hat{D} \gamma_{13}(x_1^i) \right] = \frac{\alpha}{x_1} \left[(\ell_f^{2\alpha-2} \mathcal{A})_i \right] = h^{1-\alpha} \mathbf{A} \left[\mathbf{B}(\ell_f^{2\alpha-2} \mathcal{A})'_{i-m} + \sum_{j^a=i-m+1}^{i-1} \mathbf{C}^a (\ell_f^{2\alpha-2})'_{j^a} \mathcal{A}'_{j^a} + 2(\ell_f^{2\alpha-2} \mathcal{A})'_i + \sum_{j^b=i+1}^{i+m-1} \mathbf{C}^b (\ell_f^{2\alpha-2} \mathcal{A})'_{j^b} + \mathbf{B}(\ell_f^{2\alpha-2} \mathcal{A})'_{i+m} \right], \quad (28)$$

$$\mathcal{A}_i = \frac{\alpha}{x_1} \hat{D} \gamma_{13}(x_1^i) = h^{1-\alpha} \mathbf{A} \left[\mathbf{B} \gamma'_{i-m} + \sum_{j^a=i-m+1}^{i-1} \mathbf{C}^a \gamma'_{j^a} + 2\gamma'_i + \sum_{j^b=i+1}^{i+m-1} \mathbf{C}^b \gamma'_{j^b} + \mathbf{B} \gamma'_{i+m} \right]. \quad (29)$$

It should be pointed out that the first derivatives $(\cdot)'$ in Eqs. (23 ÷ 29) are approximated by forward, backward or central difference formulae at the relevant nodes according to Eq. (30) and Table 1,

$$(\cdot)'_i = \left[-(\cdot)_{i+N_1-N_2} + (\cdot)_{i+N_1+N_2} \right] \frac{1}{2N_2 h}, \quad (30)$$

where N_1 and N_2 determine which finite difference scheme has been applied (see Table 1).

One should conclude that similarly to Eq. (19), for $\alpha = 1$ Eq. (22) returns for $N_1 = 0$ and $N_2 = \frac{1}{2}$ to the classical central difference scheme ($\mathbf{A} = \frac{1}{2}$, $\mathbf{B} = \mathbf{C}^a = \mathbf{C}^b = 0$)

where $\Phi_i = \Phi_2(x_1^i)$, $M_i = M_2(x_1^i)$, $V_i = V_3(x_1^i)$, $p_i = p_3(x_1^i)$, $\gamma_i = \gamma_{13}(x_1^i)$ and $u_i = \bar{u}_3(x_1^i)$.

Applied numerical methods have resulted in fictitious nodes ($x_1^{-8} \div x_1^{-1}$ and $x_1^{n+1} \div x_1^{n+8}$) outside the beam in addition to real nodes ($x_1^0 \div x_1^n$) - see Fig. 1. The application of the variable length scale $\ell_f(x)$, which is decreasing at the boundaries [36], results in only 8 fictitious nodes ($x_1^{-8} \div x_1^{-1}$, $x_1^{n+1} \div x_1^{n+8}$) on each side of the beam. These points are eliminated in final set of equations, by the analogy to the approach presented in [36], by equating of the finite difference approximation

$$\Phi_i = -\frac{-u_{i-1/2} + u_{i+1/2}}{h} + \gamma_i, \quad (31)$$

$$M_i = \left(-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \frac{-\gamma_{i-1/2} + \gamma_{i+1/2}}{h} \right) (EI)^*, \quad (32)$$

$$V_i = \left(-\frac{-u_{i-3/2} + 3u_{i-1/2} - 3u_{i+1/2} + u_{i+3/2}}{h^3} + \frac{\gamma_{i-1} - 2\gamma_i + \gamma_{i+1}}{h^2} \right) (EI)^*, \quad (33)$$

and

$$\frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{h^4} (EI)^* = p_i + \frac{-\gamma_{i-3/2} + 3\gamma_{i-1/2} - 3\gamma_{i+1/2} + \gamma_{i+3/2}}{h^3} (EI)^*, \quad (34)$$

