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Simple Paradoxical Replications of Sets*

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Abstract. The paradox of Banach, Tarski, and Hausdorff shows that any two bounded sets $M, N \subseteq E^3$ with non-empty interior are equidecomposable. The result remains true if M and N are replaced by collections of sets. We present quantified versions of the paradox by giving estimates for the minimal number of pieces in such decompositions. The emphasis is on replications of sets M, i.e., on the equidecomposability of M with k copies of M, $k \geq 2$. In particular, we discuss the problem of replicating the cube.

1. Introduction

The theory of equidecomposability of sets gives rise to new interpretations and solutions of some classical geometric problems. The most popular example is the task of squaring the circle, which has been solved by Laczkovich [8]. Another problem, serving as a motivation for the present paper, is the duplication of the cube, whose traditional form has essentially influenced the development of mathematics for thousands of years (see [3]). The basis for an adequate reformulation and solution of the last problem is given by the paradox of Banach, Tarski, and Hausdorff (see [1] and [6]). It shows that any cube C in Euclidean space E^3 can be decomposed into a finite number $k = l + m$ of disjoint subsets $C_1, C_2, \ldots, C_k$ such that both $\varphi_1(C_1), \varphi_2(C_2), \ldots, \varphi_l(C_l)$ and $\varphi_{l+1}(C_{l+1}), \varphi_{l+2}(C_{l+2}), \ldots, \varphi_k(C_k)$ form decompositions of C, where φ_i, $1 \leq i \leq k$, are suitable isometries of E^3. (Later we say that "C and $C + C$ are equidecomposable using k pieces".) However, the smallest possible number k in the above construction is unknown. A very rough estimate from [5] gives $k < 21, 556, 563, 000$. The goal of the present paper is to find reasonable estimates for the minimal number of pieces, not only for the duplication of the cube, but for paradoxical replications of arbitrary bounded sets

* This research is part of the author's Ph.D. thesis [12].

$M \subseteq E^3$ with non-empty interior. First we present an appropriate general version of the theorem of Banach, Tarski, and Hausdorff.

2. The Theorem of Banach, Tarski, and Hausdorff

Let $\mathcal{I}_3$ be the group of isometries of E^3. For $A, B \subseteq E^3$, we write $A \simeq B$ if A and B are congruent, i.e., $A = \varphi(B)$ for some $\varphi \in \mathcal{I}_3$. $\langle 2^{E^3} \rangle$ is to denote the free Abelian semigroup generated by the subsets of E^3. That is, the elements of $\langle 2^{E^3} \rangle$ are finite formal sums $a = A_1 + A_2 + \cdots + A_k$ of subsets A_i of E^3. For $k \geq 1$ and $A \subseteq E^3$, $k \cdot A$ is meant to be the k-fold sum $A + A + \cdots + A$. The set $\{0, 1, 2, \ldots\}$ of non-negative integers is denoted by $\mathbb{N}$. We write $\mathcal{S}_n$ for the symmetric group of all permutations of n elements.

Definition 1.

(a) Let $a = A_1 + A_2 + \cdots + A_k$, $b = B_1 + B_2 + \cdots + B_l \in \langle 2^{E^3} \rangle$, and $n \in \mathbb{N}$. The sums a and b are called *equidecomposable using n pieces* ($a \overset{n}{\simeq} b$) if there exist representations $n = r_1 + r_2 + \cdots + r_k$, $r_i \in \mathbb{N}$, and $n = s_1 + s_2 + \cdots + s_l$, $s_j \in \mathbb{N}$, decompositions

$$A_i = \bigcup_{u=r_1+\cdots+r_{i-1}+1}^{r_1+\cdots+r_i} M_u \quad \text{and} \quad B_j = \bigcup_{v=s_1+\cdots+s_{j-1}+1}^{s_1+\cdots+s_j} N_v$$

for $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, l\}$, and a permutation $\pi \in \mathcal{S}_n$ such that

$$N_v \simeq M_{\pi(v)}$$

holds true for $v \in \{1, 2, \ldots, n\}$.

(b) Two sums $a, b \in \langle 2^{E^3} \rangle$ are called *equidecomposable* ($a \overset{*}{\simeq} b$) if $a \overset{n}{\simeq} b$ is valid for some $n \in \mathbb{N}$.

Note that we use the word “decomposition” for disjoint decompositions in the sense of set theory only. This is totally different from the concept of a decomposition in elementary geometry, where for instance a closed triangle can be “decomposed” into two closed subtriangles having boundary points in common.

Roughly speaking, the relation $a \overset{n}{\simeq} b$ means that there exists a “construction set” consisting of n pieces which, on the one hand, can be used to build up all the terms of a simultaneously and, on the other hand, suffice for constructing b in the same way. The above definition shows in particular that, for any two sets $A, B \subseteq E^3$, the relations $A \overset{1}{\simeq} A$, $A + B \overset{2}{\simeq} A + B$, $A \overset{n}{\simeq} A$ for $n \geq 1$, $A \overset{1}{\simeq} A + \emptyset$, and $\emptyset \overset{0}{\simeq} k \cdot \emptyset$ for $k \geq 1$ are fulfilled.

Clearly, one can define more general concepts of equidecomposability by considering other groups instead of $\mathcal{I}_3$ possibly acting on other spaces than E^3. In particular, the following abstract Proposition 1 and Theorem 2 remain true in this much more general setting. We refer the reader to the comprehensive survey [14] for considerations of that type.

The equidecomposability of sums from the semigroup $\langle 2^{E^3} \rangle$ generalizes the classical equidecomposability $A \overset{*}{\simeq} B$ of subsets $A, B \subseteq E^3$. The following can easily be seen as for the classical concept.

Proposition 1. *The relation $\overset{*}{\simeq}$ is an equivalence relation in $\langle 2^{E^3} \rangle$. In particular*:

(a) *The relation $a \overset{k+n}{\simeq} a$ is valid for all $a = A_1 + A_2 + \cdots + A_k \in \langle 2^{E^3} \rangle$ and $n \geq 0$.*
(b) *Let $a, b \in \langle 2^{E^3} \rangle$ be sums fulfilling $a \overset{n}{\simeq} b$. Then $b \overset{n}{\simeq} a$.*
(c) *Let $a, b, c \in \langle 2^{E^3} \rangle$ be such that $a \overset{n}{\simeq} b$ and $b \overset{m}{\simeq} c$. Then $a \overset{nm}{\simeq} c$.*

We recall the well-known paradox due to Banach, Tarski, and Hausdorff (see [1], [5], and p. 29 of [14]):

Theorem 1 (Banach, Tarski, Hausdorff). *Let $M, N \subseteq E^3$ be bounded sets with non-empty interior. Then $M \overset{*}{\simeq} N$.*

The paradox will find a natural generalization concerning the equidecomposability of sums from $\langle 2^{E^3} \rangle$. However, the main task of the present paper is to quantify the result by giving reasonable upper estimates for the optimal number of pieces realizing certain equidecomposabilities.

Definition 2. Let $a, b \in \langle 2^{E^3} \rangle$ be equidecomposable. The *degree of equidecomposability* of a and b is meant to be

$$\deg(a, b) = \min\{n\colon\ a \overset{n}{\simeq} b\}.$$

The main tool for finding estimates for $\deg(a, b)$ is the Banach–Schröder–Bernstein theorem concerning the following relation in $\langle 2^{E^3} \rangle$.

Definition 3.

