Discrete Comput Geom 25:51–63 (2001)
DOI: 10.1007/s004540010066

Discrete & Computational
Geometry

Polytopes in Arrangements*

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Abstract. Consider an arrangement of n hyperplanes in $\mathbb{R}^d$. Families of convex polytopes whose boundaries are contained in the union of the hyperplanes are the subject of this paper. We aim to bound their maximum combinatorial complexity. Exact asymptotic bounds were known for the case where the polytopes are cells of the arrangement. Situations where the polytopes are pairwise openly disjoint have also been considered in the past. However, no nontrivial bound was known for the general case where the polytopes may have overlapping interiors, for $d > 2$. We analyze families of polytopes that do not share vertices. In $\mathbb{R}^3$ we show an $O(k^{1/3}n^2)$ bound on the number of faces of k such polytopes. We also discuss worst-case lower bounds and higher-dimensional versions of the problem. Among other results, we show that the maximum number of facets of k pairwise vertex-disjoint polytopes in $\mathbb{R}^d$ is $\Omega(k^{1/2}n^{d/2})$ which is a factor of $\sqrt{n}$ away from the best known upper bound in the range $n^{d-2} \le k \le n^d$. The case where $1 \le k \le n^{d-2}$ is completely resolved as a known $\Theta(kn)$ bound for cells applies here.

1. Introduction

Consider an arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{R}^d$. We say that P is a *polytope in the arrangement* $\mathcal{A}$ if P is a closed bounded d-dimensional cell in the arrangement of a subset of the hyperplanes. The *complexity* of a polytope is the total number of its faces of all dimensions. We are interested in the maximum complexity of k polytopes in a d-dimensional arrangement of n hyperplanes. In the absence of additional constraints, it is $\Theta(kn^{\lfloor d/2 \rfloor})$ by the Upper Bound Theorem for Simple Polytopes [19], as, for example, one

* The research of Boris Aronov was partially supported by a Sloan Research Fellowship.

could take k identical polytopes. (One could object that the k polytopes must be distinct, but then it is only a matter of modifying each polytope slightly to be able to distinguish them. In any case, we do not discuss the unconstrained problem in this paper.) We discuss below several different sets of conditions, some old and some apparently new, that make this question more interesting. Several special cases of the problem arise naturally and are discussed below. We denote the desired maximum complexity by $K_*(k, n, d)$ with $*$ replaced by an abbreviation of the class of polytope families over which the maximum is taken.

For clarity, we do not consider unbounded polyhedra in our analysis, although they can be accommodated by introducing additional constraints.

1.1. *Cell Families*

In recent years considerable attention has been paid to the problem of estimating the maximum total number $K_c(k, n, d)$ of faces in k distinct cells in an arrangement of n hyperplanes in $\mathbb{R}^d$; see Chapter 6 of Edelsbrunner's book [8] and a more recent survey by Halperin [11]. This is known as the *many-faces* (or *many-cells*) problem in arrangements. Besides being a challenging combinatorial question, the problem has bearing in applications such as robotics and design of geometric algorithms; see, for example, [15]. For $d = 2$, an optimal bound $K_c(k, n, 2) = \Theta(k^{2/3}n^{2/3} + n)$ is known [4], [8]. Similarly, for $d = 3$ it is known that $K_c(k, n, 3) = \Theta(k^{2/3}n + n^2)$ for $n \leq k \leq n^3$ and $K_c(k, n, 3) = \Theta(kn)$ for $k < n$, see [1] and [8]. In higher dimensions the situation is not completely resolved. For $d > 3$, an upper bound of $O(k^{1/2}n^{d/2} \log^{(\lfloor d/2\rfloor-1)/2} n)$, and a lower bound which is $\Omega(k^{1/2}n^{d/2-1/4})$ for all valid values of k and n, and reaches $\Omega(k^{1/2}n^{d/2})$ for many combinations of values, are known for $K_c(k, n, d)$; see [2] for details. If only the *facets* ($(d-1)$-faces) of the k cells are of interest, optimal bounds of $\Theta(k^{2/3}n^{d/3} + n^{d-1})$ for $n^{d-2} \leq k \leq n^d$ and $\Theta(kn)$ for $k \leq n^{d-2}$ are known [1], [9].