Table 1 The applied finite difference schemes evaluated at node x_1^i where N_1 and N_2 determine which finite difference scheme has been applied (see also Eq. (30) and Fig. 1)

	Forward $N_1 = \frac{1}{2},$ $N_2 = \frac{1}{2}$	Backward $N_1 = -\frac{1}{2},$ $N_2 = \frac{1}{2}$	Central $N_1 = 0,$ $N_2 = \frac{1}{2}$	Central $N_1 = 0,$ $N_2 = 1$
u_i'	$x_1^i = x_1^{-8}$	$x_1^i = x_1^{n+8}$	$x_1^i = x_1^{-7.5} \div x_1^{n+7.5}$	$x_1^i = x_1^{-7} \div x_1^2; x_1^{n-2} \div x_1^{n+7}$
$(\ell_f^{\alpha-1} \mathcal{A})_i'$	$x_1^i = x_1^{-6}$	$x_1^i = x_1^{n+6}$	$x_1^i = x_1^{-5} \div x_1^{n+5}$	
$(\ell_f^{2\alpha-2} \mathcal{B})_i'$	$x_1^i = x_1^{-4}$	$x_1^i = x_1^{n+4}$	$x_1^i = x_1^{-3.5} \div x_1^{n+3.5}$	$x_1^i = x_1^{-3} \div x_1^{n+3}$
$(\ell_f^{\alpha-1} \mathcal{C})_i'$	$x_1^i = x_1^{-2}$	$x_1^i = x_1^{n+2}$	$x_1^i = x_1^{-1} \div x_1^{n+1}$	
$\dot{\gamma}_i'$	$x_1^i = x_1^{-6}$	$x_1^i = x_1^{n+6}$	$x_1^i = x_1^{-5.5} \div x_1^{n+5.5}$	$x_1^i = x_1^{-5} \div x_1^2; x_1^{n-2} \div x_1^{n+5}$
$(\ell_f^{2\alpha-2} \mathcal{D})_i'$	$x_1^i = x_1^{-4}$	$x_1^i = x_1^{n+4}$	$x_1^i = x_1^{-3} \div x_1^{n+3}$	
$(\ell_f^{\alpha-1} \mathcal{B})_i'$	$x_1^i = x_1^{-2}$	$x_1^i = x_1^{n+2}$		$x_1^i = x_1^{-1} \div x_1^{n+1}$

- with the central and forward schemes for the fourth order derivative of displacement at nodes $x_1^{-6} \div x_1^{-1}$ and for the third order derivative of strain at nodes $x_1^{-4} \div x_1^1$,

$$\frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{h^4} = \frac{u_i - 4u_{i+1} + 6u_{i+2} - 4u_{i+3} + u_{i+4}}{h^4}, \quad \text{for } x_1^{-6} \div x_1^{-1},$$

$$\frac{-\dot{\gamma}_{i-2} + 2\dot{\gamma}_{i-1} - 2\dot{\gamma}_{i+1} + \dot{\gamma}_{i+2}}{2h^3} = \frac{-\dot{\gamma}_i + 3\dot{\gamma}_{i+1} - 3\dot{\gamma}_{i+2} + \dot{\gamma}_{i+3}}{h^3}, \quad \text{for } x_1^i = x_1^{-4} \div x_1^1;$$
(35)

