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Realizations of Regular Toroidal Maps of Type $\{4, 4\}$*

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Abstract. We determine and completely describe all pure realizations of the finite toroidal maps of types $\{4, 4\}_{(b,0)}$ and $\{4, 4\}_{(b,b)}$, $b \geq 2$. For large values of b, most such realizations are eight-dimensional.

1. Introduction

The theory of regular polytopes, and other regular figures with kindred structures, has a long and rich history [7, pp. 97–100]. A recent contribution of great significance in this story was Branko Grünbaum's "Regular polyhedra—old and new" [5], with its startling exhortation to abandon membranes. We vividly recall experiencing this wonderful insight in person, when Branko lectured at the University of Toronto, just over 20 years ago.

From these investigations (see [4] as well), and the work of many others, we now have available the highly distilled notion of *regular abstract polytope*, which is a combinatorial structure having the essential features of such diverse objects as the classical regular convex and star-polytopes, regular honeycombs, and regular maps on surfaces. Now while an abstract approach clarifies the general properties and construction of polytopes, it is nevertheless still interesting and useful to consider concrete realizations, for example, as symmetric objects in Euclidean space (see [6]). For example, in [1], Burgiel and Stanton describe the pure realizations of the finite, regular toroidal maps of type $\{3, 6\}$, essentially by examining the action of the group on a unitary space whose basis is identified with the vertex set of the map.

Here we investigate maps of type $\{4, 4\}$ from a somewhat different point of view, which allows us to describe real representations of the group explicitly.

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2. Regular Toroidal Maps of Type {4, 4}

Let us begin with an overview of the basic theory, referring to the survey article [7] and monograph [9] for details. An (*abstract*) *n-polytope* $\mathcal{P}$ is a partially ordered set with a strictly monotone rank function having range $\{-1, 0, \ldots, n\}$. An element $F \in \mathcal{P}$ with $\text{rank}(F) = j$ is called a *j-face*; naturally, faces of rank 0 and $n-1$ are called *vertices* and *facets*, respectively. We also require that $\mathcal{P}$ have two improper faces: a unique least face F_{-1} and a unique greatest face F_n. Furthermore, each maximal chain or *flag* in $\mathcal{P}$ must contain $n+2$ faces, and $\mathcal{P}$ should be strongly flag-connected. Finally, $\mathcal{P}$ must have a homogeneity property: whenever $F < G$ with $\text{rank}(F) = j-1$ and $\text{rank}(G) = j+1$, there are exactly two j-faces H with $F < H < G$.

The symmetry of $\mathcal{P}$ is, of course, exhibited by its *automorphism group* $\Gamma(\mathcal{P})$. In particular, $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on flags, as we henceforth assume. Now fix a base flag $\Phi = \{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$, with rank $(F_j) = j$. For $0 \leq j \leq n-1$, there is a unique flag Φ^j differing from Φ in just the rank j face; so let ρ_j be the (unique) automorphism with $(\Phi)\rho_j = \Phi^j$. In this case, $\Gamma(\mathcal{P})$ is generated by the involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$, which satisfy at least the relations

$$(\rho_i \rho_j)^{p_{ij}} = 1, \qquad 0 \leq i, j \leq n-1, \tag{1}$$

where $p_{ii} = 1$, $2 \leq p_{ij} \leq \infty$ for $i \neq j$, and $p_{ij} = 2$ for $|i-j| \geq 2$. Furthermore, an *intersection condition* on standard subgroups holds:

$$\langle \rho_i \colon i \in I \rangle \cap \langle \rho_i \colon i \in J \rangle = \langle \rho_i \colon i \in I \cap J \rangle \tag{2}$$

for all $I, J \subseteq \{0, \ldots, n-1\}$. In short, $\Gamma(\mathcal{P})$ is a certain quotient of a Coxeter group with linear diagram, and we call $\Gamma(\mathcal{P})$ a *string C-group*.

Conversely, given any group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ generated by involutions and satisfying (1) and (2), one may construct a polytope $\mathcal{P}$ with $\Gamma(\mathcal{P}) = \Gamma$ (see Theorem 2.9 of [7]).

