Comput Mech (2006) 37: 362–368
DOI 10.1007/s00466-005-0746-6

ERRATUM

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On the straightness of eigenvalue interactions

Received: 30 May 2004 / Accepted: 12 December 2004 / Published online: 8 December 2005

Unfortunately, the entire article was originally published Online First with errors. The publishers wish to apologize for this mistake. The correct article is shown here:

Abstract Solutions to eigenvalue problems come in two parts, an eigenvalue and an eigenvector, and these solution pairs occur at discrete points in the range of possible eigenvalues. Multiparameter eigenvalue problems similarly have solutions that have a dimension smaller by 1 than the space of the eigenvalues - solutions to a 2-parameter problem are discrete curves in a plane, and in general, solutions to an n-parameter problem are hypersurfaces in an n dimensional space. These curves/surfaces/hypersurfaces are *eigenvalue interaction curves* (*/surfaces*, etc.), and they might be flat. An unchanging eigenvector leads to a flat interaction, almost trivially. This paper addresses the question *if an interaction is flat (in particular, if an interaction curve is straight), what conditions does this place on the eigenvector?*

1 Introduction

Many structural problems lead to algebraic eigenvalue formulations: conventional finite element modelling of structural vibration gives eigenvalue equations

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{0}, \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the structure stiffness and mass matrices, and solutions are eigenvalue/eigenvector pairs (ω^2, $\mathbf{u}$), which are a frequency-squared and a vibration shape. Buckling similarly gives

$$(\mathbf{K} - P\mathbf{S})\mathbf{u} = \mathbf{0}, \tag{2}$$

The online version of the original article can be found at http://dx.doi.org/10.1007/s00466-005-0675-4.

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where $\mathbf{S}$ is a stability matrix showing changes in stiffness for some pattern of loading, and the solution $(P, \mathbf{u})$ is a magnitude of this loading pattern and a buckled shape.

The simply supported uniform beam/column of Fig. 1a has a lowest natural frequency of $\omega^2 = \pi^4 EI/\rho L^4$, where L is the span, EI is the rigidity and ρ is the mass per unit length, with a vibration mode as shown. The same structure loaded as in Fig. 1b buckles into the shape shown, at a lowest buckling load of $P = \pi^2 EI/L$.

Vibration of a loaded structure is governed by

$$(\mathbf{K} - P\mathbf{S} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{0}. \tag{3}$$

This is a multiparameter eigenvalue problem, with solutions (P, ω^2, $\mathbf{u}$), and eigenvalues (P, ω^2) form a curve in a P, ω^2 plot, as shown in Fig. 2 (the dimensional terms L, EI and ρ have been dropped, for simplicity). This is an eigenvalue *interaction curve*, where, for example, point A shows that a beam with an axial compression of $\pi^2/2$ vibrates with a frequency of $\omega^2 = \pi^4/2$.

The interaction curve of Fig. 2 is a perfectly straight line, which is uncommon. It is straight here because the vibration mode of Fig. 1a and the buckling mode of Fig. 1b are identical.

This paper asks the question *what is the general relation between a straight interaction curve and the eigenmode along the curve?* The answer is somewhat surprising, and more complex than might be expected. But first, some background.

2 Background

1. Interaction can be between any two (or more) eigenvalues, not only between vibration and buckling, and without loss of generality, it is written here as a buckling interaction between two different loadings

$$(\mathbf{K} - P_1\mathbf{S}_1 - P_2\mathbf{S}_2)\mathbf{u} = \mathbf{0}. \tag{4}$$

2. The argument of the introduction leads to one interaction curve. The real structure has an infinity of buckling loads,

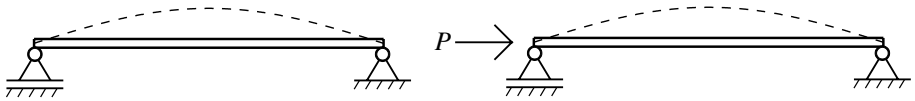


Fig. 1 Simply supported beam/column **a** Vibration **b** Buckling

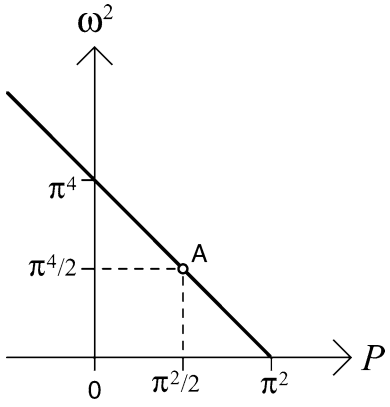


Fig. 2 Vibration-buckling interaction for the beam of Fig. 1

and an infinity of natural frequencies, and with this, an infinity of interaction curves. Algebraic models, seen in Eqs. (1)–(4) have a finite number, limited to the number of freedoms in the model. Figure 3 shows the previously used beam-column with a second independent loading at midspan, and the first few interaction curves, labelled c_1 to c_5.