- with the central and backward schemes for the fourth order derivative of displacement at nodes $x_1^{n+1} \div x_1^{n+6}$ and for the third order derivative of strain at nodes $x_1^{n-1} \div x_1^{n+4}$

$$\frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{h^4} = \frac{u_{i-4} - 4u_{i-3} + 6u_{i-2} - 4u_{i-1} + u_i}{h^4}, \quad \text{for } x_1^i = x_1^{n+1} \div x_1^{n+6},$$

$$\frac{-\dot{\gamma}_{i-2} + 2\dot{\gamma}_{i-1} - 2\dot{\gamma}_{i+1} + \dot{\gamma}_{i+2}}{2h^3} = \frac{-\dot{\gamma}_{i-3} + 3\dot{\gamma}_{i-2} - 3\dot{\gamma}_{i-1} + \dot{\gamma}_i}{h^3}, \quad \text{for } x_1^i = x_1^{n-1} \div x_1^{n+4}.$$
(36)

3.2 Parametric study

This section highlights the influence of the material parameters α and ℓ_f on the bending behaviour of the beam and compares the fractional and classical approaches to show the ability of taking into account the SE. A comparison of s-FTB and s-FEBB is provided as well to emphasize the need of considering the shear effect when the beam is thick in relation to its length. The examples of beam with following data are thereby considered: beam length $L = 100 \mu\text{m}$, width $a = 10 \mu\text{m}$ and height $b = 50 \mu\text{m}$ of rectangular cross-section, homogenized Young's modulus $E = 10 \text{ GPa}$, Poisson's ratio $\nu = 0.2$ and $h = 0.1 \mu\text{m}$. The shear correction factor for rectangular cross-section is assumed $k = 5/6$. In each of the examples, the beam was loaded with a concentrated force: at the mid-span for simply supported, fixed, and propped cantilever schemes, and at the free end for a cantilever scheme (see Fig. 2). The point load $P = 10 \mu\text{N}$ is

introduced by equivalent continuous load [28,41]

$$p_3(\xi) = \frac{k_1 k_2}{2 \tanh(k_1/2)} \frac{1}{\cosh^2[k_1(\xi - L_1)]} \frac{P}{L}, \quad (37)$$

where $k_1 = 100$ and k_2 are dimensionless parameters, $k_2 = (L_1 + 1)^{-200} + 1$ for $L_1 \in (0.0; 0.5]$ and $k_2 = (-L_1 + 2)^{-200} + 1$ for $L_1 \in (0.5; 1.0]$, $\xi = x_1/L$ is a dimensionless coordinate and L_1 is a point load position in relation to the beam length ($L_1 = 0.5$ for load at the midpoint and $L_1 = 1.0$ for load at the end of beam). The boundary conditions are summarised in Table 2. The effect of non-locality was investigated for the following parameters: $\alpha \in \{0.8, 0.6\}$ and $\ell_f^{\max} \in \{0.001L, 0.10L, 0.2L\}$ with a symmetric distribution that is smoothly decreasing at the boundaries (see Fig. 3), described by the following function [42]

$$\ell_f(x_1) = \begin{cases} \ell_f^{max} \zeta_1^3 (10 - 15\zeta_1 + 6\zeta_1^2) & \text{for } 0 \leq x_1 \leq \eta_1 \ell_f^{max} \\ \ell_f^{max} & \text{for } \eta_1 \ell_f^{max} < x_1 < L - \eta_2 \ell_f^{max} \\ \ell_f^{max} \zeta_2^3 (10 - 15\zeta_2 + 6\zeta_2^2) & \text{for } L - \eta_2 \ell_f^{max} \leq x_1 \leq L \end{cases}, \quad (38)$$

where $\zeta_1 = x_1/(\eta_1 \ell_f^{max})$, $\zeta_2 = (L - x_1)/(\eta_2 \ell_f^{max})$ and for symmetric case $\eta_1 = 1.2$, $\eta_2 = 1.2$.

The results of this study, presented in Fig. 4, show the effect of a small scale on the deflection. It can be observed that parameters α and ℓ_f influence the stiffness of the beam. With a decrease in α or an increase in ℓ_f^{max} , the deflection of the simply supported and cantilever beam increases, while for propped cantilever and fixed beams the stiffening or softening effect is observed for certain values of these parameters. For length scale ℓ_f , small in relation to L ($\ell_f^{max} = 0.001L$), the effect of non-locality disappears and the results are identical to the classic local formulation, as expected.