(a) The sums $a = A_1 + A_2 + \cdots + A_k$ and b from $\langle 2^{E^3} \rangle$ fulfil the relation $a \overset{n}{\succeq} b$, $n \in \mathbb{N}$, if there exist subsets $A_i' \subseteq A_i$, $1 \leq i \leq k$, such that the relation $a' \overset{n}{\simeq} b$ holds true for the sum $a' = A_1' + A_2' + \cdots + A_k'$.
(b) The sums $a, b \in \langle 2^{E^3} \rangle$ fulfil the relation $a \overset{*}{\succeq} b$ (a is *larger than b by decomposition*) if $a \overset{n}{\succeq} b$ is valid for some $n \in \mathbb{N}$.

The relation $\overset{*}{\succeq}$ in $\langle 2^{E^3} \rangle$ is reflexive and transitive. Indeed, $a \overset{k}{\succeq} a$ trivially holds true for any $a = A_1 + A_2 + \cdots + A_k \in \langle 2^{E^3} \rangle$. For $a, b, c \in \langle 2^{E^3} \rangle$ fulfilling $a \overset{n}{\succeq} b$ and $b \overset{m}{\succeq} c$, we have $a \overset{nm}{\succeq} c$.

The following statement is the adequate generalization of the Banach–Schröder–Bernstein theorem, which is usually formulated for subsets $A, B \in E^3$, but not for sums

$a, b \in \langle 2^{E^3} \rangle$ (see p. 25 of [14]). However, the idea of its proof is exactly the same as in [14]. (A proof including all technical details is given in [12].)

Theorem 2. *Suppose the sums $a, b \in \langle 2^{E^3} \rangle$ fulfil the relations $a \overset{n}{\succeq} b$ and $b \overset{m}{\succeq} a$. Then $a \overset{n+m}{\simeq} b$. In particular, $\overset{*}{\succeq}$ gives rise to a partial ordering of the $\overset{*}{\simeq}$-classes in $\langle 2^{E^3} \rangle$.*

The considerations concerning the paradox of Banach, Tarski, and Hausdorff start with the special case of replicating spheres and balls. We use $B(x, r)$ to denote the closed Euclidean ball centred at $x \in E^3$ with radius $r > 0$. The two-dimensional unit sphere $\mathrm{bd}(B(0, 1))$ is denoted by S^2. As usual, SO_3 denotes the group of all rotations mapping the unit ball $B(0, 1)$ onto itself. We use the following statements, which follow immediately from the proofs of Theorems 4.5 and 4.7 in [14].

Proposition 2. *Let $\rho, \sigma \in SO_3$ be generators of a free non-Abelian subgroup of rank two. Then the following are true*:

(a) *There exist sets A, B, C, D such that the sphere S^2 admits the decompositions*

$$S^2 = A \cup B \cup C \cup D = A \cup \rho(B) = C \cup \sigma(D).$$

(b) *There exist sets A', B', C', D', E' and a translation τ such that $B(0, 1)$ can be decomposed into*

$$B(0, 1) = A' \cup B' \cup C' \cup D' \cup E' = A' \cup \rho(B') = C' \cup \sigma(D') \cup \tau(E').$$

Besides that, we need the following lemma (see [4]).

Lemma 1. *Let $\rho, \sigma \in SO_3$ be rotations with the same angle α of rotation such that their axes are perpendicular and $\tan(\alpha/2)$ is transcendental. Then ρ and σ generate a free non-Abelian group of rank two.*

The above give rise to a theorem on k-fold replications of spheres, balls, and pointed balls, which includes a statement on the optimal number of pieces in the corresponding decompositions. The *pointed ball* centred at x with radius r is meant to be the set $B^p(x, r) = B(x, r) \backslash \{x\}$.

Theorem 3. *Let $x \in E^3$, $r > 0$, and $k \in \{2, 3, 4, \ldots\}$. Then the sphere $S = \mathrm{bd}(B(x, r))$ as well as the balls $B^p(x, r)$ and $B(x, r)$ admit a k-fold replication in the sense of $S \overset{*}{\simeq} k \cdot S$, $B^p(x, r) \overset{*}{\simeq} k \cdot B^p(x, r)$, and $B(x, r) \overset{*}{\simeq} k \cdot B(x, r)$, respectively. Moreover,*

$$\deg(S, k \cdot S) = \deg(B^p(x, r), k \cdot B^p(x, r)) = 2k$$

and

$$\deg(B(x, r), k \cdot B(x, r)) = 3k - 1.$$

Proof. We assume that $x = 0$ and $r = 1$ without loss of generality, i.e., $S = S^2$ and $B(x, r) = B(0, 1)$. By Lemma 1, there exist rotations $\rho, \sigma \in SO_3$ generating a free non-Abelian group of rank two.

We employ part (a) of Proposition 2 for proving $S \overset{2k}{\simeq} k \cdot S$. Let $B_{i+1} = \rho^{-(i-1)}(B)$, $C_i = \rho^{-(i-1)}(C)$, and $D_i = \rho^{-(i-1)}(D)$ for $i = 1, 2, 3, \ldots$. Obviously, B_k can be decomposed into $B_k = \rho^{-(k-1)}(\rho(B)) = \rho^{-(k-1)}(B \cup C \cup D) = B_{k+1} \cup C_k \cup D_k$. Now a simple induction with respect to k shows that S admits the following decompositions:

$$S = A \cup B_k \cup C_1 \cup C_2 \cup \cdots \cup C_{k-1} \cup D_1 \cup D_2 \cup \cdots \cup D_{k-1} \tag{1}$$

as well as

$$\begin{aligned} S &= A \cup \rho^{k-1}(B_k) \quad \text{and} \\ S &= \rho^{i-1}(C_i) \cup \sigma\rho^{i-1}(D_i) \qquad \text{for} \quad 1 \le i \le k-1. \end{aligned} \tag{2}$$

This proves $S \overset{2k}{\simeq} k \cdot S$ and, in particular, $\deg(S, k \cdot S) \le 2k$. On the other hand, $\deg(S, k \cdot S) \ge 2k$, since each of the k copies of S must consist of at least two pieces.

The treatment of the pointed ball $B^p(0, 1)$ is similar; one has to consider half-open radii $\{\lambda x\colon\ 0 < \lambda \le 1\}$ of the ball instead of points x from the sphere.

We use Proposition 2(b) for replicating the solid ball $B(x, r) = B(0, 1)$. Putting $B'_{i+1} = \rho^{-(i-1)}(B')$, $C'_i = \rho^{-(i-1)}(C')$, $D'_i = \rho^{-(i-1)}(D')$, and $E'_i = \rho^{-(i-1)}(E')$ for $i = 1, 2, 3, \ldots$, we obtain the decompositions

$$B(x, r) = A' \cup B'_k \cup C'_1 \cup \cdots \cup C'_{k-1} \cup D'_1 \cup \cdots \cup D'_{k-1} \cup E'_1 \cup \cdots \cup E'_{k-1} \tag{3}$$

as well as

$$\begin{aligned} B(x, r) &= A' \cup \rho^{k-1}(B'_k) \quad \text{and} \\ B(x, r) &= \rho^{i-1}(C'_i) \cup \sigma\rho^{i-1}(D'_i) \cup \tau\rho^{i-1}(E'_i) \qquad \text{for} \quad 1 \le i \le k-1. \end{aligned} \tag{4}$$

This gives $B(x, r) \overset{3k-1}{\simeq} k \cdot B(x, r)$ and $\deg(B(x, r), k \cdot B(x, r)) \le 3k - 1$.