Most of the following discussion concentrates on classes of polytope families that include families of arrangement cells. In particular, for such classes $K_*(k, n, d) \geq K_c(k, n, d)$. In three dimensions, in this case, $K_*(k, n, 3) = O(kn)$ as any 3-polytope in the arrangement has complexity $O(n)$, and $K_*(k, n, 3) = \Theta(kn)$ for $k = O(n)$, as it is easy then to arrange for k completely disjoint cells in the arrangement, each of complexity $\Theta(n)$. In higher dimensions, an analogous argument gives a bound of $O(kn^{\lfloor d/2\rfloor})$ on $K_*(k, n, d)$ for any k and $d \geq 2$ and $K_*(k, n, d) = \Theta(kn^{\lfloor d/2\rfloor})$ whenever d is odd and $k \leq n$. Hence for "reasonable" polytope classes we hereafter assume $k > n$ if d is odd (in particular, when $d = 3$).

1.2. *Noncell Families*

After considering cell families, the next natural question is bounding the maximum complexity $K_{dis}(k, n, d)$ of a family of polytopes in an arrangement of n hyperplanes that need not coincide with arrangement cells, but are pairwise openly disjoint. (Two sets with nonempty interiors are *openly disjoint* if their interiors do not intersect.) They correspond to (a subset of) cells in what Hershberger and Snoeyink call "erased arrangements" [16] and Dey and Shah call "convex arrangements" [7]. As already noted, $K_{\mathrm{dis}}(k, n, d) \geq K_{\mathrm{c}}(k, n, d)$. In fact, $K_{\mathrm{dis}}(k, n, 2) = \Theta(k^{2/3}n^{2/3}+n)$, surprisingly matching the asymptotic bounds for $K_{\mathrm{c}}(k, n, 2)$ [12], [14], [16]. This quantity arose in the analysis of a simple algorithm for decomposing a nonconvex polyhedron into convex pieces using planes that resolve reflex edges [3]. The algorithm can be extended to higher dimensions in a relatively straightforward manner, however requiring the complexity function $K_{\mathrm{dis}}(k, n, d)$ for precise analysis. To the best of our knowledge nothing is known for $d \geq 3$ beyond the immediate upper bound of $O(kn^{\lfloor d/2 \rfloor})$ that follows from the Upper Bound Theorem [19].

We look into a larger class of polytope families that properly includes all the classes discussed so far. We let the polytopes overlap in their interiors. As already pointed out, further restrictions are necessary to exclude trivial cases. Specifically, we consider polytope families in which polytopes are not allowed to share vertices, but overlap of faces of higher dimension is permitted. This class of polytope families, which is the subject of this paper, in essence generalizes all other families discussed above and has not been studied earlier except in $\mathbb{R}^2$. We denote the resulting complexity function $K_{\mathrm{vert}}(k, n, d)$. Katoh and Tokuyama [18] consider the related structure of a set of convex polyhedral surfaces whose facets lie on n planes and study the complexity of their k-level.

In $\mathbb{R}^2$, Halperin and Sharir show that $K_{\mathrm{vert}}(k, n, 2) = \Theta(k^{1/2}n)$ [13], a result motivated by their analysis of certain motion planning problems with three degrees of freedom. When the polygons (or convex polygonal chains) are not even permitted to overlap along edges (i.e., when polygon boundaries or polygonal chains must cross at discrete points), the respective bound is $\Theta(k^{1/3}n)$ for $k \leq n$ and $\Theta(k^{2/3}n^{2/3})$ for $n \leq k \leq n^2$. The upper and lower bounds in the first range follow from the results of Dey [5] and Eppstein [10], respectively. The bound in the second range follows from the many-faces results.

In dimensions less than four, for any polytope the number of facets and the number of faces of all dimensions are within a constant factor of each other. However, in higher dimensions this assertion no longer holds. Hence it is reasonable to distinguish between the number of facets in polytopes and the total number of faces in them. We denote the maximum number of facets by $F_*(k, n, d)$. As pointed out above, $K_*(k, n, d) = \Theta(F_*(k, n, d))$ for $d \leq 3$.

1.3. *Results*

It is known that $K_{\mathrm{vert}}(k, n, 2) = \Theta(k^{1/2}n)$ [13]. We show an upper bound of $O(k^{1/3}n^2)$ on $K_{\mathrm{vert}}(k, n, 3)$ when $n \leq k \leq n^3$. It is better than the trivial $O(kn)$ bound in the range $n^{3/2} \leq k \leq n^3$. For $k = O(n)$, $K_{\mathrm{vert}}(k, n, 3) = \Theta(kn)$, as families of vertex-disjoint polytopes generalize cell families; more precisely, $K_{\mathrm{vert}}(k, n, d) \leq K_{\mathrm{c}}(k, 2n, d)$, since by replacing each hyperplane with a pair of nearly parallel hyperplanes one can turn any family of closed bounded cells in an arrangement into a family of strictly disjoint cells. On the other hand, we show a lower bound of $\Omega(k^{1/2}n^{3/2})$ on $K_{\mathrm{vert}}(k, n, 3)$ for $n \leq k \leq n^3$. These bounds are summarized in Table 1 and Fig. 1.