3. **K** is assumed to be the stiffness of a stable structure, and is therefore positive-definite. Displacements **u** can be transformed so that **K** becomes diagonal. This can be done in many ways, but in all cases, all (diagonal) terms of **K** will be positive. For any chosen values of P_1 and P_2, $\mathbf{K} - P_1\mathbf{S}_1 - P_2\mathbf{S}_2$ can be formed and similarly diagonalised, leading to a number π of positive terms, a number ν of negative terms, and a number ζ of zero diagonal terms. These numbers are the positive, negative and zero inertias of the matrix (Parlett 1980), and are independent of how the diagonal form is found. They are obviously functions of (P_1, P_2) and allow the P_1, P_2 plane to be separated into regions. In Fig. 3b, the regions R_i are defined by the negative inertia ν, that is, for any point (P_1, P_2) in, say, R_2, when $\mathbf{K} - P_1\mathbf{S}_1 - P_2\mathbf{S}_2$ is formed and diagonalised in any way whatever, it will have exactly two negative elements. The interaction curves are the boundaries between these regions - c_i separates R_{i-1} and R_i. In passing, if any straight line is drawn starting from within R_0, it will pass through the regions R_1, R_2 etc. in increasing order - this is the basis of Sturm sequence or mode counting methods for solving the eigenvalue problem (Wilkinson 1965; Parlett 1980).
4. Suppose that an interaction curve is straight. The origin is moved to some point on this curve (which is assumed to be a point on this curve and no other). The P_1, P_2 axes are rotated to $\bar{P}_1$, $\bar{P}_2$, with $\bar{P}_1$ directed along the interaction curve, so that Eq. (4) is now $(\widehat{\mathbf{K}} - \bar{P}_1\bar{\mathbf{S}}_1 - \bar{P}_2\bar{\mathbf{S}}_2)\mathbf{u} = \mathbf{0}$. With this construction, $(\widehat{\mathbf{K}} - \bar{P}_1\bar{\mathbf{S}}_1)\mathbf{u} = \mathbf{0}\ \forall \bar{P}_1$. The subscripts and overscores are now dropped, and the problem is written

$$(\mathbf{K} - P\mathbf{S})\mathbf{u} = \mathbf{0}\ \forall P \tag{5}$$

where it is emphasised that $\mathbf{u} = \mathbf{u}(P)$. When diagonalised, **K** has exactly one zero (diagonal) element, π_K positive elements and ν_K negative elements (ν_K is 1 less than the number of the curve of interest). Further, the non-zero elements are assumed to be ± 1, with no loss of generality.

3 Results and proofs

Result 1:

The eigenmode $\mathbf{u}(P)$ along a straight interaction curve can vary with P, but only as an nth order polynomial [1] in P, and then iff both the positive and negative inertias of **K** and **S** are all $\geq n$. This means that the number of the interaction curve is at least $n+1$, and is at most $N-n$, where N is the number of freedoms in the formulation.

Proof

a. Necessity: if the eigenvector $\mathbf{u}(P)$ on a straight interaction curve is a linear combination of n independent vectors then it is an $n-1$th order polynomial in P, the curve is at least the nth, and the number of freedoms is at least $2n-1$. The following argument considers an eigenvector that is a combination of five vectors, but this is for clarity: the same arguments apply for any number.

The eigenvector $\mathbf{u}(P)$ is assumed to be the combination of five independent vectors

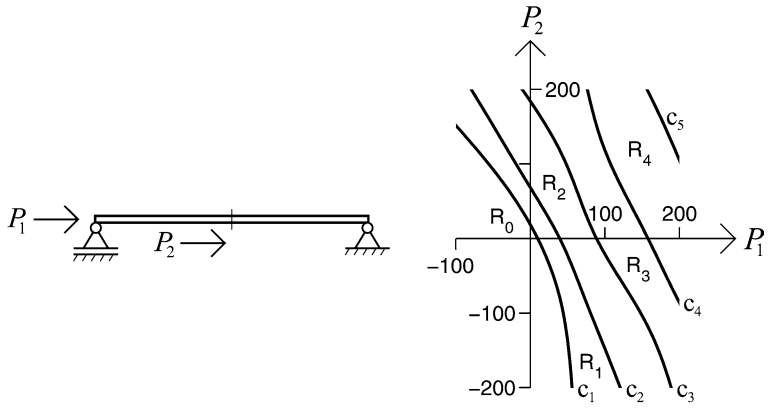
$$\mathbf{u}(P) = \alpha_1(P)\mathbf{u}_1 + \alpha_2(P)\mathbf{u}_2 + \alpha_3(P)\mathbf{u}_3 + \alpha_4(P)\mathbf{u}_4 + \alpha_5(P)\mathbf{u}_5 \tag{6}$$

where none of the $\alpha_i(P) \equiv 0$, that is, $\mathbf{u}(P)$ cannot be written as a function of fewer than five independent vectors.

Step 1. The form of Eq. (6) is a 4th order polynomial in P:

Any independent vectors expressible by Eq. (6) can be used in the basis for $\mathbf{u}(P)$. $\mathbf{u}(0)$ is $\mathbf{u}_K$, the unique null vector of **K**, and as $P \to \infty$ $\mathbf{u}(P) \to \mathbf{u}_S$, some null vector of **S** (**S** could have multiple null vectors). $\mathbf{u}_K$ and $\mathbf{u}_S$ (which are

[1] More precisely, since eigenvectors are unscaled, $\mathbf{u}(P)$ can be scaled to an nth order polynomial. The scaling factor could be a function of P.



(a) Structure (b) Interaction curves

Fig. 3 The beam column of Fig. 1, with independent loadings. **a** Structure. **b** Interaction curves

different - if $\mathbf{u}_K = \mathbf{u}_S$ then the mode $\mathbf{u}(P)$ is constant along the interaction curve, and is not a function of five vectors) are used in the basis for $\mathbf{u}(P)$, which is scaled so the multiplier of $\mathbf{u}_K$ is 1, and Eq. (6) becomes

$$\mathbf{u}(P) = \mathbf{u}_K + \alpha_1(P)\mathbf{u}_1 + \alpha_2(P)\mathbf{u}_2 + \alpha_3(P)\mathbf{u}_3 + \alpha_S(P)\mathbf{u}_S. \tag{7}$$

With this,

$$\begin{aligned}&(\mathbf{K} - P\mathbf{S})\mathbf{u}(P)\\ &= \alpha_1\mathbf{K}\mathbf{u}_1 + \alpha_2\mathbf{K}\mathbf{u}_2 + \alpha_3\mathbf{K}\mathbf{u}_3 + \alpha_S\mathbf{K}\mathbf{u}_S\\ &\quad - P\mathbf{S}\mathbf{u}_K - \alpha_1 P\mathbf{S}\mathbf{u}_2 - \alpha_2 P\mathbf{S}\mathbf{u}_2 - \alpha_3 P\mathbf{S}\mathbf{u}_3 = \mathbf{0}.\end{aligned} \tag{8}$$