As a concrete example, consider the regular tessellation $\{4, 4\}$ of the plane (by identical squares). Indeed, $\{4, 4\}$ is an infinite regular 3-polytope; and the full symmetry group $[4, 4]$ is generated by the reflections ρ_0, ρ_1, ρ_2 indicated in Fig. 1. The perpendicular, unit translations $\tau_x = \rho_1\rho_2\rho_1\rho_0$ and $\tau_y = \rho_2\rho_1\rho_0\rho_1$ generate an abelian subgroup; and we may regard $\tau_x^b \tau_y^c$ as translating the origin $(0, 0)$ to the point (b, c). For a fixed pair of integers (b, c), consider the translation subgroup $\langle \tau_x^b \tau_y^c, \tau_x^{-c} \tau_y^b \rangle$, whose fundamental region is the square with vertices

$$(0, 0), (b, c), (b-c, b+c), (-c, b).$$

Identifying opposite edges of this square, we obtain the finite toroidal map $\mathcal{P} = \{4, 4\}_{(b,c)}$, having $v = b^2 + c^2$ vertices, $2v$ edges, and v faces. In fact, $\mathcal{P}$ is also a regular 3-polytope when $b \geq 2$, $c = 0$ (and vice versa), or when $b = c \geq 2$ [3, Section 8.3]. Moreover, for $\{4, 4\}_{(b,0)}$, the automorphism group, of order $8b^2$, has the presentation

$$\begin{gathered} \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = 1, \\ (\rho_1\rho_2\rho_1\rho_0)^b = 1. \end{gathered} \tag{3}$$

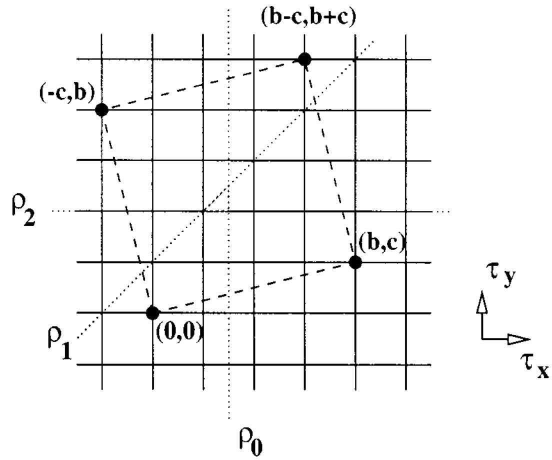

Fig. 1. The tessellation {4, 4}.

The automorphism group for $\{4,4\}_{(c,c)}$ has order $16c^2$ and the presentation

$$\begin{gathered}\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = 1,\\ (\rho_2\rho_1\rho_0)^{2c} = 1.\end{gathered} \tag{4}$$

(For simplicity we also use ρ_j to indicate the generators of these finite groups; and it is convenient, though not geometrically accurate, to still speak of τ_x, τ_y as "translations.")

Although it is fruitful, even necessary at times, to abandon concrete geometric figures (such as a torus) when thinking of an abstract polytope $\mathcal{P}$, it is nevertheless interesting to model $\mathcal{P}$ in a natural way in Euclidean n-space E, as is done in McMullen's theory of *realizations* of $\mathcal{P}$. (We assume that $\mathcal{P}$ is finite and thus modify slightly the discussion in Section 3 of [7].

Fixing an origin $o \in E$, we consider any homomorphism

$$f\colon \Gamma(\mathcal{P}) \to O(E)$$

(into the orthogonal group). Taking $R_j := (\rho_j)f$, we define the *Wythoff space* for f as

$$W := \{p \in E\colon\ pR_j = p,\ 1 \le j \le n-1\}.$$

A *realization* $P := [f, p]$ is now defined by the homomorphism f, together with a base vertex $p \in W_P := W$.

Now consider the vertex set of $\mathcal{P}$, namely $\mathcal{P}_0 := \{F_0\gamma\colon\ \gamma \in \Gamma(\mathcal{P})\}$. Then the map

$$\begin{aligned}\beta\colon \mathcal{P}_0 &\to E,\\ F_0\gamma &\mapsto p(\gamma f)\end{aligned}$$

is well defined, and each $\gamma \in \Gamma(\mathcal{P})$ thereby induces an isometric permutation on $V(P) := (\mathcal{P}_0)\beta$ (the vertex set of the realization). If $E' = \mathrm{aff}(V(P))$, then the *dimension* of the realization is $\dim(P) = \dim(E')$. Note that the linear group $G(P) := (\Gamma(\mathcal{P}))f$ leaves E' invariant.

We naturally say that two realizations of $\mathcal{P}$, say $P_j = [f_j, p_j]$ in E_j ($j = 1, 2$), are *congruent* if there is an isometry $g\colon E_1 \to E_2$ such that $(p_1)g = p_2$ and $(\gamma f_1)g = g(\gamma f_2), \forall\gamma \in \Gamma$. It is known that the congruence classes of realizations have the structure of a convex r-dimensional cone, where r is the number of diagonal classes in $\mathcal{P}$ [7, Theorem 3.8]. (A *diagonal* is an unordered pair of distinct vertices in $\mathcal{P}_0$.) If the jth diagonal class is represented by $p, q_j \in V(P)$, and $\| p - q_j \|^2 = \delta_j$, then P is determined by the diagonal vector $\triangle(P) = (\delta_1, \ldots, \delta_r)$.