In $\mathbb{R}^d$, $d > 3$, we show several lower bounds on the two complexity functions. For example, we show an $\Omega(k^{1/2}n^{d/2})$ bound on $K_{\mathrm{vert}}(k, n, d)$, unless $k \leq n$ and d is odd. This bound is slightly stronger than the best known lower bound on $K_{\mathrm{c}}(k, n, d)$ [2].

Table 1. Bounds in $\mathbb{R}^3$.

	Upper bound	Lower bound
$K_{\text{vert}}(k, n, 3)$	$\Theta(kn)$, $k \le n$	
	$O(kn)$, $n < k < n^{3/2}$	$\Omega(k^{1/2}n^{3/2})$, $k > n$
	$O(k^{1/3}n^2)$, $n^{3/2} \le k \le n^3$	

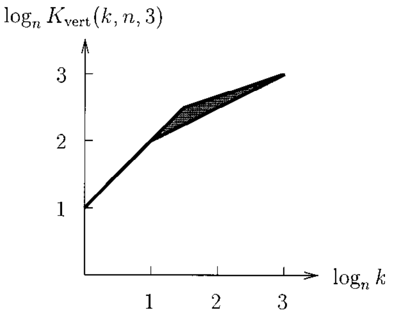

Fig. 1. Bounds on $K_{\text{vert}}(k, n, 3)$. The shaded area indicates the gap between best known lower and upper bounds.

Table 2. Bounds in $\mathbb{R}^d$.

	Upper bound	Lower bound
$F_{\text{vert}}(k, n, d)$	$\Theta(kn)$, $k \le n^{d-2}$	
	$O(\min\{kn, n^d\})$, $n^{d-2} \le k \le n^d$	$\Omega(k^{1/2}n^{d/2})$, $n^{d-2} \le k \le n^d$
$K_{\text{vert}}(k, n, d)$	$\Theta(kn^{\lfloor d/2 \rfloor})$, $k \le n$ and odd d	
	$O(\min\{kn^{\lfloor d/2 \rfloor}, n^d\})$	$\Omega(k^{1/2}n^{d/2})$, $n \le k \le n^d$

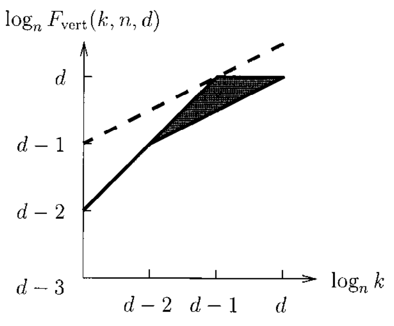

Fig. 2. Bounds on $F_{\text{vert}}(k, n, d)$. The dashed line indicates the weaker $O(k^{1/2}n^{d/2+1/2})$ bound.

The same expression also bounds $F_{\mathrm{vert}}(k, n, d)$ from below for $k \geq n^{d-2}$ and all d. The latter lower bound is close to an upper bound $O(k^{1/2}n^{d/2+1/2})$ that follows from the straightforward upper bound $O(\min\{kn, n^d\})$ on $F_{\mathrm{vert}}(k, n, d)$. Nontrivial upper bounds on the two quantities remain elusive. Table 2 and Fig. 2 summarize our results in d dimensions.

The paper is organized as follows. Section 2 presents preliminary definitions and assumptions. Section 3 proves an $O(k^{1/3}n^2)$ upper bound and an $\Omega(k^{1/2}n^{3/2})$ lower bound on $K_{\mathrm{vert}}(k, n, 3)$. Section 4 presents results in $\mathbb{R}^d$ and finally we conclude in Section 5. For completeness we attach an appendix containing a simple proof of a projective version of a well-known crossing result that we use in our upper bound arguments.