Multiplying Eq. 8 by $\mathbf{u}_i^T\mathbf{K}$, $i = 1, 2, 3, S$ gives

$$\begin{bmatrix} \mathbf{u}_1^T\mathbf{K}\mathbf{K}\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{K}\mathbf{K}\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{K}\mathbf{K}\mathbf{u}_3 & \mathbf{u}_1^T\mathbf{K}\mathbf{K}\mathbf{u}_S \\ \mathbf{u}_2^T\mathbf{K}\mathbf{K}\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{K}\mathbf{K}\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{K}\mathbf{K}\mathbf{u}_3 & \mathbf{u}_2^T\mathbf{K}\mathbf{K}\mathbf{u}_S \\ \mathbf{u}_3^T\mathbf{K}\mathbf{K}\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{K}\mathbf{K}\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{K}\mathbf{K}\mathbf{u}_3 & \mathbf{u}_3^T\mathbf{K}\mathbf{K}\mathbf{u}_S \\ \mathbf{u}_S^T\mathbf{K}\mathbf{K}\mathbf{u}_1 & \mathbf{u}_S^T\mathbf{K}\mathbf{K}\mathbf{u}_2 & \mathbf{u}_S^T\mathbf{K}\mathbf{K}\mathbf{u}_3 & \mathbf{u}_S^T\mathbf{K}\mathbf{K}\mathbf{u}_S \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_S \end{bmatrix}$$

$$- P \begin{bmatrix} \mathbf{u}_1^T\mathbf{K}\mathbf{S}\mathbf{u}_K \\ \mathbf{u}_2^T\mathbf{K}\mathbf{S}\mathbf{u}_K \\ \mathbf{u}_3^T\mathbf{K}\mathbf{S}\mathbf{u}_K \\ \mathbf{u}_S^T\mathbf{K}\mathbf{S}\mathbf{u}_K \end{bmatrix} \tag{9}$$

$$- P \begin{bmatrix} \mathbf{u}_1^T\mathbf{K}\mathbf{S}\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{K}\mathbf{S}\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{K}\mathbf{S}\mathbf{u}_3 \\ \mathbf{u}_2^T\mathbf{K}\mathbf{S}\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{K}\mathbf{S}\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{K}\mathbf{S}\mathbf{u}_3 \\ \mathbf{u}_3^T\mathbf{K}\mathbf{S}\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{K}\mathbf{S}\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{K}\mathbf{S}\mathbf{u}_3 \\ \mathbf{u}_S^T\mathbf{K}\mathbf{S}\mathbf{u}_1 & \mathbf{u}_S^T\mathbf{K}\mathbf{S}\mathbf{u}_2 & \mathbf{u}_S^T\mathbf{K}\mathbf{S}\mathbf{u}_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0}$$

or

$$\mathbf{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_S \end{bmatrix} = P \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{10}$$

The a_i are constants of no particular interest, and $\mathbf{A}$ is a matrix with 1st order polynomials in P for its first three columns, and constants in its last column (these polynomial orders are the maximum possible - they could be lower). It follows that all terms of the adjoint of $\mathbf{A}$ (Aitken 1956) [2] are polynomials of 3rd order, or lower.

When Eq. (10) is multiplied by adj($\mathbf{A}$)

$$\det(\mathbf{A}) \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_S \end{bmatrix} = \mathbf{v}, \tag{11}$$

where $\mathbf{v}$ is a vector of 4 polynomials in P, each of 4th order, at most. Next, a term of $\det(\mathbf{A})\alpha_K$ is added to both vectors in Eq. (11) as a new first row, where $\alpha_K(P)$ is the multiplier of $\mathbf{u}_K$ in Eq. (7), which is $\alpha_K = 1$. Thus

$$\det(\mathbf{A}) \begin{bmatrix} \alpha_K \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_S \end{bmatrix} = \begin{bmatrix} \det(\mathbf{A}) \\ \\ \mathbf{v} \\ \\ \\ \end{bmatrix}. \tag{12}$$

The multipliers α_i give an eigenvector, which has arbitrary scale, and can be taken as the vector on the right of Eq. (12). Det($\mathbf{A}$) is a 3rd order polynomial, allowing Eq. (6) to be written as a polynomial, 4th order at most,

$$\mathbf{u}(P) = \mathbf{u}_0 + P\mathbf{u}_1 + \frac{P^2}{2!}\mathbf{u}_2 + \frac{P^3}{3!}\mathbf{u}_3 + \frac{P^4}{4!}\mathbf{u}_4. \tag{13}$$

For it to be a function of five independent vectors, all $\mathbf{u}_i \neq \mathbf{0}$, and $\mathbf{u}(P)$ is a polynomial of 4th order, not lower.

This argument has a potential flaw: det($\mathbf{A}$) is a polynomial of order 3 at most, and could be the zero polynomial, invalidating Eqs. (11) and (12). $\mathbf{A} = \mathbf{A}(P)$, and $\mathbf{A}(0)$ is the square matrix of Eq. (9), where all terms are scalar products of $\mathbf{K}\mathbf{u}_i$, $i = 1, 2, 3,$ S, and $\det(\mathbf{A}(0)) = 0$ iff these are dependent vectors. A dependence $\Sigma\beta_i\mathbf{K}\mathbf{u}_i = \mathbf{0}$ is $\mathbf{K}(\Sigma\beta_i\mathbf{u}_i) = \mathbf{0}$, so $\Sigma\beta_i\mathbf{u}_i$ is a null vector of $\mathbf{K}$. But $\mathbf{K}$ has only one null vector, $\mathbf{u}_K$, so $\mathbf{u}_i$, $i = K, 1, 2, 3, S$ are dependent, contrary to the construction of Eq. (7). $\mathbf{K}\mathbf{u}_i$, $i = 1, 2, 3, S$ are independent and $\det(\mathbf{A}(0)) \neq 0$. Det($\mathbf{A}$) is not the zero polynomial.