Moreover, in the above examples, the length to height ratio is $L/b = 2$, and it is visible that for this beam geometry, the deflections predicted by s-FTB and s-FEBB differ significantly. Beam deflections predicted by s-FTB and s-FEBB theories in the wider range of $L/b = 2 \div 26$ are compared in Fig. 5. The maximum deflections are higher for s-FTB than the s-FEBB model, and similarly to the classic approach, the differences are noticeable for thick beams and negligible for slender beams. Moreover, the inclusion of the non-locality makes the shear effect more significant than in the classic approach. It is especially visible for the fixed beam scheme, where, for example, for the ratio $L/b = 2$, the maximum deflection for the Timoshenko beam is about 3.85 times greater than for the Euler–Bernoulli beam in the classical approach ($\alpha = 1$) and 4.85 in fractional approach with $\alpha = 0.6$ and $\ell_f^{max} = 0.1L$. Therefore, the shear effect for small scale thick beam definitely should not be ignored and should even be considered for a higher ratio L/b than in local theory. However, for a higher L/b ratio, the ratio of deflections $\frac{u^{s-FTB}}{u^{s-FEBB}}$ reaches a value of 1.0. For this reason, for long beams, the s-FTB model can be reduced to the s-FEBB model without losing the correctness of the results - which is analogous to local Timoshenko and Euler–Bernoulli beams.

4 Experimental validation

To show the effectiveness of the developed model, it was validated based on the microcantilever bending experiment presented in [43]. The tested samples were made of a SU-8 polymer with the following material parameters: the bulk value of Young's modulus $E_b = 6.9$ GPa and Poisson's ratio $\nu = 0.22$. These cantilevers had a length in the range of $L = 80 \div 850 \mu\text{m}$ and a rectangular cross-section with a

width of $a = 82 \mu\text{m}$ and $a = 122 \mu\text{m}$, and a height of $b = 8.4 \mu\text{m}$ and $b = 14.4 \mu\text{m}$. These data allow for the analysis of non-local effects depending not only on the cross-section dimensions but also on the length of the beam. As shown in Fig. 8, decreasing the length of the beam causes a decrease in modulus of elasticity compared to the bulk value. Conversely, the decrease of the cross-section dimensions is manifested by the increase of Young's modulus. Moreover, we took this increase into account by analogy to the results obtained for iPP+0.2% PACS polymer samples. Figure 6 shows the distribution of elastic modulus in cross-section measured for samples from the nanoindentation test, while Young's modulus from the tensile test is 1.75 ± 0.1 GPa (results derived from [44]). This map of Young's modulus indicates that it is not constant in the whole cross-section, but there is a core with other characteristics. Based on this information, we include the change of elastic modulus in x_3 direction by the core-shell (CS) model (see Fig. 7) with a stiffer core ($E_c = f E_b$) of thickness c and a bulk shell (E_b), where f means the ratio of the core Young's modulus E_s and bulk Young's modulus E_b . The stiffness of the core-shell model can be expressed by the effective bending stiffness

$$E_e I_e = E_c I_c + E_b I_b, \quad (39)$$

where E_e is the effective Young's modulus and I_e , I_c and I_b are the moments of inertia of the effective (homogeneous) cross-section, the core, and the bulk shell respectively,

$$I_e = \frac{ab^3}{12}, \quad I_c = \frac{ac^3}{12}, \quad I_b = \frac{ab^3 - ac^3}{12}. \quad (40)$$