2. Preliminaries

Let $\mathcal{A} = \mathcal{A}(\Pi)$ denote the arrangement of a set Π of n hyperplanes in $\mathbb{R}^d$. We assume that $\mathcal{A}$ is a *simple* arrangement, i.e., every set of i hyperplanes meets in a flat of dimension exactly $d - i$, for $i = 1, \ldots, d$, and does not have a point in common if $i > d$. A sufficiently small perturbation of the hyperplanes replaces any polytope in $\mathcal{A}$ by a polytope with at least as large a complexity, in a simple arrangement. (More precisely, consider a point p in the interior of the polytope P. P is the (closed) cell of a subarrangement of $\mathcal{A}$ containing the point p. Perturbing the hyperplanes slightly yields a new subarrangement in which p marks a cell P' whose complexity is at least as large as the complexity of P. Two perturbed polytopes do not share vertices if the original ones did not.) Let $P_1, P_2, \ldots, P_k$ be k polytopes in $\mathcal{A}$; the polytopes need not be cells of $\mathcal{A}$. In $\mathbb{R}^3$, the *complexity* of $\{P_1, P_2, \ldots, P_k\}$, i.e., the total number of their faces, is proportional to the number of their vertices, since the boundary of each P_i is a plane graph.

For our analysis in $\mathbb{R}^3$ we use the standard duality between points and planes. (For technical reasons, here we view $\mathbb{R}^3$ as a subset of the three-dimensional real projective space $\mathbb{RP}^3$.) It maps a plane π expressed in homogeneous coordinates (x, y, z, w) as $ax + by + cz + dw = 0$ to the dual point π^* with homogeneous coordinates (a, b, c, d) in $\mathbb{RP}^3$. Conversely, a point p: (a, b, c, d) is mapped to the dual plane p^*: $ax + by + cz + dw = 0$.

3. Bounds in $\mathbb{R}^3$

In this section we derive an upper bound $O(k^{1/3}n^2)$ on $K_{\mathrm{vert}}(k, n, 3)$ using the duality between crossings among triangles and common tangents among polytopes. A lower bound $\Omega(k^{1/2}n^{3/2})$ is proved using the "lifting" technique detailed in [8].

3.1. *Crossings*

Given three non-collinear points in the three-dimensional real projective space $\mathbb{RP}^3$, consider the projective plane spanned by them. In the plane, draw the three lines defined by pairs of the points and consider the resulting projective arrangement. It contains four

2-faces, each bounded by three edges and three vertices. We refer to each such 2-face as a (*projective*) *triangle spanned by the three points*. Two triangles (*properly*) *cross* if and only if they are vertex-disjoint and have a nonempty intersection.

Consider the set Π^* of points dual to the planes of Π. Each vertex v of P_i can be associated with a projective triangle σ_v spanned by the three points dual to the three planes of Π incident with v, as follows. The three points span a projective plane and induce a three-line arrangement in it. This arrangement contains four triangular faces, one of which consists precisely of points dual to primal planes tangent to P at v. Let σ_v denote this face.

Let T denote the set of all dual projective triangles σ_v space that correspond to vertices v of $P_1, P_2, \ldots, P_k$. If X denotes the total number of crossings (i.e., pairs of triangles that cross) in T, then by a result of [6] $X \geq ct^4/n^6$ for some positive absolute constant c whenever $t = |T| \geq 3n^2$; see the Appendix for a self-contained proof of this statement for projective triangles. Our goal is to establish an upper bound on X and use the above inequality to obtain an upper bound on t.

3.2. *Common Tangents*

To establish an upper bound on X it is sufficient to count the number of polytope vertex pairs supporting planes tangent to two of the polytopes. In $\mathbb{R}^3$ we use the standard notion of tangency. Namely, we say that a plane π is *tangent* to a polytope P at vertex v if π passes through v but avoids the interior of P. Consider vertices u and v of P_i and $P_j \neq P_i$, respectively. Let $u = \pi_1^u \cap \pi_2^u \cap \pi_3^u$ and $v = \pi_1^v \cap \pi_2^v \cap \pi_3^v$, with $\pi_i^u, \pi_j^v \in \Pi$. Suppose σ_u and σ_v cross, i.e., σ_u and σ_v are vertex disjoint and meet; the assumption that $\mathcal{A}$ is simple guarantees the absence of "accidental" collinearities and/or coplanarities among the points π_i^{u*}, π_j^{v*}. Note that, $\pi_i^u \neq \pi_j^v$ for any i, j, i.e., u and v do not lie on a common plane of Π. A point in $\sigma_u \cap \sigma_v$ corresponds to a dual plane π tangent to P_i at u and to P_j at v. Each crossing pair (σ_u, σ_v) in T is thus associated with a unique pair of vertices u, v (of different polytopes) with a common tangent plane through them, but with no plane of Π passing through both points. Therefore, an upper bound on the number of such pairs, over all pairs of polytopes, provides an upper bound on X as well.

Two openly disjoint polytopes can have *outer* and *inner* tangent planes, the former keeping both polytopes on the same side and the latter separating them. Polytopes with overlapping interiors have only outer tangents.