[2] In this reference, an *adjoint* matrix is known by the alternative name *adjugate*.

Step 2. The vectors $\mathbf{u}_i$ satisfy $\mathbf{u}_i^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = \mathbf{0}$ and $\mathbf{u}_i^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = \mathbf{0}\ \forall\ i, j$:

Writing $(\mathbf{K} - P\mathbf{S})\mathbf{u}(P) = \mathbf{0}$ with $\mathbf{u}(P)$ as in Eq. (13), differentiating repeatedly with respect to P, and evaluating the resulting expressions at $P = 0$ produces the sequence

$$\mathbf{K}\mathbf{u}_0 = \mathbf{0} \tag{14a}$$

$$\mathbf{K}\mathbf{u}_1 - \mathbf{S}\mathbf{u}_0 = \mathbf{0} \tag{14b}$$

$$\mathbf{K}\mathbf{u}_2 - 2\mathbf{S}\mathbf{u}_1 = \mathbf{0} \tag{14c}$$

$$\mathbf{K}\mathbf{u}_3 - 3\mathbf{S}\mathbf{u}_2 = \mathbf{0} \tag{14d}$$

$$\mathbf{K}\mathbf{u}_4 - 4\mathbf{S}\mathbf{u}_3 = \mathbf{0} \tag{14e}$$

$$\mathbf{S}\mathbf{u}_4 = \mathbf{0}. \tag{14f}$$

Since all $\mathbf{u}_i$ are non-zero, Eqs. (14a) and (14f) identify $\mathbf{u}_0$ as $\mathbf{u}_{\mathrm{K}}$ and $\mathbf{u}_4$ as $\mathbf{u}_{\mathrm{S}}$.

The first and last of these equations are multiplied by $\mathbf{u}_j^{\mathrm{T}}$, $j = 0, \ldots, 4$, giving

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{K}\mathbf{u}_0 = 0 \text{ and } \mathbf{u}_0^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = 0, \quad j = 0, \ldots, 4 \tag{15a}$$

and

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{S}\mathbf{u}_4 = 0 \text{ and } \mathbf{u}_4^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = 0, \quad j = 0, \ldots, 4 \tag{15b}$$

(in each case, the second equation follows from the first through the symmetry of $\mathbf{K}$ or $\mathbf{S}$ - these symmetries are understood without further mention).

Eqs. (14b–e) are now multiplied by $\mathbf{u}_4^{\mathrm{T}}$ and $\mathbf{u}_0^{\mathrm{T}}$, giving

$$\mathbf{u}_4^{\mathrm{T}}\mathbf{K}\mathbf{u}_j - j\mathbf{u}_4^{\mathrm{T}}\mathbf{S}\mathbf{u}_{j-1} = 0, \quad j = 1, \ldots, 4 \tag{16a}$$

and

$$\mathbf{u}_0^{\mathrm{T}}\mathbf{K}\mathbf{u}_j - j\mathbf{u}_0^{\mathrm{T}}\mathbf{S}\mathbf{u}_{j-1} = 0, \quad j = 1, \ldots, 4. \tag{16b}$$

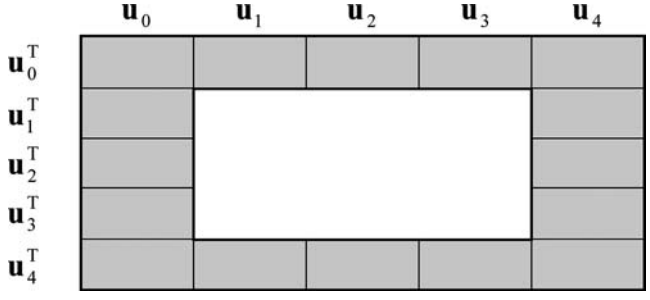
Equations (15) and (16) now give

$$\mathbf{u}_4^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = 0, \quad j = 1, \ldots, 4 \tag{17a}$$

and

$$\mathbf{u}_0^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = 0, \quad j = 0, \ldots, 3. \tag{17b}$$

This establishes the scalar products $\mathbf{u}_i^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = 0$ and $\mathbf{u}_i^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = 0$ for the shaded cells in the following table:



Next, the second and second last of Eqs. (14) are multiplied by $\mathbf{u}_j^{\mathrm{T}}$, $j = 1, \ldots, 3$, giving

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{K}\mathbf{u}_1 - \mathbf{u}_j^{\mathrm{T}}\mathbf{S}\mathbf{u}_0 = 0, \quad j = 1, \ldots, 3 \tag{18a}$$

and

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{K}\mathbf{u}_4 - 4\mathbf{u}_j^{\mathrm{T}}\mathbf{S}\mathbf{u}_3 = 0, \quad j = 1, \ldots, 3. \tag{18b}$$

From Eqs. (17) and (18)

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{K}\mathbf{u}_1 = 0, \quad j = 1, \ldots, 3, \tag{19a}$$

and

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{S}\mathbf{u}_3 = 0, \quad j = 1, \ldots, 3. \tag{19b}$$