The inverse inequality $\deg(B(x, r), k \cdot B(x, r)) \ge 3k - 1$ is based on an idea from the paper by Robinson [13], who has originally shown that the smallest possible number of pieces in the duplication $B(x, r) \overset{*}{\simeq} 2 \cdot B(x, r)$ of solid balls is five. In fact, he has proved the following (see also pp. 40–41 of [14]): *There do not exist any four disjoint subsets* A_1, A_2, A_3, A_4 *of* $B(0, 1)$ *and isometries* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{I}_3$ *such that both* $\alpha_1(A_1) \cup \alpha_2(A_2)$ *and* $\alpha_3(A_3) \cup \alpha_4(A_4)$ *cover* $B(0, 1)$. Consequently, in any realization of the equidecomposability $B(x, r) \overset{*}{\simeq} k \cdot B(x, r)$ at most one term of the sum $k \cdot B(x, r)$ is decomposed into two pieces, whereas the others need at least three. Hence, $\deg(B(x, r), k \cdot B(x, r)) \ge 2 + 3(k - 1) = 3k - 1$. This completes the proof. $\square$

We remark that one can similarly show the equidecomposability of $E^3 \setminus \{x\}$ and $k \cdot (E^3 \setminus \{x\})$ with $\deg(E^3 \setminus \{x\}, k \cdot (E^3 \setminus \{x\})) = 2k$ for all $x \in E^3$ and $k \ge 2$. The considerations of solid balls can be continued to $E^3 \overset{3k-1}{\simeq} k \cdot E^3$.

All estimates to follow for the degree of equidecomposabilities $a \overset{*}{\simeq} b$ make use of the Banach–Schröder–Bernstein theorem (Theorem 2). That is, we shall show $a \overset{n}{\succeq} b$ and $b \overset{m}{\succeq} a$ in order to conclude that $\deg(a, b) \leq n + m$. Proposition 3 is a useful tool for proofs of that type.

Proposition 3. *Let $a = A_1 + A_2 + \cdots + A_k, b = B_1 + B_2 + \cdots + B_l \in \langle 2^{E^3} \rangle$, and $n \in \mathbb{N}$. Suppose that there exist pairwise disjoint subsets $M_1, M_2, \ldots, M_n$ of A_1, a representation $n = s_1 + s_2 + \cdots + s_l$, $s_j \in \mathbb{N}$, and isometries $\varphi_v \in \mathcal{I}_3$, $1 \leq v \leq n$, such that the terms B_j, $1 \leq j \leq l$, of the sum b are covered by*

$$B_j \subseteq \bigcup_{v=s_1+\cdots+s_{j-1}+1}^{s_1+\cdots+s_j} \varphi_v(M_v).$$

Then $a \overset{n}{\succeq} b$.

Proof. Obviously, there exist suitable subsets $M'_v \subseteq M_v$ such that the sets B_j admit decompositions

$$B_j = \bigcup_{v=s_1+\cdots+s_{j-1}+1}^{s_1+\cdots+s_j} \varphi_v(M'_v).$$

Thus we have $A'_1 \overset{n}{\simeq} b$ with $A'_1 = M'_1 \cup M'_2 \cup \cdots \cup M'_n$. Putting $A'_i = \emptyset$ for $2 \leq i \leq k$ and $a' = A'_1 + A'_2 + \cdots + A'_k$ we obtain $a' \overset{n}{\simeq} b$. By Definition 3, this shows that $a \overset{n}{\succeq} b$. □

The following quantified generalization of Theorem 1 rests on coverings of bounded sets by balls. A quantity describing optimal coverings of that type is given by *Kolmogoroff's entropy function* (see p. 4 of [7]). For bounded sets $M \subseteq E^3$ and radii $r > 0$, the function $\mathcal{N}_r(M)$ is defined by

$$\mathcal{N}_r(M) = \min\{m \in \mathbb{N}\colon\ M \text{ can be covered by } m \text{ closed balls of radius } r\}.$$

Theorem 4. *Let $a = A_1 + A_2 + \cdots + A_k$ and $b = B_1 + B_2 + \cdots + B_l$ be sums from $\langle 2^{E^3} \rangle$ such that all sets A_i and B_j are bounded and that at least one term of the sum a as well as one term of b has inner points. Then $a \overset{*}{\simeq} b$.*

If r and s are the radii of closed balls contained in a term of a and of b, respectively, then

$$\deg(a, b) \leq 3 \left(\sum_{i=1}^{k} \mathcal{N}_s(A_i) + \sum_{j=1}^{l} \mathcal{N}_r(B_j) \right) - 2.$$

Proof. Without loss of generality, we assume that $B(x_0, r) \subseteq A_1$. By the definition of $\mathcal{N}_r(B_j)$, the sets B_j, $1 \leq j \leq l$, can be simultaneously covered by $h = \mathcal{N}_r(B_1) + \mathcal{N}_r(B_2) + \cdots + \mathcal{N}_r(B_l)$ translates of $B(x_0, r)$. On the other hand, by Theorem 3, there exist $3h-1$ disjoint subsets $M_1, M_2, \ldots, M_{3h-1} \subseteq B(x_0, r)$ such that suitable isometric images of them form decompositions of the above-mentioned h translates of $B(x_0, r)$. Consequently, the images of the $3h-1$ disjoint subsets $M_v \subseteq A_1$ give rise to coverings of the terms B_j of the sum b in the sense of Proposition 3. Thus we obtain $a \overset{3h-1}{\succeq} b$, i.e.,

$$a \overset{n}{\succeq} b \qquad \text{with} \quad n = 3\left(\sum_{j=1}^{l} \mathcal{N}_r(B_j)\right) - 1.$$

Similar arguments yield

$$b \overset{m}{\succeq} a \qquad \text{with} \quad m = 3\left(\sum_{i=1}^{k} \mathcal{N}_s(A_i)\right) - 1.$$

Now Theorem 2 shows that $a \overset{m+n}{\simeq} b$ and, in particular,

$$\deg(a, b) \leq m + n = 3\left(\sum_{i=1}^{k} \mathcal{N}_s(A_i) + \sum_{j=1}^{l} \mathcal{N}_r(B_j)\right) - 2. \qquad \square$$

We finish this section with a corollary based on the following upper estimate of Kolmogoroff's entropy function $\mathcal{N}_r(M)$.

Proposition 4. *Let $M \subseteq E^3$ be a bounded set whose circumball is of radius R and let $r > 0$ be a real number. Then*

$$\mathcal{N}_r(M) < \min\left\{\left(\sqrt{3}\frac{R}{r} + 1\right)^3, \frac{\sqrt{3}\pi}{2}\left(\frac{R}{r} + 2\right)^3\right\}.$$

Proof. Let $B(x_0, R)$ be the circumball of M. It is contained in a cube C whose edges are of length $2R$. Clearly, C can be covered by h^3 subcubes whose edges have length $2R/h$, where the integer h is chosen such that $\sqrt{3}R/r \leq h < \sqrt{3}R/r + 1$. Any of these subcubes has a circumscribed ball of radius r, since $\sqrt{3}R/h \leq r$. This shows that M admits a covering by h^3 balls of radius r. Thus

$$\mathcal{N}_r(M) \leq h^3 < \left(\sqrt{3}\frac{R}{r} + 1\right)^3.$$

For proving the second estimate we use a covering of E^3 by a lattice of cubes with edges of length $2r/\sqrt{3}$. M is covered by those cubes whose intersection with the circumball $B(x_0, R)$ is non-empty. Let j be the number of these cubes. Obviously, they all are contained in $B(x_0, R + 2r)$. We can estimate their volume by

$$j \cdot \left(\frac{2r}{\sqrt{3}}\right)^3 < \mathrm{vol}(B(x_0, R + 2r)) = \frac{4\pi}{3}(R + 2r)^3.$$

Any of the j cubes can be covered by a ball of radius r. Thus we obtain

$$\mathcal{N}_r(M) \leq j < \left(\frac{\sqrt{3}}{2r}\right)^3 \cdot \frac{4\pi}{3}(R+2r)^3 = \frac{\sqrt{3}\pi}{2}\left(\frac{R}{r}+2\right)^3.$$

This completes the proof of Proposition 4. □

It can easily be seen that the minimum considered in Proposition 4 is given by $(\sqrt{3}R/r+1)^3$ if $R/r \leq 5.3$ and by $(\sqrt{3}\pi/2)\,(R/r+2)^3$ if $R/r \geq 5.4$.