We first deal with outer tangents. Let t_i denote the number of vertices of P_i. Two vertices u and v (not on a common plane of Π) of P_i and P_j, respectively, support a common tangent plane *only if* uv is an edge of the convex hull of $P_i \cup P_j$. Certainly, this hull cannot have more than $t_i + t_j$ vertices and $3(t_i + t_j)$ edges.

Now consider a pair (P_i, P_j) of openly disjoint polytopes. Suppose there is a plane π strictly separating them. After performing a projective transformation that maps π to the plane at infinity, we obtain two new polytopes with the property that their *outer* common tangents are exactly the *inner* common tangents of the original pair, and vice versa. Hence P_i, P_j also have no more than $3(t_i + t_j)$ pairs of vertices supporting inner common tangents, for a total of at most $6(t_i + t_j)$ pairs of vertices supporting common tangents altogether. We can disregard the case where P_i and P_j touch, that is their interiors are disjoint while their closures meet. Due to the simplicity of $\mathcal{A}$, two such polytopes cannot have an inner common tangent plane through two vertices that do not share a plane of $\mathcal{A}$.

As observed above the total number of (inner and outer) common-tangent-defining vertex pairs, over all pairs (P_i, P_j), is an upper bound on X. Thus

$$\begin{aligned} X &\le \sum_{i \ne j} 6(t_i + t_j) \\ &= 6 \sum_{1 \le j \le k} \sum_{i \ne j} t_i \\ &< 6kt, \end{aligned}$$

with the first summation taken over all unequal pairs of indices $i, j = 1, \ldots, k$.

3.3. *Tangents and Crossings*

Now we are ready to prove an upper bound in $\mathbb{R}^3$.

Theorem 1. *$K_{\mathrm{vert}}(k, n, 3)$ is $O(k^{1/3}n^2)$ which is better than $O(kn)$ in the range $n^{3/2} < k \le n^3$.*

Proof. In the dual we have $t = |T|$ triangles with vertices from a fixed set of n points. Since the total number of crossings among these triangles is $O(kt)$ by the argument in Section 3.2, we have $ct^4/n^6 \le X \le O(kt)$ for $t \ge 3n^2$. This immediately gives $t = O(k^{1/3}n^2)$. Since the total complexity of $\{P_i\}_i$ is $O(t)$, the bound follows. □

The condition of vertex-disjointness on polytopes cannot be completely removed since otherwise one may consider k copies of the same polytope and a $\Theta(kn)$ bound is obvious. However, this restriction can be relaxed to require only that if two polytopes share a vertex, then their interiors do not intersect. Indeed, if this is the case, one can replace each plane of Π by an almost parallel slab with the two planes meeting far away from all polytope features. This process only doubles the number of planes eliminating shared vertices. The claim follows.

Corollary 2. *The bound in Theorem* 1 *applies to a collection of k convex polytopes in an arrangement of n planes, such that any two polytopes are either vertex disjoint or openly disjoint.*

3.4. *Lower Bound in* $\mathbb{R}^3$

In this section we use the "lifting" technique of [8] to deduce a lower bound on $K_{\mathrm{vert}}(k, n, 3)$ from its two-dimensional analogue. Halperin and Sharir [13] construct a set of k convex polygons that are vertex disjoint, are drawn from an arrangement of n lines, and whose total complexity is $\Omega(k^{1/2}n)$ for $1 \le k \le n^2$. For completeness, we briefly outline their construction. Form an arrangement of $n/k^{1/2}$ lines in which one face has the form of a regular polygon with all the lines appearing on its boundary. Now replace each of the lines by $k^{1/2}$ essentially parallel lines. They intersect near each polygon corner in $k^{1/2} \times k^{1/2}$ points. It is not difficult now to construct k overlapping, closed convex chains each of length $n/k^{1/2}$ and each turning at a different one of these k arrangement vertices, near each polygon corner.

Theorem 3. *$K_{\mathrm{vert}}(k, n, 3)$ is $\Omega(k^{1/2}n^{3/2})$ for $n \leq k \leq n^3$.*

Proof. Let $p = 4k/n$ where $k \geq n$. Consider a horizontal plane with an arrangement of $n/2 - 1$ lines where p pairwise vertex-disjoint convex polygons have complexity $\Omega(p^{1/2}n)$. "Lift" this arrangement to an arrangement of $n/2 - 1$ vertical planes in which there are p prism-shaped polyhedra of total complexity $\Omega(p^{1/2}n)$ that do not share vertical edges. Add $n/2 + 1$ horizontal planes, so that each of the p prisms is cut into $n/2 + 2$ subprisms, two unbounded and $n/2$ bounded. Picking every other bounded subprism from each prism, we obtain a set of $pn/4 = k$ vertex-disjoint convex polytopes with boundaries in n planes. They must have $\Omega((p^{1/2}n)n) = \Omega(p^{1/2}n^2)$ complexity. Using $p = \Theta(k/n)$, we get the desired bound. $\square$