Equations 14c,d are now multiplied by $\mathbf{u}_3^{\mathrm{T}}$ and $\mathbf{u}_1^{\mathrm{T}}$, giving

$$\mathbf{u}_3^{\mathrm{T}}\mathbf{K}\mathbf{u}_j - j\,\mathbf{u}_3^{\mathrm{T}}\mathbf{S}\mathbf{u}_{j-1} = 0, \quad j = 2, \ldots, 3, \tag{20a}$$

and

$$\mathbf{u}_1^{\mathrm{T}}\mathbf{K}\mathbf{u}_j - j\,\mathbf{u}_1^{\mathrm{T}}\mathbf{S}\mathbf{u}_{j-1} = 0, \quad j = 2, \ldots, 3. \tag{20b}$$

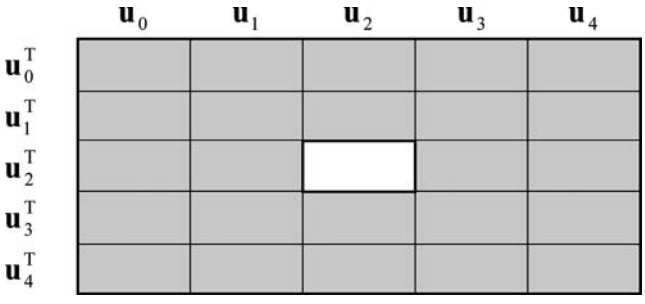
From Eqs. (19) and (20)

$$\mathbf{u}_3^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = 0, \quad j = 2, \ldots, 3 \tag{21a}$$

and

$$\mathbf{u}_1^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = 0, \quad j = 1, \ldots, 2. \tag{21b}$$

Scalar products $\mathbf{u}_i^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = 0$ and $\mathbf{u}_i^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = 0$ are now established for the shaded cells shown below



Recursive use of this process fills all cells, establishing

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{K}\mathbf{u}_j = 0 \text{ and } \mathbf{u}_i^{\mathrm{T}}\mathbf{S}\mathbf{u}_j = 0 \quad \forall\, i, j. \tag{22}$$

Step 3. Both the positive and negative inertias of $\mathbf{K}$ must be at least the order of the polynomial in Eq. (13):

$\mathbf{K}$ has zero, positive and negative inertias ζ_{K}, π_{K} and ν_{K}, and since the origin is assumed to be at a point on an interaction curve which is a single solution, $\zeta_{\mathrm{K}} = 1$. Coordinates $\mathbf{u}$ are chosen so that $\mathbf{K}$ has the form

$$\mathbf{K} = \begin{bmatrix} 0 & & \\ & \mathbf{I}_\pi & \\ & & -\mathbf{I}_\nu \end{bmatrix}. \tag{23}$$

The vectors $\mathbf{u}_0 \cdots \mathbf{u}_4$ are orthonormalised through the Gramm-Schmidt algorithm, giving a new set $\bar{\mathbf{u}}_0 \cdots \bar{\mathbf{u}}_4$ [3]. These new

[3] The relevant property of the orthonormalisation is that $\bar{\mathbf{u}}_0 = \bar{\mathbf{u}}_0(\mathbf{u}_0)$, $\bar{\mathbf{u}}_1 = \bar{\mathbf{u}}_1(\bar{\mathbf{u}}_0, \mathbf{u}_1)$, $\bar{\mathbf{u}}_2 = \bar{\mathbf{u}}_2(\bar{\mathbf{u}}_0, \bar{\mathbf{u}}_1, \mathbf{u}_2)$, etc.

vectors are linearly independent, and because they are linear combinations of $\mathbf{u}_0, \ldots, \mathbf{u}_4$, they satisfy Eq. (22). Thus

$$\bar{\mathbf{u}}_i^{\mathrm{T}} \bar{\mathbf{u}}_i = 1, \quad i = 0, \ldots, 4 \tag{24a}$$

$$\bar{\mathbf{u}}_i^{\mathrm{T}} \bar{\mathbf{u}}_j = 0, \quad i, j = 0, \ldots, 4, \quad i \neq j \tag{24b}$$

$$\bar{\mathbf{u}}_i^{\mathrm{T}} \mathbf{K} \bar{\mathbf{u}}_j = 0, \quad i, j = 0, \ldots, 4. \tag{24c}$$

$\mathbf{u}_0$ is the unique null vector of $\mathbf{K}$, $\mathbf{u}_0 = [1\ 0\ 0 \cdots\ 0]$, and given the orthonormalisation algorithm, the first vector $\bar{\mathbf{u}}_0$ is the same as the original $\mathbf{u}_0$ (scaled to a unit vector, if necessary), so that

$$\bar{\mathbf{u}}_0 = [1\ 0\ 0 \cdots\ 0]. \tag{25}$$

The vectors $\bar{\mathbf{u}}_i$ are now broken into partitions of length $\zeta_K = 1, \pi_K$ and ν_K, written as

$$\bar{\mathbf{u}}_i = \begin{bmatrix} \bar{u}_i^{\zeta} \\ \bar{\mathbf{u}}_i^{\pi} \\ \bar{\mathbf{u}}_i^{\nu} \end{bmatrix}. \tag{26}$$

From Eqs. (24b) and (25),

$$\bar{u}_i^{\zeta} = 0, \quad i = 1, \ldots, 4, \tag{27}$$

and with this, Eqs. (24) give

$$\bar{\mathbf{u}}_i^{\pi\mathrm{T}} \bar{\mathbf{u}}_i^{\pi} + \bar{\mathbf{u}}_i^{\nu\mathrm{T}} \bar{\mathbf{u}}_i^{\nu} = 1, \quad i = 1, \ldots, 4 \tag{28a}$$

$$\bar{\mathbf{u}}_i^{\pi\mathrm{T}} \bar{\mathbf{u}}_j^{\pi} + \bar{\mathbf{u}}_i^{\nu\mathrm{T}} \bar{\mathbf{u}}_j^{\nu} = 0, \quad i, j = 1, \ldots, 4, \quad i \neq j \tag{28b}$$

$$\bar{\mathbf{u}}_i^{\pi\mathrm{T}} \bar{\mathbf{u}}_j^{\pi} - \bar{\mathbf{u}}_i^{\nu\mathrm{T}} \bar{\mathbf{u}}_j^{\nu} = 0, \quad i, j = 1, \ldots, 4. \tag{28c}$$