Then, the effective Young's modulus is

$$E_e = E_b \left(1 + (f - 1) \frac{c^3}{b^3} \right). \quad (41)$$

It was found that for SU-8 polymer rectangular samples the height of core zone is $c = 7.0 \mu\text{m}$, which has a Young modulus $f = 1.91$ times higher than the bulk value, regardless of sample size. Subsequently, using the Eq. (41), the effective elastic modulus is 7.62 GPa for sample of dimensions $a = 122 \mu\text{m}$, $b = 14.4 \mu\text{m}$ and 10.52 GPa for sample of dimensions $a = 82 \mu\text{m}$, $b = 8.4 \mu\text{m}$. Then, the parameters of s-FTB model were identified as $\alpha = 0.75$ and $\ell_f^{max} = 5 \mu\text{m}$ with asymmetric distribution (cf. Fig. 3) described by Eq. (38), where $\eta_1 = 1.2$, $\eta_2 = 2.0$.

Fig. 2 Static schemes of the beams used in the parametric study under arbitrary transverse load: **a** simply supported, **b** fixed and **c** propped cantilever—with mid-span point load; and **d** cantilever—with end point load

Fig. 3 Assumed distributions of the length scale $\ell_f(x_1)$ along the beam length: symmetric case used in parametric study and asymmetric case used in validation [see also Eq. (38)]

Table 2 Boundary conditions applied for the static schemes presented in Fig. 2 (see also Eq. (30) and Fig. 1)

Beam type	Boundary conditions		
Simply supported	$\bar{u}_3(x_1^0) = 0, \quad M_2(x_1^0) = 0,$	$u_0 = 0, \quad -\mathcal{B}_0 + \mathcal{A}_0 = 0$	
	$\bar{u}_3(x_1^n) = 0, \quad M_2(x_1^n) = 0$	$u_n = 0, \quad -\mathcal{B}_n + \mathcal{A}_n = 0$	
Fixed	$\bar{u}_3(x_1^0) = 0, \quad \Phi_2(x_1^0) = 0$	$u_0 = 0, \quad -(\ell_f^{\alpha-1})_0 \mathcal{A}_0 + (-C_0 + \mathcal{B}_0) \frac{(EI)^*}{k(GA)^*} = 0$	
	$\bar{u}_3(x_1^n) = 0, \quad \Phi_2(x_1^n) = 0$	$u_n = 0, \quad -(\ell_f^{\alpha-1})_n \mathcal{A}_n + (-C_n + \mathcal{B}_n) \frac{(EI)^*}{k(GA)^*} = 0$	
Propped cantilever	$\bar{u}_3(x_1^0) = 0, \quad \Phi_2(x_1^0) = 0$	$u_0 = 0, \quad -(\ell_f^{\alpha-1})_0 \mathcal{A}_0 + (-C_0 + \mathcal{B}_0) \frac{(EI)^*}{k(GA)^*} = 0$	
	$\bar{u}_3(x_1^n) = 0, \quad M_2(x_1^n) = 0$	$u_n = 0, \quad -\mathcal{B}_n + \mathcal{A}_n = 0$	
Cantilever	$\bar{u}_3(x_1^0) = 0, \quad \Phi_2(x_1^0) = 0$	$u_0 = 0, \quad -(\ell_f^{\alpha-1})_0 \mathcal{A}_0 + (-C_0 + \mathcal{B}_0) \frac{(EI)^*}{k(GA)^*} = 0$	
	$M_2(x_1^n) = 0, \quad V_3(x_1^n) = 0$	$-\mathcal{B}_n + \mathcal{A}_n = 0, \quad -C_n + \mathcal{B}_n = 0$	

The validation results are shown in Fig. 8, where the elastic modulus was calculated according to Timoshenko beam theory for a cantilever with a point load at the free end

$$E = \left(\frac{FL^3}{3I} + \frac{2(1+\nu)FL}{kA} \right) \frac{1}{\bar{u}_3}. \quad (42)$$

It is observed, that the s-FTB model results agree very well with the measurement data. In addition, the results from the classical Timoshenko beam (CTB) model are plotted in Fig. 8, demonstrating that the classical approach is not suitable for describing the behavior of small scale beams.