4. Higher-Dimensional Bounds

The "lifting" technique of the previous section can be generalized to higher dimensions to derive a lower bound on $F_{\mathrm{vert}}(k, n, d)$. Another approach based on taking product spaces is used to derive a lower bound on $K_{\mathrm{vert}}(k, n, d)$. We do not have satisfactory upper bounds on the asymptotic behavior of these complexity functions though we suspect that our lower bounds are close to optimal. In $\mathbb{R}^d$, $d > 3$, the lower bound $\Omega(k^{1/2}n^{d/2-1/4})$ on $K_{\mathrm{c}}(k, n, d)$ [2] immediately applies to $K_{\mathrm{vert}}(k, n, d)$. We show a better lower bound $\Omega(k^{1/2}n^{d/2})$ on $K_{\mathrm{vert}}(k, n, d)$.

We first discuss upper bounds, starting with a straightforward observation:

Observation 4. *$K_{\mathrm{vert}}(k, n, d)$ and $F_{\mathrm{vert}}(k, n, d)$ are $O(n^d)$.* $\square$

Proof. Since $F_*(k, n, d) < K_*(k, n, d)$, it is sufficient to argue for $K_{\mathrm{vert}}(k, n, d)$.

Consider a family of k pairwise vertex-disjoint polytopes in an arrangement of n hyperplanes in general position in $\mathbb{R}^d$. The polytopes cannot have more than $O(n^d)$ vertices altogether, as each vertex of the arrangement can be used at most once. Each vertex may be incident to at most one polytope of the family and hence to at most 2^d faces of all dimensions. $\square$

First, we derive bounds on $F_{\mathrm{vert}}(k, n, d)$. If $k \leq n^{d-2}$, one can easily construct an arrangement of n hyperplanes with k disjoint closed cells, each bounded by $\Theta(n)$ facets [8]. This implies a trivial tight bound of $\Theta(kn)$ for $k \leq n^{d-2}$ on $F_{\mathrm{vert}}(k, n, d)$. So only the case $k \geq n^{d-2}$ is interesting. We prove an $\Omega(k^{1/2}n^{d/2})$ lower bound on $F_{\mathrm{vert}}(k, n, d)$ for $k \geq n^{d-2}$ and $d \geq 3$. Interestingly enough, this bound is close to the upper bound

$O(k^{1/2}n^{d/2+1/2})$ on $F_{\mathrm{vert}}(k, n, d)$. The latter bound is weaker than the immediate bound of $F_{\mathrm{vert}}(k, n, d) = O(\min\{kn, n^d\})$ that follows from Observation 4 and the fact that no polytope has more than n facets.

The lower bound on $F_{\mathrm{vert}}(k, n, d)$ is derived by generalizing the $\Omega(k^{1/2}n)$ lower bound of Halperin and Sharir [13] on $F_{\mathrm{vert}}(k, n, 2)$ to a higher dimension using the same "lifting" method as in Theorem 3.

Theorem 5. *$F_{\mathrm{vert}}(k, n, d)$ is $\Omega(k^{1/2}n^{d/2})$ for $k \geq n^{d-2}$ and $d \geq 3$.*

Proof. Let $p = 4k/n \geq n^{d-3}$. In a horizontal hyperplane construct inductively an arrangement of $n/2 - 1$ $(d-2)$-flats with p pairwise vertex-disjoint $(d-1)$-polytopes with $c = \Omega(p^{1/2}n^{(d-1)/2})$ facets. "Lift" this arrangement to an arrangement of $n/2 - 1$ vertical hyperplanes in which there are p prism-shaped d-polyhedra with a total of c facets. Add $n/2 + 1$ horizontal hyperplanes to obtain $n/2 + 2$ subprisms from each of the p prisms. Picking every other bounded subprism from each prism, we obtain a set of $pn/4 = k$ pairwise vertex-disjoint convex d-polytopes with boundaries in n hyperplanes. They must have $\Omega(cn) = \Omega(p^{1/2}n^{(d+1)/2})$ facets. Using $p = \Theta(k/n)$, we obtain the claimed bound. □