From these

$$\bar{\mathbf{u}}_i^{\pi\mathrm{T}}\, \bar{\mathbf{u}}_i^{\pi} = 1/2, \quad i = 1, \ldots, 4 \tag{29a}$$

$$\bar{\mathbf{u}}_i^{\pi\mathrm{T}}\, \bar{\mathbf{u}}_j^{\pi} = 0, \quad i, j = 1, \ldots, 4, \quad i \neq j \tag{29b}$$

and

$$\bar{\mathbf{u}}_i^{\nu\mathrm{T}} \bar{\mathbf{u}}_i^{\nu} = 1/2, \quad i = 1, \ldots, 4 \tag{30a}$$

$$\bar{\mathbf{u}}_i^{\nu\mathrm{T}} \bar{\mathbf{u}}_j^{\nu} = 0, \quad i, j = 1, \ldots, 4, \quad i \neq j. \tag{30b}$$

The partitions $\bar{\mathbf{u}}_j^{\pi}$, $j = 1, \ldots, 4$ are orthogonal, as are the partitions $\bar{\mathbf{u}}_j^{\nu}$. Each partition has four independent vectors, so $\pi_{\mathrm{K}} \geq 4$ and $\nu_{\mathrm{K}} \geq 4$. The number of freedoms in the problem must be at least 9. Further, the buckling surface must be at least the 5th.

This completes the necessity proof.

The preceding arguments were demonstrated with $\mathbf{u}(P)$ as a combination of five independent vectors, but nothing in the argument is specific to this size. In general, if $\mathbf{u}(P)$ is a combination of n independent vectors then it is (essentially) a polynomial of order $n-1$, the buckling surface must be at least the nth, and the number of freedoms is at least $2n-1$.

b. Existence, or sufficiency: Eq. (5) has solutions with varying $\mathbf{u}(P)$, with the properties given above.

Consider

$$\mathbf{K} = [0]\,; \mathbf{S} = -[0]\,; \mathbf{u}_1 = [-1]\,. \tag{31}$$

Equation (5) is satisfied with $\mathbf{u}(P) = \mathbf{u}_1 = \text{constant}$, and since the size of the formulation is 1, the interaction curve is both the first and the last.

Next consider

$$\mathbf{K} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}; \quad \mathbf{S} = -\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{32a}$$

and the vectors

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \tag{32b}$$

$\mathbf{K}$ has $\zeta = 1$, and clearly, $\mathbf{u}_1$ is the single null vector. Equally clearly, $\mathbf{u}_2$ is a null vector of S. The eigenmode is

$$\mathbf{u}(P) = \mathbf{u}_1 + P\mathbf{u}_2 \tag{33}$$

which is confirmed by

$$\begin{aligned} &\left(\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} + P \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) \\ &\left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + P \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) \\ &\quad = \mathbf{0} + P \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + P \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + P^2 \mathbf{0} = \mathbf{0}. \end{aligned} \tag{34}$$

A system with $N = 3, \zeta = 1, \pi = 1$ and $\nu = 1$ can produce a straight line interaction curve with a linearly changing eigenvector. Since $\mathbf{K}$ has a negative inertia of 1, the interaction curve is c_2, and because it has a positive inertia of 1, it is c_{N-1}.

In the same way that Eq. (31) produces Eqs. (32), Eqs. (32) are augmented to

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix}; \\ \mathbf{S} &= -\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{aligned} \tag{35a}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{35b}$$

As before, $\mathbf{u}_1$ is the unique null vector of $\mathbf{K}$, and $\mathbf{u}_3$ is a null vector of $\mathbf{S}$. The eigenmode is

$$\mathbf{u}(P) = \mathbf{u}_1 + P\mathbf{u}_2 + P^2\mathbf{u}_3 \tag{36}$$

which is readily confirmed by forming $(\mathbf{K} - P\mathbf{S})\mathbf{u}(P)$.

A system with $N = 5, \zeta = 1, \pi = 2$ and $\nu = 2$ can produce a straight line interaction curve with a quadratically changing eigenvector. From the inertias of $\mathbf{K}$, the interaction curve is both c_3 and c_{N-2}

The step from Eq. (31) to Eqs. (32), or Eqs. (32) to Eqs. (35) does not depend on the size of the equations. It is a general step from size $2n-1$ to $2n+1$, providing a proof that a system with $N = 2n+1, \zeta = 1, \pi = n$ and $\nu = n$ can produce a straight line interaction curve with an eigenvector changing as an nth order polynomial in P.

c. The inertias of $\mathbf{S}$.

A diagonalised $\mathbf{K}$ was used in Step 3 of the necessity proof above, but the displacements $\mathbf{u}$ could have been chosen so that $\mathbf{S}$ is diagonal. One difference between diagonalising $\mathbf{K}$ and $\mathbf{S}$ is that $\mathbf{K}$ has a nullity of $\zeta_{\mathrm{K}} = 1$, but the nullity of $\mathbf{S}$ could be higher.

A diagonalised $\mathbf{S}$ has the form

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_\zeta & & \\ & \mathbf{I}_\pi & \\ & & -\mathbf{I}_\nu \end{bmatrix}. \tag{37}$$

For demonstration, assume that $\mathbf{S}$ has a nullity of $\zeta_{\mathrm{S}} = 3$, and therefore has 3 independent null vectors.