5 Conclusions

In the presented work, the space-Fractional Timoshenko beam theory has been developed from space-Fractional Euler–Bernoulli beam theory by including the shear deformation. The effect of non-locality for bending of different beam types was studied and the results of the space-Fractional Timoshenko and space-Fractional Euler–Bernoulli beam theories were compared. The validation of the model was also provided. From these analyzes, the following conclusions are drawn:

Fig. 4 Deflection of beam predicted by s-FTB and s-FEBB models for: **a** the simply supported, **b** the fixed and **c** the propped cantilever schemes with mid-span point load; and **d** the cantilever scheme with end point load, for $\alpha \in \{0.8, 0.6\}$, $\ell_f^{\max} \in \{0.001L, 0.1L, 0.2L\}$ and symmetric ℓ_f distribution (cf. Fig. 3)

- the numerical algorithm has been enriched with the shear effect while keeping the possibility of including any boundary conditions, any transverse load, and variable length scale,
- parameters α and ℓ_f control the stiffness of the fractional beam,
- the shear effect is more significant in non-local beams, therefore it should be considered even with more slender beams than in a local approach,
- the current model is adequate for modeling small scale short beams,
- the validation confirms the applicability of the presented model.

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Appendix

The equilibrium equations of fractional beam are derived from the principle of virtual work $\delta U = \delta W$, where δU is the variation in the internal energy

Table 3 Boundary conditions applied for the static schemes presented in Fig. 2

Beam type	Boundary conditions
Simply supported	$\bar{u}_3(x_1^0) = 0, \quad M_2(x_1^0) = \ell_f^{\alpha-1}(x_1^0) \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1}(x_1^0) \frac{\alpha}{x_1} \bar{u}_3(x_1^0) \right) + \frac{\alpha}{x_1} \hat{\gamma}_{13}(x_1^0) \right] (EI)^* = 0$ $\bar{u}_3(x_1^n) = 0, \quad M_2(x_1^n) = \ell_f^{\alpha-1}(x_1^n) \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1}(x_1^n) \frac{\alpha}{x_1} \bar{u}_3(x_1^n) \right) + \frac{\alpha}{x_1} \hat{\gamma}_{13}(x_1^n) \right] (EI)^* = 0$
Fixed	$\bar{u}_3(x_1^0) = 0, \quad \Phi_2(x_1^0) = -\ell_f^{\alpha-1}(x_1^0) \frac{\alpha}{x_1} \bar{u}_3(x_1^0) + \hat{\gamma}_{13}(x_1^0) = 0,$ $\bar{u}_3(x_1^n) = 0, \quad \Phi_2(x_1^n) = -\ell_f^{\alpha-1}(x_1^n) \frac{\alpha}{x_1} \bar{u}_3(x_1^n) + \hat{\gamma}_{13}(x_1^n) = 0$
Propped cantilever	$\bar{u}_3(x_1^0) = 0, \quad \Phi_2(x_1^0) = -\ell_f^{\alpha-1}(x_1^0) \frac{\alpha}{x_1} \bar{u}_3(x_1^0) + \hat{\gamma}_{13}(x_1^0) = 0,$ $\bar{u}_3(x_1^n) = 0, \quad M_2(x_1^n) = \ell_f^{\alpha-1}(x_1^n) \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1}(x_1^n) \frac{\alpha}{x_1} \bar{u}_3(x_1^n) \right) + \frac{\alpha}{x_1} \hat{\gamma}_{13}(x_1^n) \right] (EI)^* = 0$
Cantilever	$\bar{u}_3(x_1^0) = 0, \quad \Phi_2(x_1^0) = 0,$ $M_2(x_1^n) = \ell_f^{\alpha-1}(x_1^n) \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1}(x_1^n) \frac{\alpha}{x_1} \bar{u}_3(x_1^n) \right) + \frac{\alpha}{x_1} \hat{\gamma}_{13}(x_1^n) \right] (EI)^* = 0$ $V_3(x_1^n) = \frac{\alpha}{x_1} \left\{ \ell_f^{2\alpha-2}(x_1^n) \left[-\frac{\alpha}{x_1} \left(\ell_f^{\alpha-1}(x_1^n) \frac{\alpha}{x_1} \bar{u}_3(x_1^n) \right) + \frac{\alpha}{x_1} \hat{\gamma}_{13}(x_1^n) \right] \right\} (EI)^* = 0$