Corollary 1. *Let $a = A_1 + A_2 + \cdots + A_k$ and $b = B_1 + B_2 + \cdots + B_l$ be sums from $\langle 2^{E^3}\rangle$ such that all sets A_i and B_j are bounded and that at least one term of the sum a as well as one term of b has inner points. Moreover, let r and s be the radii of closed balls contained in a term of a and of b, respectively.*

(a) *If $R_1, R_2, \ldots, R_k$ and $S_1, S_2, \ldots, S_l$ are the radii of the circumballs of the sets $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_l$, respectively, then*

$$\begin{aligned}\deg(a,b) < 3\Bigg(&\sum_{i=1}^{k}\min\left\{\left(\sqrt{3}\frac{R_i}{s}+1\right)^3, \frac{\sqrt{3}\pi}{2}\left(\frac{R_i}{s}+2\right)^3\right\}\\ &+\sum_{j=1}^{l}\min\left\{\left(\sqrt{3}\frac{S_j}{r}+1\right)^3, \frac{\sqrt{3}\pi}{2}\left(\frac{S_j}{r}+2\right)^3\right\}\Bigg)-2.\end{aligned}$$

(b) *If R and S are the largest radii of the circumballs of the k terms of the sum a and of the l terms of b, respectively, then*

$$\begin{aligned}\deg(a,b) < 3\Bigg(&k\cdot\min\left\{\left(\sqrt{3}\frac{R}{s}+1\right)^3, \frac{\sqrt{3}\pi}{2}\left(\frac{R}{s}+2\right)^3\right\}\\ &+l\cdot\min\left\{\left(\sqrt{3}\frac{S}{r}+1\right)^3, \frac{\sqrt{3}\pi}{2}\left(\frac{S}{r}+2\right)^3\right\}\Bigg)-2.\end{aligned}$$

We remark that in [5] the following very rough estimate for the degree of equidecomposability of sets $M, N \subseteq E^3$ is given: if M and N contain a ball of radius r in their intersection and if their union is covered by a ball of radius R, then $\deg(M,N) \leq (1+2420(R/r)^6)^2$. The second part of Corollary 1 gives rise to the essentially better estimate $\deg(M,N) < 6\cdot\min\{(\sqrt{3}(R/r)+1)^3, (\sqrt{3}\pi/2)(R/r+2)^3\} - 2$.

3. Replicating Sets

Therorem 4 shows that any bounded set $M \subseteq E^3$ with non-empty interior admits a k-fold replication in the sense of $M \overset{*}{\simeq} k\cdot M$ for all $k \geq 2$. Moreover, if M contains a closed ball of radius r, then

$$\deg(M, k\cdot M) \leq 3(k+1)\cdot\mathcal{N}_r(M) - 2. \tag{5}$$

In this section we sharpen this estimate. First we present a statement resting on coverings by balls as the claim of Theorem 4.

Theorem 5. *Let $M \subseteq E^3$ be a bounded set with non-empty interior and let $k \geq 2$ be an integer.*

(a) *If M can be covered by m translates of a closed ball being a subset of M, then*

$$\deg(M, k \cdot M) \leq 3m(k-1) + 3.$$

(b) *If M can be covered by m translates of a pointed ball being a subset of M, then*

$$\deg(M, k \cdot M) \leq 2m(k-1) + 3.$$

Proof. Let the ball $B(x,r)$ be a subset of M such that M admits a covering

$$M \subseteq \tau_1(B(x,r)) \cup \tau_2(B(x,r)) \cup \cdots \cup \tau_m(B(x,r)) \tag{6}$$

with suitable translations $\tau_i \in \mathcal{I}_3$. By Theorem 3, we have the relation $B(x,r) \overset{3m(k-1)+2}{\simeq} (m(k-1)+1) \cdot B(x,r)$. The particular structure of this equidecomposability is described by formuals (3) and (4) in the proof of Theorem 3. Accordingly, there exist a decomposition

$$B(x,r) = A' \cup M_2 \cup M_3 \cup \cdots \cup M_{3m(k-1)+2} \tag{7}$$

and isometries $\varphi_u \in \mathcal{I}_3$, $2 \leq u \leq 3m(k-1)+2$, such that

$$B(x,r) = A' \cup \varphi_2(M_2) \tag{8}$$

and

$$B(x,r) = \bigcup_{u=3v}^{3v+2} \varphi_u(M_u) \tag{9}$$

for $1 \leq v \leq m(k-1)$.

We put $M_1 = (M \setminus B(x,r)) \cup A'$. By (7), we obtain the decomposition

$$M = M_1 \cup M_2 \cup \cdots \cup M_{3m(k-1)+2}.$$

Moreover, the first term M of the sum $k \cdot M$ has the representation

$$M = (M \setminus B(x,r)) \cup A' \cup \varphi_2(M_2) = M_1 \cup \varphi_2(M_2)$$

by (8). Finally, by (6) and (9), the $k-1$ remaining terms of $k \cdot M$ admit the coverings

$$M \subseteq \bigcup_{i=1}^{m} \bigcup_{u=3((j-1)m+i)}^{3((j-1)m+i)+2} \tau_i \varphi_u(M_u)$$

for $1 \leq j \leq k-1$. Thus all terms of $k \cdot M$ are covered by isometric images of the disjoint sets $M_u \subseteq M$, $1 \leq u \leq 3m(k-1)+2$. Now Proposition 3 yields $M \overset{3m(k-1)+2}{\succeq} k \cdot M$. The counterpart $k \cdot M \overset{1}{\succeq} M$ is trivial. Applying Theorem 2 we conclude that $M \overset{3m(k-1)+3}{\simeq} k \cdot M$. Hence part (a) of Theorem 5 is proved.

Part (b) can be shown in the same way by the aid of the statement from Theorem 3 on replications of pointed balls. □

Theorem 5 sharpens the estimate (5) obtained by the general Theorem 4 in so far as we get

$$\deg(M, k \cdot M) \leq 3(k-1) \cdot \mathcal{N}_r(M) + 3.$$

Proposition 4 gives rise to the following.

Corollary 2. *Let $M \subseteq E^3$ be a bounded set with non-empty interior and let $k \geq 2$ be an integer. If M contains a closed ball of radius r and if R is the radius of the circumball of M, then*

$$\deg(M, k \cdot M) \leq 3(k-1) \cdot \min\left\{\left(\sqrt{3}\frac{R}{r}+1\right)^3, \frac{\sqrt{3}\pi}{2}\left(\frac{R}{r}+2\right)^3\right\} + 3.$$

As mentioned above, the present paper is motivated by the question for a paradoxical duplication of a cube with a small number of pieces. We illustrate the result attainable by Theorem 5. We have to find the smallest possible m which can be used when applying Theorem 5 to the case of replicating the cube.