Now if $G(P)$ acts reducibly on E', then in a natural way P is congruent to a *blend* of lower dimensional realizations, say Q and R, and we write $P \equiv Q \# R$ (see Section 3.1.4 of [7]). On the other hand, if this does not happen, i.e., if $G(P)$ acts irreducibly on E', then P is said to be a *pure* realization. The fact that diagonal vectors of pure realizations span the extreme rays in the realization cone is crucial to McMullen's proof of the fundamental numerical results outlined below.

For $v = |\mathcal{P}_0|$, let $\overline{E}$ be $(v-1)$-dimensional Euclidean space. Clearly, $\mathcal{P}$ has a *simplex realization* T in $\overline{E}$, obtained by letting $\Gamma(\mathcal{P})$ act in a natural way on the vertex set $V(T)$ of a regular simplex in $\overline{E}$. Let $\overline{w} = \dim(W_T)$.

Take d_G to be the degree and w_G to be the Wythoff space dimension for each of the (finitely many) distinct, irreducible representations G of $\Gamma(\mathcal{P})$, excluding cases with $w_G = 0$, and let $\Gamma_0 = \langle \rho_1, \ldots, \rho_{n-1} \rangle$ in $\Gamma(\mathcal{P})$.

Theorem 2.1 [7, Section 3.2]. *With the notation above and summing over distinct irreducible represenations of* $\Gamma(\mathcal{P})$, *we have*:

(a) $\sum_G w_G d_G = v - 1 = |\{\Gamma_0\sigma\colon \sigma \in \Gamma\backslash\Gamma_0\}|$.
(b) $\sum_G \frac{1}{2} w_G(w_G + 1) = r = |\{\Gamma_0\sigma\Gamma_0 \cup \Gamma_0\sigma^{-1}\Gamma_0\colon \sigma \in \Gamma\backslash\Gamma_0\}|$.
(c) $\sum_G w_G^2 = \overline{w} = |\{\Gamma_0\sigma\Gamma_0\colon \sigma \in \Gamma\backslash\Gamma_0\}|$.

These character-like results are extremely useful in classifying the full range of pure realizations for a finite regular polytope $\mathcal{P}$.

3. General Pure Realizations for $\{4, 4\}_{(b,0)}$

Throughout this section, $b \geq 2$ is a fixed positive integer and $\mathcal{P} = \{4, 4\}_{(b,0)}$. Thus $\mathcal{P}$ has automorphism group $\Gamma = \langle \rho_0, \rho_1, \rho_2 \rangle$, whose "translation" subgroup is generated by $\tau_x = \rho_1\rho_2\rho_1\rho_0$ and $\tau_y = \rho_2\rho_1\rho_0\rho_1$.

We begin our construction with the Coxeter group K whose diagram is shown in Fig. 2. (The generators $r_0, r_1, \ldots, r_7$ of K correspond to the nodes as labeled. As usual, when $b = 2$ the branches in the diagram are removed.) Thus K is a direct product of dihedral groups. We also require two outer automorphism λ, μ for K, as suggested in Fig. 2: μ interchanges r_0 and r_2, etc., while λ interchanges r_0 and r_6, etc. By adjoining λ, μ to K we obtain a group K' of order $64b^4$. Evidently, this extension to K' is an example of *twisting* (see pp. 205–206 of [8]). To motivate our construction, it is helpful

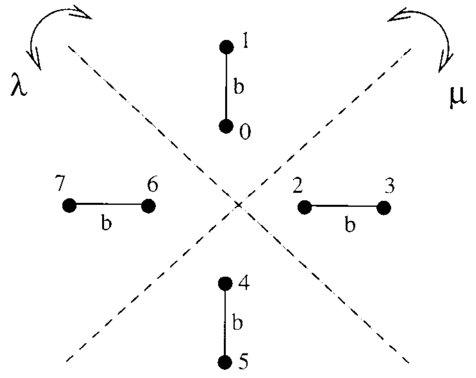

Fig. 2. The group K.

to note that the group for $\{4, 4\}_{(b,0)}$ can be defined by applying the twisting operation

$$(r_0, r_1, r_2, r_3; \mu) \to (r_0, \mu, r_3)$$

to a subdiagram in Fig. 2. (We thank the referee for this comment.)

We can faithfully represent K' as a group of orthogonal transformations on the Euclidean space $E = \mathbb{R}^8$, endowed with the usual inner product. In fact, if $e_0, \ldots, e_7$ is the usual basis, we may suppose for $j = 0, 2, 4, 6$ that r_j has root e_j; and for $j = 1, 3, 5, 7$ that r_j has root $\cos(\pi/b)e_{j-1} + \sin(\pi/b)e_j$. It follows that μ is the linear map permuting e_0 and e_2, e_1 and e_3, e_4 and e_6, e_5 and e_7; λ acts similarly on basis vectors.