Only one of these null vectors is in the subspace spanned by $\mathbf{u}_0 \ldots \mathbf{u}_4$. Otherwise, if another null vector of $\mathbf{S}$ were a linear combination of $\mathbf{u}_0 \ldots \mathbf{u}_4$, it could be included in the basis of Eq. (7). Let it be $\mathbf{u}_3$. With $\mathbf{S}\mathbf{u}_3 = \mathbf{0}$, the last two columns of the matrix $\mathbf{A}$ in Eq. (10) would be constant, and the vector $\mathbf{v}$ in Eq. (11) would be cubic at most. Equation (13) would also be cubic at most, and could not have five independent vectors.

The two null vectors outside $\mathbf{u}_0 \ldots \mathbf{u}_4$ are added to $\mathbf{u}_0 \ldots \mathbf{u}_4$ to form a set $\mathbf{u}_0 \ldots \mathbf{u}_6$. As in the previous step 3, $\mathbf{u}_0 \ldots \mathbf{u}_6$ is orthonormalised to $\bar{\mathbf{u}}_0 \ldots \bar{\mathbf{u}}_6$ by the same algorithm as before, but in the order $\mathbf{u}_6$ to $\mathbf{u}_0$. Following the arguments of that section, the $\bar{\mathbf{u}}_i$ satisfy Eqs. (24), except that $\mathbf{S}$ replaces $\mathbf{K}$ in Eq. (24c), and with $\bar{\mathbf{u}}_i$ written

$$\bar{\mathbf{u}}_i = \begin{bmatrix} \bar{\mathbf{u}}_i^\zeta \\ \bar{\mathbf{u}}_i^\pi \\ \bar{\mathbf{u}}_i^\nu \end{bmatrix}, \tag{38}$$

$\bar{\mathbf{u}}_i^{\mathrm{T}} = [\,\bar{\mathbf{u}}_i^{\zeta\mathrm{T}}\ \mathbf{0}\ \mathbf{0}\,]$, $i = 6, \ldots, 4$ and $\bar{\mathbf{u}}_i^{\mathrm{T}} = [\mathbf{0}\,\bar{\mathbf{u}}_i^{\pi\mathrm{T}}\bar{\mathbf{u}}_i^{\nu\mathrm{T}}]$, $i = 3, \ldots, 0$, with the same conclusion that $\pi_{\mathrm{S}} \geq 4$ and $\nu_{\mathrm{S}} \geq 4$. The only difference between here and Step 3 is that the total number of freedoms is now increased to $N \geq \zeta_{\mathrm{S}} + \pi_{\mathrm{S}} + \nu_{\mathrm{S}}, \geq 11$ in this example.

Result 2:

If $n+2$ solutions to Eq. (4) are different points lying in a straight line of the P_1, P_2 plane, and the eigenvectors at these points can be described by an nth order polynomial in distance along this line, then the connecting line is an eigenvalue interaction, with the eigenvector described by the same nth order polynomial.

The following proof uses a 2nd order polynomial for simplicity, but the argument applies to any order n.

Consider four different points A, B, C and D which lie in a straight line in a P_1, P_2 plane. All are solutions to Eq. (4) (but no assumption is made about any connections by interaction curves). Further, the eigenvectors $\mathbf{u}^{\mathrm{A}} \ldots \mathbf{u}^{\mathrm{D}}$ at these points are related by the quadratic

$$\mathbf{u}(\alpha) = \mathbf{u}_0 + \alpha\mathbf{u}_1 + \frac{\alpha^2}{2!}\mathbf{u}_2 \tag{39}$$

where α is a measure of distance along the line AD. As before, the origin is moved to some point on the line, and the axes are transformed so that P is directed along this line, making P a suitable distance measure. Thus for $\alpha \equiv P = P^{\mathrm{A}}$

$$\begin{aligned} \mathbf{0} &= (\mathbf{K} - P^{\mathrm{A}}\mathbf{S})\left(\mathbf{u}_0 + P^{\mathrm{A}}\mathbf{u}_1 + \frac{(P^{\mathrm{A}})^2}{2!}\mathbf{u}_2\right) \\ &= \mathbf{K}\mathbf{u}_0 + P^{\mathrm{A}}(\mathbf{K}\mathbf{u}_1 - \mathbf{S}\mathbf{u}_0) \\ &\quad + (P^{\mathrm{A}})^2(\mathbf{K}\mathbf{u}_2 - 2\mathbf{S}\mathbf{u}_1)/2! + (P^{\mathrm{A}})^3(-3\mathbf{S}\mathbf{u}_2/3!). \end{aligned} \tag{40}$$

When written at all 4 points, Eq. (40) gives

$$\begin{bmatrix} 1 & P^{\mathrm{A}} & (P^{\mathrm{A}})^2 & (P^{\mathrm{A}})^3 \\ 1 & P^{\mathrm{B}} & (P^{\mathrm{B}})^2 & (P^{\mathrm{B}})^3 \\ 1 & P^{\mathrm{C}} & (P^{\mathrm{C}})^2 & (P^{\mathrm{C}})^3 \\ 1 & P^{\mathrm{D}} & (P^{\mathrm{D}})^2 & (P^{\mathrm{D}})^3 \end{bmatrix} \begin{bmatrix} (\mathbf{K}\mathbf{u}_0)^{\mathrm{T}} \\ (\mathbf{K}\mathbf{u}_1 - \mathbf{S}\mathbf{u}_0)^{\mathrm{T}} \\ (\mathbf{K}\mathbf{u}_2 - 2\mathbf{S}\mathbf{u}_1)^{\mathrm{T}}/2! \\ (-3\mathbf{S}\mathbf{u}_2)^{\mathrm{T}}/3! \end{bmatrix} = \mathbf{0} \tag{41}$$

and since the points A,. . . ,D are different,

$$\begin{bmatrix} (\mathbf{K}\mathbf{u}_0)^{\mathrm{T}} \\ (\mathbf{K}\mathbf{u}_1 - \mathbf{S}\mathbf{u}_0)^{\mathrm{T}} \\ (\mathbf{K}\mathbf{u}_2 - 2\mathbf{S}\mathbf{u}_1)^{\mathrm{T}}/2! \\ (-3\mathbf{S}\mathbf{u}_2)^{\mathrm{T}}/3! \end{bmatrix} = \mathbf{0}. \tag{42}$$