Proposition 5. *Any cube $C \subseteq E^3$ can be covered by eight translates of a suitable pointed ball which is contained in C. A covering by seven translates of a solid closed ball being a subset of C does not exist.*

Proof. We consider the cube $C = [-1, 1]^3$. Of course, C is covered by the eight pointed balls $B^p(x, 1)$, $x = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$.

Now let C be covered by m solid balls of radius 1. We have to show that $m \geq 8$. Any of the m balls can contain at most two vertices of C. Accordingly, $m = m_0 + m_1 + m_2$, m_i denoting the number of balls containing exactly i vertices. For $x \in \mathrm{vert}(C)$, let $h_1(x)$ be the number of those balls which cover x but no other vertex, and let $h_2(x)$ be the number of those containing x and one additional vertex. Obviously,

$$m \geq m_1 + m_2 = \sum_{x \in \mathrm{vert} C} (h_1(x) + \tfrac{1}{2}h_2(x)).$$

Next we observe that $h_1(x) \geq 1$ or $h_2(x) = 3$ for any $x \in \mathrm{vert}(C)$. Indeed, if $h_1(x) = 0$, then x together with some neighbourhood of x in C must be covered by balls containing an additional vertex besides x. Each of these balls must contain one of the three edges starting in x as a diameter. Clearly, all three balls of that kind are needed to cover a neighbourhood of x, and therefore $h_2(x) = 3$.

Thus we obtain $h_1(x) + \frac{1}{2}h_2(x) \geq 1$ for all vertices x, and the estimate given above can be continued to

$$m \geq \sum_{x \in \mathrm{vert} C} 1 = 8,$$

which completes the proof. (Moreover it is shown that the optimal covering of C consists of eight balls each containing exactly one vertex.) □

We can apply both parts of Theorem 5 with $m = 8$. Obviously, part (b) gives the better estimate: for any cube $C \subseteq E^3$ and any integer $k \geq 2$,

$$\deg(C, k \cdot C) \leq 16(k-1) + 3, \tag{10}$$

in particular, $\deg(C, C + C) \leq 19$ for duplicating the cube.

The remainder of this section deals with a second theorem concerning the degree of replications $M \overset{*}{\simeq} k \cdot M$.

Theorem 6. *Let $M \subseteq E^3$ be a bounded set such that its closure* $\mathrm{cl}(M)$ *can be covered by m isometric images of its interior* $\mathrm{int}(M)$. *Then*

$$\deg(M, k \cdot M) \leq 2mk + 1$$

for all integers $k \geq 2$.

Before giving the proof we demonstrate the power of Theorem 6 compared with that of Theorem 5. The crucial number m in Theorem 6 is given by a covering of $\mathrm{cl}(M)$ by images of $\mathrm{int}(M)$, whereas in Theorem 5 the set M has to be covered by translates of a ball $B(x, r)$ contained in M. Obviously, this new m can be chosen smaller than that from Theorem 5 for many reasonable sets M, since $\mathrm{int}(M)$ is usually much larger than $B(x, r)$. The next section gives some interesting applications of Theorem 6. We present *universal* estimates of $\deg(M, k \cdot M)$ for sets M belonging to certain classes of sets.

We give an example by applying Theorem 6 to the replication of cubes. In Section 5 we prove that any cube $C \subseteq E^3$ admits a covering by four isometric images of its interior (see Proposition 7). Thus we obtain the estimate

$$\deg(C, k \cdot C) \leq 8k + 1 \tag{11}$$

for all $k \geq 2$, in particular, $\deg(C, C + C) \leq 17$. This obviously improves the above estimate (10) inferred from Theorem 5.

However, in some cases Theorem 5 gives the better result. Assume for instance that M is the union of two closed balls. Then Theorem 5, part (a), applies with $m = 2$, i.e., $\deg(M, k \cdot M) \leq 6k - 3$. On the other hand, the smallest m in Theorem 6 is four. This gives rise to a worse inequality $\deg(M, k \cdot M) \leq 8k + 1$.

In preparation for the proof of Theorem 6, we show that any set M fulfilling the assumptions of the theorem can be covered by m images of an *inner parallel set*. The inner parallel set M_{-r} of $M \subseteq E^3$ to the distance $r > 0$ is defined by

$$M_{-r} = \{x \in M\colon\ B(x, r) \subseteq M\}.$$

Proposition 6. *Let $M \subseteq E^3$ be a bounded set whose closure* $\mathrm{cl}(M)$ *admits a covering by m isometric images of its interior* $\mathrm{int}(M)$. *Then there exists some $\varepsilon > 0$ such that M can be covered by m isometric images of the inner parallel set $M_{-\varepsilon}$.*

Proof. There exist Euclidean motions $\varphi_i \in \mathcal{I}_3$, $1 \le i \le m$, such that

$$\mathrm{cl}(M) \subseteq \varphi_1(\mathrm{int}(M)) \cup \varphi_2(\mathrm{int}(M)) \cup \cdots \cup \varphi_m(\mathrm{int}(M)).$$

We define a function $f\colon \mathrm{cl}(M) \to \mathbb{R}$ by

$$f(x) = \max_{1 \le i \le m} d(x, (\varphi_i(\mathrm{int}(M)))^{\mathrm{c}}),$$

d denoting the Euclidean distance, $(\varphi_i(\mathrm{int}(M)))^{\mathrm{c}}$ the complement of $\varphi_i(\mathrm{int}(M))$, and $d(x, (\varphi_i(\mathrm{int}(M)))^{\mathrm{c}}) = \inf\{d(x, y)\colon y \in (\varphi_i(\mathrm{int}(M)))^{\mathrm{c}}\}$. Obviously, f is positive and continuous on the compact set $\mathrm{cl}(M)$ and therefore attains its minimum $\delta > 0$ at some point $x_0 \in \mathrm{cl}(M)$. Putting $\varepsilon = \delta/2$ we obtain the asserted covering

$$M \subseteq \varphi_1(M_{-\varepsilon}) \cup \varphi_2(M_{-\varepsilon}) \cup \cdots \cup \varphi_m(M_{-\varepsilon}).$$

Indeed, any $x \in M$ fulfils the inequality $f(x) > \varepsilon$. Thus $d(x, (\varphi_i(\mathrm{int}(M)))^{\mathrm{c}}) > \varepsilon$ for some $i \in \{1, 2, \ldots, m\}$, which means that $B(x, \varepsilon) \subseteq \varphi_i(\mathrm{int}(M))$. Accordingly, $x \in (\varphi_i(\mathrm{int}(M)))_{-\varepsilon} = \varphi_i(\mathrm{int}(M)_{-\varepsilon}) \subseteq \varphi_i(M_{-\varepsilon})$. This proves Proposition 6. $\square$

Proof of Theorem 6. By Proposition 6, there exist $\varepsilon > 0$ and isometries $\varphi_i \in \mathcal{I}_3$, $1 \le i \le m$, with

$$M \subseteq \varphi_1(M_{-\varepsilon}) \cup \varphi_2(M_{-\varepsilon}) \cup \cdots \cup \varphi_m(M_{-\varepsilon}). \tag{12}$$