We can now define the group of main interest to us, which we found after a considerable amount of experimentation involving some well-known four-dimensional tori [2, Section 4.5] and certain computations in *GAP*:

Definition. For integers ℓ, m satisfying $0 \le \ell, m \le b - 1$, let $G_{\ell,m}$ be the subgroup of K' generated by

$$\begin{aligned} g_0 &= \mu(r_0r_1)^\ell r_0(r_2r_3)^\ell r_2(r_4r_5)^m r_4(r_6r_7)^m r_6, \\ g_1 &= \lambda\mu r_0 r_4, \\ g_2 &= \mu. \end{aligned}$$

Noting that we compose linear mappings left to right, it is easy to verify that

$$g_0^2 = g_1^2 = g_2^2 = (g_0g_1)^4 = (g_1g_2)^4 = (g_0g_2)^2 = (g_1g_2g_1g_0)^b = I,$$

where I denotes the identity on E. Hence $G_{\ell,m}$ is the image of Γ under the homomorphism

$$f\colon \Gamma \to G_{\ell,m}$$

which sends each $\rho_j \to g_j$. Since $r_{j-1}r_j$ has period b ($j = 1, 3, 5, 7$), we may treat ℓ, m as residues (mod b).

We require the basic "translations" $g_x = g_1g_2g_1g_0$ and $g_y = g_2g_1g_0g_1$. Now for any integers j, k, we may write $g_x^j g_y^k$ as a matrix (with respect to the usual basis):

$$g_x^j g_y^k = \left[R\left(\frac{2\pi}{b}(j\ell - km)\right); R\left(\frac{2\pi}{b}(j\ell + km)\right); \right. \\ \left. R\left(\frac{2\pi}{b}(jm - k\ell)\right); R\left(\frac{2\pi}{b}(jm + k\ell)\right) \right]. \tag{5}$$

In this 8×8 block diagonal matrix, $R(\theta)$ denotes the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Since the relations in (3) imply that τ_x and τ_y commute, it follows that every $g \in G_{\ell,m}$ can be written as

$$g = hg_x^j g_y^k,$$

where $h \in \langle g_1, g_2 \rangle$. It is now readily verified that *all* elements of $G_{\ell,m}$ other than "translations" have trace 0, whereas

$$\operatorname{trace}\big(g_x^j g_y^k\big) = 4\left[\cos\left(\frac{2\pi}{b} j\ell\right)\cos\left(\frac{2\pi}{b} km\right) + \cos\left(\frac{2\pi}{b} jm\right)\cos\left(\frac{2\pi}{b}(k\ell)\right)\right]. \tag{6}$$

Next, it is easy to verify that the Wythoff space $W_{\ell,m}$ for $G_{\ell,m}$ is spanned by

$$p = e_1 + e_3 + e_5 + e_7. \tag{7}$$

We, therefore, have the ingredients necessary to define a realization $P_{\ell,m} := [f, p]$ (depending on (ℓ, m)) for the toroidal map $\mathcal{P} = \{4, 4\}_{(b,0)}$.

Notice that g_0 fixes p only if $\ell = m = 0$, in which case $V(P_{0,0}) = \{p\}$, so that $P_{0,0}$ is a trivial realization for $\mathcal{P}$. Hence $\dim(P_{0,0}) = 0$, although it is still useful to consider $G_{0,0}$ as a trivial linear group of degree $\deg(G_{0,0}) = 1$ acting irreducibly on $E' = \mathbb{R}p$ (the linear subspace of E spanned by p).

Otherwise, when $0 < \ell$ or $0 < m$, $o \in E$ is the unique point fixed by $G_{\ell,m}$ and so $V(P_{\ell,m})$ affinely spans a linear subspace E' of E. In this case, $\dim(P_{\ell,m}) = \dim(E') = \deg(G_{\ell,m}) > 0$. (The dimension of the realization coincides with the degree of the *induced* representation on E'.)

In summary, in all cases E' denotes the linear subspace of E spanned by $V(P_{\ell,m})$, and we usually consider $G_{\ell,m}$ as acting on E'.

We assemble several properties of this realization into

Theorem 3.1. *For integers* $0 \le \ell, m \le b - 1$ $(b \ge 2)$, *let* $P_{\ell,m}$ *be the realization in* E' *described above for* $\{4, 4\}_{(b,0)}$. *Then* $P_{\ell,m}$ *has the following properties*:

(a) $P_{\ell,m} \equiv P_{m,\ell}$ *and* $P_{\ell,m} \equiv P_{-\ell,m}$. (*Recall that* ℓ, m *can be taken* (*mod* b).)
(b) *Assume* (*as we may, by* (*a*)) *that* $0 \le m \le \ell \le b/2$. *Then each* $P_{\ell,m}$ *is a pure realization with*
   (i) $\dim(P_{\ell,m}) = 8$, *if* $0 < m < \ell < b/2$. *Each* g_j *has trace* 0 (*acting on* $E = E'$).