In the above proof we required $k \geq n^{d-2}$ to carry the induction through dimensions. This is not an artifact of our proof but an intrinsic requirement, for we have observed earlier that $F_{\mathrm{vert}}(k, n, d)$ is $\Theta(kn)$ when $k = O(n^{d-2})$. It follows from the next theorem that this is not true for $K_{\mathrm{vert}}(k, n, d)$. In fact, the lower bound $\Omega(k^{1/2}n^{d/2})$ applies to $K_{\mathrm{vert}}(k, n, d)$ for a larger range of k. Recall that Aronov et al. [2] have considered maximum cell family complexity $K_{\mathrm{c}}(k, n, d)$, and constructed a lower bound somewhat smaller than $\Theta(k^{1/2}n^{d/2})$ and an upper bound slightly larger than it. It is conceivable that $K_{\mathrm{c}}(k, n, d) = \Theta(k^{1/2}n^{d/2})$. We show a lower bound of $\Omega(k^{1/2}n^{d/2})$ on $K_{\mathrm{vert}}(k, n, d)$, which is better than the (best currently known) corresponding lower bound on $K_{\mathrm{c}}(k, n, d)$. The proof again uses the $\Omega(k^{1/2}n)$ lower bound of Halperin and Sharir [13].

Theorem 6. *$K_{\mathrm{vert}}(k, n, d)$ is $\Omega(k^{1/2}n^{d/2})$ for $d \geq 2$, unless d is odd and $k \leq n$.*

Proof. Suppose d is even. Let $p = k^{2/d}$ and $q = 2n/d$. Construct a family of p pairwise vertex-disjoint convex polygons in an arrangement of q lines in the plane with $\Omega(p^{1/2}q)$ vertices, as described in [13]. Place $d/2$ copies of this construction in $d/2$ orthogonal 2-flats which together span $\mathbb{R}^d$. Consider one such flat π and one of the q lines we drew there. Extend it to a hyperplane by a Cartesian product with the orthogonal complement of π. This yields a set of $qd/2 = n$ hyperplanes altogether. We obtain $k = p^{d/2}$ polytopes in the resulting arrangement by taking all possible Cartesian products of $d/2$ polygons, one from each 2-flat. It is easy to see that no vertex is shared between two resulting polytopes and the number of their vertices is $(\Omega(p^{1/2}q))^{d/2} = \Omega(k^{1/2}n^{d/2})$ satisfying the claimed bound.

Now suppose d is odd and $k > n$. As in Theorem 5, use the bound in the immediately lower even dimension and "lift" it to the desired bound in the claimed range. □

5. Conclusions

In this paper we discussed the complexity of families of polytopes in an arrangement of hyperplanes. Unlike most previous work, we allow the polytopes to overlap, but do not permit sharing vertices. Except in $\mathbb{R}^2$ these families have not been studied in erstwhile literature with the same generality as considered here. We obtain several bounds in three and higher dimensions. The exact asymptotic complexity of these quantities is still to be determined, however. Moreover, several other classes of families of polytopes in hyperplane arrangements deserve investigation, such as those generalizing non-overlapping convex curve families studied by Dey [5] and Eppstein [10]. In addition, we recently learned that Sharir and Smorodinsky [20] have produced a tight $\Theta(k^{1/2}n^{3/2})$ bound on the maximum complexity of $k > n$ polytopes with non-overlapping edges, in an arrangement of n planes in $\mathbb{R}^3$.

Appendix

Throughout this appendix, S is a set of n points in general position in $\mathbb{RP}^3$—no three on a line, no four on a plane. Consider three non-collinear points $v, w, u \in S$. They define a projective plane $\pi \subset \mathbb{RP}^3$. A (*projective*) *triangle with vertices* u, v, w is defined, as in Section 3.1, as a 2-face in the two-dimensional line arrangement formed in π by the three lines uv, vw, uw. Three non-collinear points define four triangles. Similarly, two distinct points define two (projective straight-line) segments. Given two distinct triangles, we say that they *cross properly* if they intersect and are vertex-disjoint, and *improperly* if their interiors intersect, but the triangles share vertices. Let T be a collection of triangles with vertices from S, and put $t = |T|$.

In this Appendix we present a self-contained proof of the following theorem. An affine version of this theorem was originally proven by Dey and Edelsbrunner [6]. An affine version of the simpler proof given below was communicated to the authors at different times by Jiří Matoušek and by Emo Welzl, but to our knowledge it has not appeared in print. A related proof is presented in [17].

Theorem 7. *Given a set S of n points in $\mathbb{RP}^3$ in general position and a set T of $t = \Omega(n^2)$ triangles spanned by the points, there are $\Omega(t^4/n^6)$ pairs of properly crossing triangles in T.*

We prove this statement by a series of assertions on the number and type of pairs of crossing triangles in T.