Now we verify the existence of a decomposition

$$M = M_1 \cup M_2 \cup \cdots \cup M_{2mk}$$

of M and of motions $\psi_u \in \mathcal{I}_3$, $1 \le u \le 2mk$, such that the inclusions

$$M_{-\varepsilon} \subseteq \psi_{2j-1}(M_{2j-1}) \cup \psi_{2j}(M_{2j}) \qquad \text{for} \quad 1 \le j \le mk \tag{13}$$

hold true. Without loss of generality, we assume that M is a subset of the pointed ball $B^p(0, 1)$. Recall that the proof of Theorem 3 concerning the replication of spheres and pointed balls has shown the following: for any two rotations $\rho, \sigma \in SO_3$ generating a free non-Abelian subgroup of rank two, there are decompositions

$$B^p(0, 1) = A \cup B_{mk} \cup C_1 \cup C_2 \cup \cdots \cup C_{mk-1} \cup D_1 \cup D_2 \cup \cdots \cup D_{mk-1}$$

as well as

$$\begin{aligned} B^p(0, 1) &= A \cup \rho^{mk-1}(B_{mk}) \quad \text{and} \\ B^p(0, 1) &= \rho^{j-1}(C_j) \cup \sigma\rho^{j-1}(D_j) \qquad \text{for} \quad 1 \le j \le mk-1 \end{aligned}$$

of $B^p(0, 1)$ (see formulas (1) and (2)). By Lemma 1, we can choose ρ and σ such that all rotations ρ^{j-1}, $1 \le j \le mk$, and $\sigma\rho^{j-1}$, $1 \le j \le mk-1$, have an angle of rotation less than ε. Thus we have shown: there exist decompositions

$$B^p(0, 1) = H_1 \cup H_2 \cup \cdots \cup H_{2mk} \tag{14}$$

and

$$B^p(0,1) = \psi_{2j-1}(H_{2j-1}) \cup \psi_{2j}(H_{2j}) \qquad \text{for} \quad 1 \le j \le mk, \tag{15}$$

where $\psi_u \in SO_3$, $1 \le u \le 2mk$, are rotations whose angles of rotation are less than ε. This restriction for the angles makes sure that $d(x, \psi_u^{-1}(x)) < \varepsilon$ for all $x \in B^p(0,1)$ and, in particular, for all $x \in M$. Thus we obtain $M_{-\varepsilon} = \{x \in E^3\colon\ B(x,\varepsilon) \subseteq M\} \subseteq \{x \in E^3\colon\ \psi_u^{-1}(x) \in M\} = \psi_u(M)$, i.e.,

$$M_{-\varepsilon} \subseteq \psi_u(M) \qquad \text{for} \quad 1 \le u \le 2mk. \tag{16}$$

We put $M_u = H_u \cap M$, $1 \le u \le 2mk$. By (14), $M \subseteq B^p(0,1)$ is decomposed into

$$M = M_1 \cup M_2 \cup \cdots \cup M_{2mk}.$$

By the help of (15) and (16), we obtain the coverings

$$\begin{aligned}
M_{-\varepsilon} &= B^p(0,1) \cap M_{-\varepsilon} \\
&= (\psi_{2j-1}(H_{2j-1}) \cup \psi_{2j}(H_{2j})) \cap M_{-\varepsilon} \\
&= (\psi_{2j-1}(H_{2j-1}) \cap M_{-\varepsilon}) \cup (\psi_{2j}(H_{2j}) \cap M_{-\varepsilon}) \\
&\subseteq (\psi_{2j-1}(H_{2j-1}) \cap \psi_{2j-1}(M)) \cup (\psi_{2j}(H_{2j}) \cap \psi_{2j}(M)) \\
&= \psi_{2j-1}(M_{2j-1}) \cup \psi_{2j}(M_{2j})
\end{aligned}$$

for $1 \le j \le mk$. This completes the verification of formula (13).

By (12) and (13), there exist coverings

$$M \subseteq \bigcup_{i=1}^{m} (\varphi_i \psi_{2(h-1)m+2i-1}(M_{2(h-1)m+2i-1}) \cup \varphi_i \psi_{2(h-1)m+2i}(M_{2(h-1)m+2i}))$$

for $1 \le h \le k$. Thus the k terms of the sum $k \cdot M$ are covered by images of the pairwise disjoint subsets $M_u \subseteq M$, $1 \le u \le 2mk$. By Proposition 3, we get $M \overset{2mk}{\succeq} k \cdot M$. Applying Theorem 2 to this relation and the trivial counterpart $k \cdot M \overset{1}{\succeq} M$, we obtain $M \overset{2mk+1}{\simeq} k \cdot M$. This proves our claim. □

4. Applications of Theorem 6

If one wants to apply Theorem 6 to a bounded set M with non-empty interior one immediately is lead to the following question: How many images of $\mathrm{int}(M)$ are needed to cover $\mathrm{cl}(M)$? Although the smallest possible number is not easy to find in general, there exist useful results from combinatorial geometry which give rise to reasonable universal estimates for certain classes of sets M.

Most of the combinatorial statements are given for convex bodies K. Recall that a boundary point x of K is called *regular* if there exists only one supporting hyperplane of K through x.

Lemma 2. *Let $K \subseteq E^3$ be a compact convex set with non-empty interior and let m denote the smallest integer such that K can be covered by m translates of* $\mathrm{int}(K)$. *Then the following estimates are valid*:

(a) $m \leq 16$.
(b) $m \leq 8$ *if K is centrally symmetric*.
(c) $m \leq 6$ *if K is of constant width*.
(d) $m = 4$ *if K has at most four non-regular boundary points*.

Proof. On p. 262 of [2] it is shown that the following three problems are equivalent for all sets K of the above type: What is the smallest number m such that K can be covered by m translates of $\mathrm{int}(K)$ (Levi)? What is the smallest $m \in \mathbb{N}$ such that m smaller homothetic copies of K suffice to cover K (Gohberg, Markus, Hadwiger)? What is the smallest m such that the whole boundary of K can be illuminated by m directions (Boltyanski)? (For historical details as well as for recent developements and more general results concerning these problems we refer the reader to [2] and to the references given there.) Thus parts (b) and (c) of the lemma are equivalent to Lassak's results on coverings by homothetic copies from his papers [9] and [10]. The remaining statements rest on corresponding theorems on the illumination problem (see [11] and p. 280 of [2]). □

Now Theorem 6 immediately leads to the following conclusions.

Theorem 7. *Let $K \subseteq E^3$ be a bounded convex set with non-empty interior and let $k \geq 2$ be an integer. Then the following estimates are valid*:

(a) $\deg(K, k \cdot K) \leq 32k + 1$.
(b) $\deg(K, k \cdot K) \leq 16k + 1$ *if K is centrally symmetric*.
(c) $\deg(K, k \cdot K) \leq 12k + 1$ *if K is of constant width*.
(d) $\deg(K, k \cdot K) \leq 8k + 1$ *if K has at most four non-regular boundary points*.

A second lemma of combinatorial type can be shown for certain Minkowski sums $M = N \oplus K$. We present this statement not only for the three-dimensional case, since we did not find it in the literature and it could be of independent interest besides its application in the present paper. Note that the Minkowski sums considered in Lemma 3 and Corollary 3 are not necessarily bounded, closed, or convex. However, the idea is taken from a theorem on convex bodies (see Theorem 34.8 of [2]).