(ii) $\dim(P_{\ell,m}) = 4$, *if* $0 = m < \ell < b/2$. *Here* g_0, g_2 *act as reflections and* g_1 *as a half-turn on the* 4*-space* E'.
(iii) $\dim(P_{m,m}) = 4$, *if* $0 < m = \ell < b/2$. *Here* g_0, g_2 *act as half-turns and* g_1 *as a reflection on the* 4*-space* E'.
(iv) $\dim(P_{b/2,m}) = 4$, *if* $0 < m < \ell = b/2$ (*for* b *even*). *Here* g_0 *acts as a rotatory reflection* (*trace*(g_0) $= -2$), g_1 *as a half-turn and* g_2 *as a reflection on the* 4*-space* E'.
(v) $\dim(P_{b/2}, 0) = 2$ (*for* b *even*). *In this collapse to a square*, g_0, g_1 *act as reflections and* g_2 *as the identity on the* 2*-space* E'.
(vi) $\dim(P_{b/2,b/2}) = 1$ (*for* b *even*). *In this collapse to a segment*, g_0 *acts as a reflection and* g_1, g_2 *as the identity on the* 1*-space* E'.
(vii) $\dim(P_{0,0}) = 0$ (*the trivial realization*).

(c) *The realizations enumerated in part* (*b*) *are mutually incongruent*.
(d) *For* $(\ell, m) \neq (0, 0)$, *let* $d = \gcd(\ell, m, b)$ *and let*

$$\varepsilon = \begin{cases} 2, & \textit{if } b/d \textit{ is even, and both } \ell/d \textit{ and } m/d \textit{ are odd;} \\ 1, & \textit{otherwise.} \end{cases}$$

Then

$$\left|V(P_{\ell,m})\right| = \frac{b^2}{\varepsilon d^2}.$$

(*Of course, the trivial realization* $P_{(0,0)}$ *has one vertex*.)

Proof. (a) The indicated congruences of realizations follow from conjugating the g_j's by $\lambda\mu$ and by $\mu r_0 r_2$, respectively.

(b) From character theory, a complex representation $f\colon \Gamma \to G$ is irreducible iff its character norm equals 1 [10, p. 69]. In the present (real) case, it follows at least that $G_{\ell,m} = (\Gamma)f$ acts irreducibly on the eight-dimensional space E if

$$1 = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \operatorname{trace}(\gamma(f))\overline{\operatorname{trace}(\gamma(f))}.$$

However, by (6) and the related remarks, the preceding sum is

$$= \frac{16}{8b^2} \sum_{j,k=0}^{b-1} \left[\cos\left(\frac{2\pi}{b} j\ell\right)\cos\left(\frac{2\pi}{b}km\right) + \cos\left(\frac{2\pi}{b}k\ell\right)\cos\left(\frac{2\pi}{b}jm\right)\right]^2$$
$$= \frac{4}{b^2}[s(\ell,\ell)s(m,m) + s(\ell,m)^2],$$

where

$$s(p,q) = \sum_{j=0}^{b-1} \cos\left(\frac{2\pi pj}{b}\right)\cos\left(\frac{2\pi qj}{b}\right)$$
$$= \begin{cases} b, & \text{if} \quad p \equiv q \equiv 0 \quad \text{or} \quad p \equiv q \equiv b/2 \pmod b; \\ b/2, & \text{if} \quad p \equiv \pm q \quad \text{but} \quad p \not\equiv 0 \quad \text{or} \quad b/2 \pmod b; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, as we claim in (i), $P_{\ell,m}$ is indeed a pure eight-dimensional realization if $0 < m < \ell < b/2$.

For general irreducible real representations, the character norm may exceed 1. Even so, in the present instance, $G_{\ell,m}$ is reducible for other values of the parameters ℓ, m. For example, in part (ii) with $0 = m < \ell < b/2$, we find that E' is an invariant four-dimensional subspace of E with basis

$$\{e_0 + e_2, e_1 + e_3, e_5 + e_7, e_6 - e_4\}.$$

(Note that the base vertex $p \in E'$.) The induced generators $\tilde{g}_0, \tilde{g}_2$ are reflections (trace $=$ 2) whereas $\tilde{g}_1$ is a half-turn (trace $= 0$). Thus some nontranslations have nonvanishing trace. Nevertheless, a similar and straightforward calculation of the character norm shows that $P_{\ell,0}$ $(0 < \ell < b/2)$ is pure four-dimensional.

The remaining cases (iii)–(vii) follow similarly.