Thus, $\forall P$,

$$\begin{aligned} &\mathbf{K}\mathbf{u}_0 + P(\mathbf{K}\mathbf{u}_1 - \mathbf{S}\mathbf{u}_0) \\ &\quad + P^2(\mathbf{K}\mathbf{u}_2 - 2\mathbf{S}\mathbf{u}_1)/2! - P^3 3\mathbf{S}\mathbf{u}_2/3! \\ &\quad = (\mathbf{K} - P\mathbf{S})(\mathbf{u}_0 + P\mathbf{u}_1 + \frac{P^2}{2!}\mathbf{u}_2) = \mathbf{0}, \end{aligned} \tag{43}$$

showing that the line through A, B, C and D is an eigenvalue interaction, with the eigenvector given by Eq. 39 for all values of P, not just $P^{\mathrm{A}}, \ldots, P^{\mathrm{D}}$.

It follows immediately that the polynomial $\mathbf{u}(P)$ cannot change along a straight line - if, for example, $\mathbf{u}(P)$ is constant for part of a straight line interaction curve, it is constant for the entire line, extending to $\pm\infty$.

4 Examples

1. The eigenvalue equation

$$\left(\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} - P_1\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - P_2\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}\right)\mathbf{u} = \mathbf{0} \tag{44}$$

has interaction curves as seen in Fig. 4.

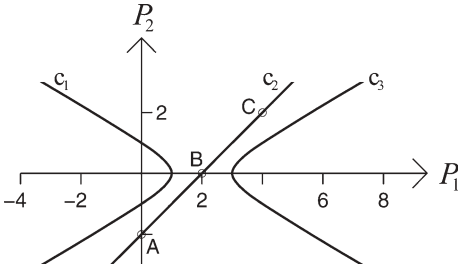


Fig. 4 Interaction curves for the eigenvalue problem of Eq. (44)

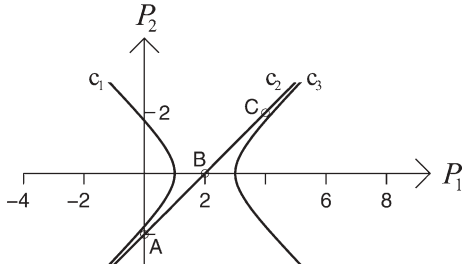


Fig. 5 Interaction curves for the eigenvalue problem of Eq. (46)

The second interaction curve is straight, and in the direction $P_1 = P_2$. The mode $\mathbf{u}(P)$ varies along this line as a linear function of distance ($\mathbf{u}^A$, $\mathbf{u}^B$ and $\mathbf{u}^C$ are different, but can be scaled so that $\mathbf{u}^B = \mathbf{u}^A + \mathbf{u}^C$), and this is the maximum complexity possible, contingent on $\zeta = 1$, $\pi = 1$ and $\nu = 1$ for both **K** and **S**, when **K** is set up at any point on the line, and **S** is in the direction of the line.

These inertias are now checked. **K** has these values, fairly clearly, but those of **S** are less obvious. Since the direction of the line is $P_1 = P_2$, **S** in this direction is $\mathbf{S}_1 + \mathbf{S}_2$ (or some multiple of this);

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}. \tag{45}$$

Inertias of this are $\zeta = 1$, $\pi = 1$ and $\nu = 1$, as anticipated.

2. The eigenvalue equation

$$\left(\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} - P_1 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - P_2 \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right) \mathbf{u} = \mathbf{0} \tag{46}$$

has interaction curves as seen in Fig. 5.

As in the previous example, the second interaction curve is straight, in the direction $P_1 = P_2$. The **S** matrix in this direction is

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \tag{47}$$

Inertias of this matrix are $\zeta = 2$, $\pi = 1$ and $\nu = 0$. Mode variation along a straight interaction curve is limited to a polynomial, of order no higher than the least positive or negative inertia of either **K** or **S** (where **K** is the complete matrix of Eq. (46), evaluated at some point (P_1, P_2) on c_2, and **S** is the matrix of Eq. (47)), and since $\nu_S = 0$, the mode along c_2 must be constant. This mode is $\mathbf{u} = [0\ 1\ 0]^T$. Further, **u** along c_2 must be capable of expressing null vectors of both **K** and **S**, and since it is constant, is a null vector of both. Checking this, **u** is clearly a null vector of **S**. To show that it is a null vector of **K**, this matrix is first formed at a generic point on c_2, where $P_1 = 2 + P_2$,

$$\mathbf{K}(P_2) = \begin{bmatrix} -1 - P_2 & 0 & P_2 \\ 0 & 0 & 0 \\ P_2 & 0 & 1 - P_2 \end{bmatrix}, \tag{48}$$

when it is now seen that $\mathbf{K}(P_2)\mathbf{u} = \mathbf{0}\ \forall\ P_2$, and since $\zeta_K = 1$, this is the sole null vector.

5 Closure

Solutions to eigenvalue problems are in two parts, the eigenvalue and eigenvector. With multiparameter problems, the eigenvalue part of the solution is described by interaction curves in an eigenvalue space, and every such eigenvalue solution has an associated eigenvector. If all points on a curve have the same eigenvector then the curve is necessarily a straight line, but the converse is far more complex. If an interaction curve is straight, then the eigenvector is restricted to polynomial variation with distance along this line, and the order of this polynomial is limited by both the curve number and the size of the eigenvalue formulation. There is a further restriction: the direction of the line implies a matrix **S**, and the order of the polynomial is also limited by the inertias of this matrix.

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