Lemma 3. *Let $M \subseteq E^n$ be a Minkowski sum $M = N \oplus K$ of two sets $N, K \subseteq E^n$ such that at least one of them is bounded. If the closure* $\mathrm{cl}(K)$ *of K can be covered by m translates of the interior* $\mathrm{int}(K)$, *then* $\mathrm{cl}(M)$ *can be covered by m translates of* $\mathrm{int}(M)$, *too*.

Proof. One can easily verify the inclusions

$$\mathrm{cl}(M) \subseteq \mathrm{cl}(N) \oplus \mathrm{cl}(K) \quad \text{and} \quad \mathrm{cl}(N) \oplus \mathrm{int}(K) \subseteq \mathrm{int}(M).$$

According to the assumption there exist m vectors $t_1, t_2, \ldots, t_m \in E^n$ such that $\mathrm{cl}(K) \subseteq \bigcup_{i=1}^m (\mathrm{int}(K) + t_i)$. Thus we obtain

$$\begin{aligned}\mathrm{cl}(M) \;&\subseteq\; \mathrm{cl}(N) \oplus \mathrm{cl}(K) \subseteq \mathrm{cl}(N) \oplus \left(\bigcup_{i=1}^m (\mathrm{int}(K) + t_i) \right) \\ &= \bigcup_{i=1}^m \big(\mathrm{cl}(N) \oplus \mathrm{int}(K) + t_i\big) \subseteq \bigcup_{i=1}^m (\mathrm{int}(M) + t_i).\end{aligned}$$

Hence $\mathrm{cl}(M)$ is covered by m translates of $\mathrm{int}(M)$. □

An important subclass of sets considered in Lemma 3 consists of the so-called *parallel sets* M. That is, M is a Minkowski sum $M = N \oplus B$ of some set $N \subseteq E^n$ and a closed ball B. Lemma 3 yields a universal estimate for all parallel sets, since any ball $B \subseteq E^n$ can be covered by $n + 1$ translates of its interior.

Corollary 3. *The closure of any parallel set in E^n can be covered by $n + 1$ translates of its interior.*

The three-dimensional cases of Lemma 3 and of Corollary 3 give rise to another application of Theorem 6.

Theorem 8. *Let $M = N \oplus K$ be a Minkowski sum of two bounded sets $N, K \subseteq E^3$ such that the closure* $\mathrm{cl}(K)$ *can be covered by m translates of* $\mathrm{int}(K)$. *Then*

$$\deg(M, k \cdot M) \leq 2mk + 1$$

for all integers $k \geq 2$.

In particular, any bounded parallel set $M \subseteq E^3$ fulfils the estimate

$$\deg(M, k \cdot M) \leq 8k + 1$$

for all $k \geq 2$.

5. Replicating the Cube

Now we come back to the problem of duplicating (or replicating) the cube. The best result obtained by the general theorems given above has been presented in formula (11). In this section we improve this estimate by proving a particular theorem concerning the cube. We have to use the special geometric structure of the cube. The following proposition prepares the forthcoming considerations.

Proposition 7. *For any cube $C \subseteq E^3$ whose edges are of length s, there exists a real number $\varepsilon > 0$ such that*

(a) *C can be covered by four isometric images of the inner parallel set $C_{-\varepsilon}$ and*

(b) *C can be covered by two isometric images of $C_{-\varepsilon}$ and one rectangular parallelotope of size* $(s-2\varepsilon)\times(s-2\varepsilon)\times s$.

Proof. Without loss of generality, we consider the cube $C=[-1,1]^3$ with $s=2$. We put $\varepsilon=\frac{1}{3}(1-(2\sqrt{2}/3))$. It can easily be seen that the four points $(2\sqrt{2}/3)(1,1,-\frac{1}{2})$, $(2\sqrt{2}/3)(\frac{1}{2},-1,-1)$, $(2\sqrt{2}/3)(-1,-1,\frac{1}{2})$, and $(2\sqrt{2}/3)(-\frac{1}{2},1,1)$ are the vertices of a square S whose edges are of length 2 and which is completely contained in the cube $[-2\sqrt{2}/3,2\sqrt{2}/3]^3=C_{-3\varepsilon}$. Consequently, the rectangular parallelotope P with base S and altitude 2ε is a subset of $C_{-\varepsilon}$.

The proof of part (a) is based on the covering

$$\begin{aligned} C &= [-1,1-2\varepsilon]^3\cup([1-2\varepsilon,1]\times[-1,1]^2)\\ &\quad\cup([-1,1]\times[1-2\varepsilon,1]\times[-1,1])\cup([-1,1]^2\times[1-2\varepsilon,1]). \end{aligned}$$

The first set is a translate of $C_{-\varepsilon}$. The additional three sets are congruent with P and therefore contained in suitable images of $C_{-\varepsilon}$.

The second assertion (b) can be verified by

$$\begin{aligned} C &= ([-1,1-2\varepsilon]^2\times[-1,1])\cup([1-2\varepsilon,1]\times[-1,1]^2)\\ &\quad\cup([-1,1]\times[1-2\varepsilon,1]\times[-1,1]). \end{aligned}$$

The first set in the covering is a rectangular parallelotope of size $(s-2\varepsilon)\times(s-2\varepsilon)\times s$, whereas the two remaining parallelotopes again are subsets of images of $C_{-\varepsilon}$. $\square$

Theorem 9. *Let $C\subseteq E^3$ be a cube and let $k\geq 2$ be an integer. Then*

$$2k\leq \deg(C,k\cdot C)\leq 8k-3.$$

Proof. The left-hand inequality is trivial.

The proof of the upper estimate is similar to that of Theorem 6. Without loss of generality, we restrict our considerations to the cube $C=[\frac{1}{4},\frac{1}{2}]^3$, which is a subset of the pointed ball $B^p(0,1)$ and whose edges are parallel to the coordinate axes (see Fig. 1). We apply Proposition 7 to the cube C with $s=\frac{1}{4}$. Hence there exist some $\varepsilon>0$ and isometries $\varphi_i\in\mathcal{I}_3$, $1\leq i\leq 7$, such that

$$C\subseteq\varphi_1(C_{-\varepsilon})\cup\varphi_2(C_{-\varepsilon})\cup\varphi_3(C_{-\varepsilon})\cup\varphi_4(C_{-\varepsilon}) \tag{17}$$

and

$$C\subseteq\varphi_5([0,\tfrac{1}{4}-2\varepsilon]^2\times[0,\tfrac{1}{4}])\cup\varphi_6(C_{-\varepsilon})\cup\varphi_7(C_{-\varepsilon}). \tag{18}$$

The proof of $B^p(0,1)\overset{8k-4}{\simeq}(4k-2)\cdot B^p(0,1)$ in Theorem 3 has shown the following: for any two rotations $\rho,\sigma\in SO_3$ generating a free non-Abelian subgroup of rank two there exist decompositions

$$B^p(0,1)=A\cup B_{4k-2}\cup C_1\cup\cdots\cup C_{4k-3}\cup D_1\cup\cdots\cup D_{4k-3} \tag{19}$$

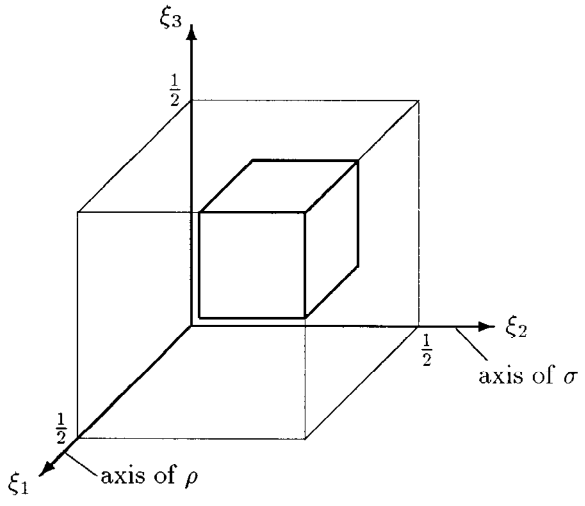

Fig. 1. Position of C in E^3.