(c) Given the information concerning dimension and trace, it is clear that realizations from different classes (i)–(vii) are incongruent. So consider, for example, two realizations in class (i), for pairs (ℓ_1, m_1) and (ℓ_2, m_2). If these realizations were congruent, corresponding translations $g_x^j g_y^k$ would have equal traces, as described in (6). Taking $(j, k) = (1, 0)$ and $(1, 1)$, and recalling that $0 < m_j < \ell_j < b/2$, we soon find that $m_1 = m_2$ and $\ell_1 = \ell_2$. Cases (ii)–(vii) follow similarly.

(d) The "translation" subgroup acts transitively on $V(P_{\ell,m})$. A close look at (5) and (7) shows that the number of vertices is the same as the order of the subgroup of $(Z_b^4, +)$ generated by $[\ell, \ell, m, m]$ and $[-m, m, -\ell, \ell]$. Having found the elementary divisors of the 2×4 integer matrix with these rows, we may readily compute $|V(P_{\ell,m})|$. $\square$

A remarkable consequence of Theorem 3.1 is that much of the data for these pure realizations of $\{4, 4\}_{(b,0)}$ is neatly encoded in a picture of the polytope as a toroidal map. The cases $b = 4$ and $b = 5$, which are typical enough, are shown in Fig. 3. In Fig. 3 we interpret (ℓ, m) as coordinates (mod b) for a typical vertex in the grid. Also indicated are two adjacent lines of symmetry for the grid (and the toroidal map). Theorem 3.1(a),(c) can now be interpreted as asserting that

- each distinct pure realization $P_{\ell,m}$ corresponds to exactly one vertex in the wedge-shaped fundamental region enclosed by the two mirrors in the grid. (The corresponding vertices have been clearly indicated by a circle, or box.)

Just as interesting we observe from Theorem 3.1(b) that

- the dimension of each nontrivial pure realization $P_{\ell,m}$ equals the size of the orbit of the corresponding grid vertex, under the dihedral group generated by the two grid symmetries. (This dimension appears as a label inside each circle. The corner box, which indicates the trivial realization, also fits into this scheme, if we replace "dimension" by "degree.")

For example, up to similarity $\{4, 4\}_{(5,0)}$ has exactly one eight-dimensional pure realization of type $P_{\ell,m}$, while $\{4, 4\}_{(4,0)}$ has none.

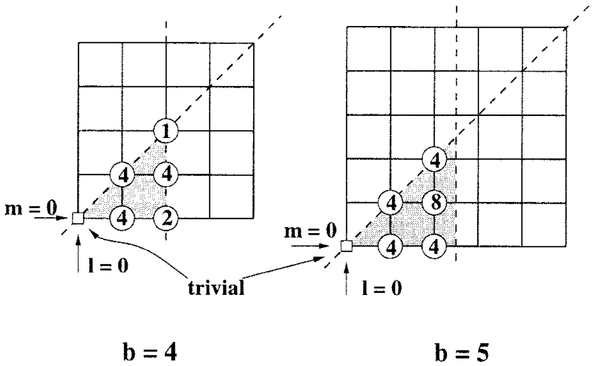

Fig. 3. Pure realizations of $\{4, 4\}_{(b,0)}$ encoded in the polytope itself.

In a sense, the polytope acts as a kind of discrete "moduli space" for its own set of pure realizations. As yet, we lack a more insightful proof of this. (The same situation occurs for maps of type {3, 6}, as explained in [1].)

We must still, however, check that *every* pure realization of $\{4, 4\}_{(b,0)}$ is (similar to) a realization of type $P_{\ell,m}$. To do so, we first refer to Fig. 1, noting that any two vertices in the square tessellation {4, 4} can be interchanged by a symmetry of the whole tessellation (e.g., by some half-turn). Passing to the quotient polytope $\{4, 4\}_{(b,0)}$, with group Γ, and identifying its vertices with the cosets $\Gamma_0\sigma$, we conclude that for each $\sigma \in \Gamma$ there exists $\lambda \in \Gamma$ such that $(\Gamma_0, \Gamma_0\sigma)\lambda = (\Gamma_0\sigma, \Gamma_0)$. Hence

$$\Gamma_0\sigma\Gamma_0 = \Gamma_0\sigma^{-1}\Gamma_0, \qquad \forall\sigma \in \Gamma.$$

It follows from Theorem 2.1(b),(c) that $r = \overline{w}$. Thus for each distinct, nontrivial irreducible representation $\Gamma \to G$, the Wythoff dimension $w_G = 1$ (see Theorem 19 of [6]). Furthermore, there are just r inequivalent, pure, nontrivial realizations. However, $\Gamma_0\sigma\Gamma_0$ can be identified with the Γ_0-orbit of the vertex $\Gamma_0\sigma$, so that by Theorem 2.1(c), $r = \overline{w}$ is just the number of orbits under the action of Γ_0 on the vertex set of $\{4, 4\}_{(b,0)}$ (excluding the base vertex Γ_0 itself).