Fig. 5 Comparison of maximum deflection for s-FTB and s-FEBB models for: the simply supported scheme, the fixed scheme and the propped cantilever scheme with mid-span point load and the cantilever scheme with end point load, for $\alpha = 0.6$, $\ell_f^{\max} = 0.10L$ and symmetric ℓ_f distribution **Fig. 6** Local elasticity modulus map from nanoindentation test for polymer (iPP+0.2% PACS) sample, reprinted from [44] with permission of John Wiley and Sons **Fig. 7** Core-shell model of rectangular sample consisting of: the stiffer core of a Young's modulus E_c and thickness of c ; shell of a bulk Young's modulus E_b **Fig. 8** The comparison of experimental measurements [43] (elastic modulus) of SU-8 polymer microcantilevers loaded at the free end, vs. the results of s-FTB model and CTB model

$$\begin{aligned}\delta U &= \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \int_L \int_A (\delta \varepsilon_{11}^\alpha \sigma_{11} + 2\delta \varepsilon_{13}^\alpha \sigma_{13}) dA dL = \\ &= \int_L \int_A [x_3 \ell_f^{\alpha-1} \sigma_{11} \frac{\alpha}{x_1} \bar{D} \Phi_2 + (\delta \Phi_2 \frac{\alpha}{x_3} \bar{D} x_3 + \frac{\alpha}{x_1} \bar{D} \bar{u}_3) \ell_f^{\alpha-1} \sigma_{13}] dA dL\end{aligned}\quad (43)$$

and introducing Eqs. (10), (11) and $\ell_f^{\alpha-1} \frac{\alpha}{x_3} \bar{D} x_3 = 1$,

$$\delta U = \int_L [M_2 \ell_f^{\alpha-1} \frac{\alpha}{x_1} \bar{D} \delta \Phi_2 + (\delta \Phi_2 + \ell_f^{\alpha-1} \frac{\alpha}{x_1} \bar{D} \bar{u}_3) V_3] dL, \quad (44)$$

and δW is the variation in the external work

$$\delta W = \int_L p_3 \delta \bar{u}_3 dL. \quad (45)$$

To obtain the equilibrium equations, we look for a minimum of the functional $\mathcal{J}$

$$\begin{aligned}\delta \mathcal{J} &= \delta U - \delta W \\ &= \int_L [(M_2 \ell_f^{\alpha-1} \frac{\alpha}{x_1} \bar{D} \delta \Phi_2 + \delta \Phi_2 V_3) + (\ell_f^{\alpha-1} V_3 \frac{\alpha}{x_1} \bar{D} \bar{u}_3 - p_3 \delta \bar{u}_3)] dL = 0,\end{aligned}\quad (46)$$

which, using the fractional Euler–Lagrange equation [45,46], provides

$$V_3 - \frac{\alpha}{x_1} \bar{D} (M_2 \ell_f^{\alpha-1}) = 0 \rightarrow V_3 = \frac{\alpha}{x_1} \bar{D} (M_2 \ell_f^{\alpha-1}), \quad (47)$$

$$p_3 + \frac{\alpha}{x_1} \bar{D} (V_3 \ell_f^{\alpha-1}) = 0 \rightarrow p_3 = -\frac{\alpha}{x_1} \bar{D} (V_3 \ell_f^{\alpha-1}). \quad (48)$$

The boundary conditions for selected beam types are presented in Table 3.

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