as well as

$$B^p(0,1) = A \cup \rho^{4k-3}(B_{4k-2}), \qquad B^p(0,1) = C_1 \cup \sigma(D_1) \tag{20}$$

and

$$B^p(0,1) = \rho^{j-1}(C_j) \cup \sigma\rho^{j-1}(D_j) \qquad \text{for} \quad 2 \le j \le 4k-3 \tag{21}$$

of the pointed ball $B^p(0,1)$ (see formulas (1) and (2)). By Lemma 1, we can choose ρ and σ such that the axes of ρ and σ coincide with the first and the second coordinate axis, respectively, and such that all the angles of the rotations ρ^{j-1}, $2 \le j \le 4k-2$, and $\sigma\rho^{j-1}$, $1 \le j \le 4k-3$, are less than ε.

We put $M_1 = A \cap C$, $M_2 = B_{4k-2} \cap C$, $M_{2j+1} = C_j \cap C$, and $M_{2j+2} = D_j \cap C$ for $1 \le j \le 4k-3$. By (19), we obtain a decomposition

$$C = M_1 \cup M_2 \cup \cdots \cup M_{8k-4} \tag{22}$$

of the cube C. As in the proof of Theorem 6, (21) combined with the restriction of the angles of rotation gives rise to the representations

$$C_{-\varepsilon} \subseteq \rho^{j-1}(M_{2j+1}) \cup \sigma\rho^{j-1}(M_{2j+2}) \qquad \text{for} \quad 2 \le j \le 4k-3. \tag{23}$$

Next we show that a rectangular parallelotope of size $(\frac{1}{4} - 2\varepsilon) \times (\frac{1}{4} - 2\varepsilon) \times \frac{1}{4}$ is covered by $M_1 \cup \rho^{4k-3}(M_2)$:

$$([\tfrac{1}{4}, \tfrac{1}{2}] \times [\tfrac{1}{4} + \varepsilon, \tfrac{1}{2} - \varepsilon]^2) \subseteq M_1 \cup \rho^{4k-3}(M_2). \tag{24}$$

We have $([\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{4} + \varepsilon, \frac{1}{2} - \varepsilon]^2) \subseteq \rho^{4k-3}(C)$, since ρ^{4k-3} is a rotation around the first coordinate axis whose angle of rotation is less than ε. By (20), we can conclude that

$$\begin{aligned}([\tfrac{1}{4},\tfrac{1}{2}]\times[\tfrac{1}{4}+\varepsilon,\tfrac{1}{2}-\varepsilon]^2) &= B^p(0,1)\cap([\tfrac{1}{4},\tfrac{1}{2}]\times[\tfrac{1}{4}+\varepsilon,\tfrac{1}{2}-\varepsilon]^2)\\ &= (A\cap([\tfrac{1}{4},\tfrac{1}{2}]\times[\tfrac{1}{4}+\varepsilon,\tfrac{1}{2}-\varepsilon]^2))\\ &\quad\cup(\rho^{4k-3}(B_{4k-2})\cap([\tfrac{1}{4},\tfrac{1}{2}]\times[\tfrac{1}{4}+\varepsilon,\tfrac{1}{2}-\varepsilon]^2))\\ &\subseteq (A\cap C)\cup(\rho^{4k-3}(B_{4k-2})\cap\rho^{4k-3}(C))\\ &= M_1\cup\rho^{4k-3}(M_2),\end{aligned}$$

which proves the inclusion (24). Similar arguments show that the second equation of (20) yields

$$([\tfrac{1}{4}+\varepsilon,\tfrac{1}{2}-\varepsilon]\times[\tfrac{1}{4},\tfrac{1}{2}]\times[\tfrac{1}{4}+\varepsilon,\tfrac{1}{2}-\varepsilon])\subseteq M_3\cup\sigma(M_4). \tag{25}$$

Now we use the pairwise disjoint subsets $M_u \subseteq C$, $1\le u\le 8k-4$, from the decomposition (22) to cover the k terms C of the sum $k\cdot C$. Let $\psi_1\in\mathcal{I}_3$ be a motion mapping the parallelotope from fromula (24) onto the congruent one from (18). Then, by (23),

$$C\subseteq\varphi_5\psi_1(M_1)\cup\varphi_5\psi_1\rho^{4k-3}(M_2)\cup\varphi_6\rho(M_5)\cup\varphi_6\sigma\rho(M_6)\cup\varphi_7\rho^2(M_7)\cup\varphi_7\sigma\rho^2(M_8).$$

Similarly, (18), (25), and (23) with a suitable isometry $\psi_2\in\mathcal{I}_3$ give rise to the covering

$$C\subseteq\varphi_5\psi_2(M_3)\cup\varphi_5\psi_2\sigma(M_4)\cup\varphi_6\rho^3(M_9)\cup\varphi_6\sigma\rho^3(M_{10})\cup\varphi_7\rho^4(M_{11})\cup\varphi_7\sigma\rho^4(M_{12}).$$

Finally, the remaining $k-2$ terms of $k\cdot C$ admit the coverings

$$C\subseteq\bigcup_{i=1}^{4}(\varphi_i\rho^{4h+i}(M_{8h+2i+3})\cup\varphi_i\sigma\rho^{4h+i}(M_{8h+2i+4}))$$

for $1\le h\le k-2$ according to (17) and (23). Thus we can apply Proposition 3 and obtain $C\overset{8k-4}{\succeq} k\cdot C$. By Theorem 2, this and the trivial relation $k\cdot C\overset{1}{\succeq} C$ imply that $C\overset{8k-3}{\simeq} k\cdot C$. This proves the desired inequality $\deg(C,k\cdot C)\le 8k-3$. □

In particular, Theorem 9 states that

$$4\le\deg(C,C+C)\le 13. \tag{26}$$

This is the sharpest estimate for the smallest possible number of pieces in a paradoxical duplication of a cube which we were able to derive.

6. Concluding Remarks

We remark that some problems remain open. Although Theorems 6 and 9 use finer arguments than simple coverings by balls as Theorem 5 does, they only give estimates for the optimal numbers $\deg(M,k\cdot M)$ and $\deg(C,k\cdot C)$, respectively. Even in the "simple" particular case of duplicating the cube our methods have not led farther than to the above result (26). Can this estimate be improved?

We did not consider paradoxical decompositions in higher dimensions. The third dimension is the natural one for the Banach–Tarski paradox. However, many results from the three-dimensional case could be generalized.

Considerations of equidecomposabilities with respect to other groups of transformations possibly acting on spaces different from the Euclidean one can be found in [14]. One can regard the present paper with its almost pure geometric methods as a counterpart to the abstract algebraic extensions of the classical paradox.

The paradox of Banach, Tarski, and Hausdorff concerning equidecomposabilities in dimension three and higher has found a modern two-dimensional counterpart in Laczkovich's positive solution of Tarski's Circle-Squaring Problem (see [8]). We finish this paper by posing the corresponding deep question: What is the minimal number of pieces in an equidecomposability of a circle and a square in the sense of Laczkovich?

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