A little thought shows that we may reinterpret Γ_0 as the group generated by the grid symmetries indicated in Fig. 3. Thus *every* pure realization of $\{4, 4\}_{(b,0)}$ corresponds to a unique vertex in the wedge; and r, the number of such vertices, is readily calculated. Summarizing we have

Theorem 3.2. *For $b \geq 2$, each pure realization of the polytope $\{4, 4\}_{(b,0)}$ is of type $P_{\ell,m}$. The number of nontrivial pure realizations is therefore*

$$r = \frac{1}{2}\left\lfloor\frac{b}{2}\right\rfloor\left(\left\lfloor\frac{b}{2}\right\rfloor + 3\right).$$

4. The Pure Realizations for $\{4, 4\}_{(c,c)}$

Using the presentation (4) for the automorphism group $\Gamma(\{4, 4\}_{(c,c)})$, one readily verifies that

$$1 = (\rho_1\rho_2\rho_1\rho_0)^{2c}.$$

It follows that there is an epimorphism

$$\Gamma\left(\{4, 4\}_{(2c,0)}\right) \to \Gamma\left(\{4, 4\}_{(c,c)}\right),$$

which preserves the distinguished generators, so that $\{4, 4\}_{(2c,0)}$ *covers* $\{4, 4\}_{(c,c)}$ [7, Section 2.1.2]. Moreover, every pure realization P of $\{4, 4\}_{(c,c)}$ is thus *some* pure realization $P_{\ell,m}$ of $\{4, 4\}_{(2c,0)}$, by Theorem 3.2. Indeed, by (4), $P_{\ell,m}$ will be a realization for $\{4, 4\}_{(c,c)}$ just when $g_x^c g_y^c = I$. Taking $j = k = c$ and $b = 2c$ in (5), we see that this happens precisely when $\ell \equiv m \pmod 2$.

Looking again at the $b \times b$ grid which parametrizes the $P_{\ell,m}$ (as in Figure 3), we observe that

- for $b = 2c$, and starting with the trivial vertex, *alternate* vertices in the fundamental wedge of the $b \times b$ grid describe inequivalent, pure realizations of $\{4, 4\}_{(c,c)}$.

The case $c = 3$ (i.e., $b = 6$) is indicated in Fig. 4.

It is easy to check that the number of alternate, nontrivial grid vertices in the wedge is $r' = c + \lfloor c^2/4 \rfloor$. On the other hand, an analysis similar to that preceding Theorem 3.2 shows that r' also equals the total number of inequivalent, nontrivial pure realizations. We thus have

Theorem 4.1. *For $c \geq 2$, the pure realizations of the polytope $\{4, 4\}_{(c,c)}$ are exactly the realizations $P_{\ell,m}$, with $\ell \equiv m \pmod 2$. The number of such nontrivial pure realizations is thus*

$$r' = c + \left\lfloor \frac{c^2}{4} \right\rfloor.$$

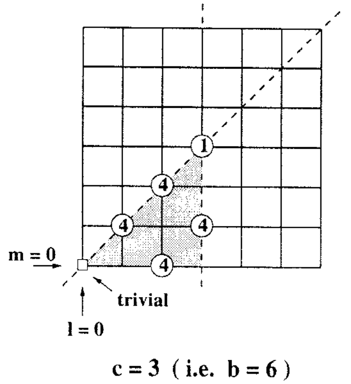

Fig. 4. Pure realizations of $\{4, 4\}_{(c,c)}$ encoded in the polytope $\{4, 4\}_{(2c,0)}$.

5. Tori Inscribed in the 4-Cube

We conclude by examining the polyhedra $\{4,4\}_{(4,0)}$ and $\{4,4\}_{(2,2)}$, which are particularly attractive, since their pure realizations in four-dimensional Euclidean space E can be usefully visualized with the help of the 4-cube {4, 3, 3} and the regular cross-polytope {3, 3, 4} situated in E.

Referring to the grid on the left in Fig. 3, we begin with the realization $P_{1,0}$ for $\{4,4\}_{(4,0)}$. In this case the "translations" g_x and g_y are simple rotations of period four about orthogonal planes in E. Thus, by Theorem 3.1(d), the 16 vertices of the realization are in fact the vertices of the double prism $\{4\} \times \{4\}$, namely, the 4-cube {4, 3, 3} [2, p. 37]. Moreover, the 32 edges of the polyhedron are faithfully represented by (all) edges of the convex polytope. Now recall that each facet of the 4-cube belongs (in three ways) to a belt of four, in which consecutive facets share a (square) face. Indeed, all eight facets lie in two such complementary belts. If we discard the eight intermediary squares from these two belts, we are left with the 16 square faces of the polyhedron, as realized by $P_{1,0}$. (A view of this familiar realization of the polyhedron as an actual map is hidden in Fig. 6 below; see also Figure 4.2B on p. 31 of [2].)