Lemma 8. *If $t > n^2$, there are two triangles whose interiors intersect.*

Proof. As each triangle is incident to three points, there is a point $p \in S$ incident to $3t/n > 3n$ triangles. Drawing a sufficiently small sphere around p and intersecting it with segments connecting p to other points of S and with triangles of T incident to p, we obtain a graph on $n - 1$ vertices with more than $3n$ edges on the sphere. This graph has too many edges to be planar, so some two edges intersect in their interiors; an edge cannot pass through a vertex by the general position assumption. Therefore some two triangles incident to p have nondisjoint interiors (and share a vertex). $\square$

Lemma 9. *If $t > 5n^2/2$, T contains a pair of properly crossing triangles.*

Proof. Suppose the interiors of two triangles meet. The intersection of the triangles lies in the intersection line of the respective planes containing them. Consider a segment s (there might be more than one) of an intersection of the two projective triangles that meets both interiors. We claim that s cannot be delimited by two triangle vertices, so a nonvertex endpoint of s is a point where an edge of one triangle pierces (i.e., meets the interior of) the other. Indeed, suppose both endpoints of s are vertices. If they belong to the same triangle, the interior of s lies completely on the triangle boundary or outside of it, contradicting the assumption that s meets both triangle interiors. If they belong to different triangles, we have a vertex of one triangle lying in the other, contradicting the general position assumption.

We now turn to the proof of the lemma. Suppose all triangle crossings are improper. As argued above, any time two triangle interiors meet, an edge pierces a triangle. Let E be the set of such edges. We claim that every edge $e \in E$ can be incident with at most three triangles of T. Indeed, by the definition of E, e pierces some $\Delta \in T$. If e is incident with four or more triangles, at least one of them does not share a vertex with Δ and thus crosses it properly.

We now delete the at most $3\binom{n}{2} < 3n^2/2$ triangles incident on edges in E. What is left is a set of triangles that do not cross at all (otherwise, there would be an edge crossing a triangle and we have deleted all triangles incident on such edges). By Lemma 8 we are left with at most n^2 triangles, so $t < 3n^2/2 + n^2 = 5n^2/2$, completing the proof of the lemma. $\square$

Let X be the number of pairs of properly crossing triangles in T. By repeatedly removing one of a crossing pair of triangles that is guaranteed to exist by the previous lemma, we deduce

Lemma 10. *The number X of properly crossing pairs is always at least $t - 5n^2/2$.*

Proof. For $t \leq 5n^2/2$, the statement holds vacuously. For larger values of t, at least one pair exists by Lemma 9, so deleting one triangle of the pair removes at least one properly crossing pair, and the result follows by induction on t. $\square$

Finally, we are ready to prove Theorem 7.

Proof of Theorem 7. We now use a standard random sampling argument, see, for example [21] and [17]. Given a collection of n points and t triangles spanned by them, we select a sample $S' \subset S$ of the points, with each point selected independently with probability q, which is to be fixed below. The sample has n' points and t' triangles (a triangle of T is selected only if it is spanned by three selected points). Let X be the number of properly crossing triangle pairs in S and let X' be the corresponding number for S'. By Lemma 10, $X' > t' - 5(n')^2/2$. By linearity of expectation this means that $E[X'] > E[t'] - (\frac{5}{2})E[(n')^2]$. A simple calculation gives $E[X'] = Xq^6$, $E[t'] = tq^3$, $E[(n')^2] = nq + n(n-1)q^2$, so the inequality can be rewritten as

$$X > \frac{t}{q^3} - \frac{5n}{2q^5} - \frac{5n^2}{2q^4} + \frac{5n}{2q^4} > \frac{t}{q^3} - \frac{5n}{2q^5} - \frac{5n^2}{2q^4}.$$

We put $q = 3n^2/t$ and obtain, for $3n^2 \le t \le \binom{n}{3} \le n^3/6$,

$$\begin{aligned} X &> \frac{t^4}{27n^6} - \frac{5t^5}{486n^9} - \frac{5t^4}{162n^6} \\ &= \frac{t^4}{27n^6}\left(\frac{1}{6} - \frac{5t}{18n^3}\right) \\ &\ge \frac{t^4}{27n^6}\left(\frac{1}{6} - \frac{5}{108}\right) \\ &= \frac{1}{27} \cdot \frac{13}{108} \cdot \frac{t^4}{n^6}. \end{aligned}$$

□

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Received September 20, 1999, *and in revised form March* 10, 2000. *Online publication September* 6, 2000.