Like $P_{1,0}$, the faithful realization $P_{2,1}$ also employs all 16 vertices of the 4-cube. Now, however, the edges of $\{4,4\}_{(4,0)}$ joining these vertices are realized by the 32 "main" diagonals of the eight cubical facets of the 4-cube (see Fig. 5). Furthermore, each face of the polyhedron $\{4,4\}_{(4,0)}$ is here realized by a skew quadrilateral following—and inscribed in—one of the above-mentioned belts of four facets.

Finally, we consider the realization $P_{1,1}$, in which the edges of $\{4,4\}_{(4,0)}$ are represented as certain diagonals of the square faces in the 4-cube. Consequently, this realization is not faithful and involves a 2 : 1 collapse of the vertices of the polyhedron onto a set of eight alternate vertices in the 4-cube (Fig. 6). Indeed, these eight vertices are the vertices of a regular cross-polytope {3, 3, 4} inscribed in the original 4-cube. Referring to Theorem 4.1, we observe that $P_{1,1}$ in fact provides a faithful realization of the regular polyhedron $\{4,4\}_{(2,2)}$, which is doubly covered by the original $\{4,4\}_{(4,0)}$.

In Fig. 7 we give a more symmetrical view of the cross-polytope itself, where we observe that the 16 edges of $\{4,4\}_{(2,2)}$ are realized as just those edges of the cross-polytope which remain after removing those in two orthogonal equatorial squares (indicated by

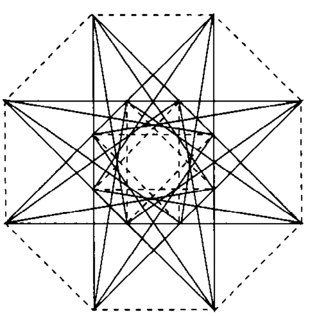

Fig. 5. A view of the realization $P_{2,1}$ for $\{4,4\}_{(4,0)}$.

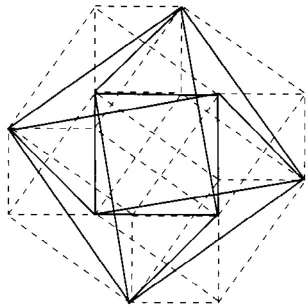

Fig. 6. The realization $P_{1,1}$ for $\{4, 4\}_{(4,0)}$.

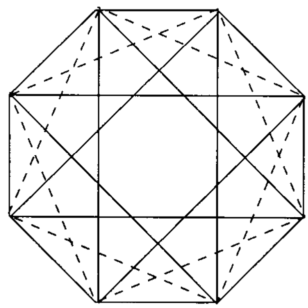

Fig. 7. The faithful realization $P_{1,1}$ for $\{4, 4\}_{(2,2)}$.

dotted lines). Last, we note that adjacent faces of $\{4, 4\}_{(2,2)}$ appear here as Petrie polygons for two tetrahedral facets of $\{3, 3, 4\}$, lying opposite one another along a common edge.

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References

1. H. Burgiel and D. Stanton, Realizations of regular abstract polyhedra of types {3, 6} and {6, 3}, *Discrete Comput. Geom.*, this issue.
2. H.S.M. Coxeter, *Regular Complex Polytopes*, 2nd edn., Cambridge University Press, Cambridge (1991).
3. H.S.M. Coxeter and W.O.J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, New York (1972).
4. B. Grünbaum, Regularity of graphs, complexes and designs, in *Colloques CNRS, Problèmes Combinatoires et Theorie des Graphes*, CNRS, Orsay, (1976), pp. 191–197.
5. B. Grünbaum, Regular polyhedra—old and new, *Aequationes Math.* **16** (1977), 1–20.
6. P. McMullen, Realizations of regular polytopes, *Aequationes Math.* **37** (1989), 38–56.

7. P. McMullen, Modern developments in regular polytopes, in T. Bisztriczky et al. (eds), *Polytopes*: *Abstract, Convex and Computational*, *Proc*. *NATO Advanced Study Institute*, NATO ASI Series C, Vol. 440, Kluwer, Dordrecht (1994), pp. 97–124.
8. P. McMullen and E. Schulte, Regular polytopes of type {4, 4, 3} and {4, 4, 4}, *Combinatorica* **12**(2) (1992), 203–220.
9. P. McMullen and E. Schulte, Abstract regular polytopes (manuscript in preparation).
10. E.B. Vinberg, *Linear Representations of Groups*, Birkhäuser Verlag, Basel (1989).

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