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Perfectly Packing an Equilateral Triangle by Equilateral Triangles of Sidelengths $n^{-1/2-\epsilon}$

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Abstract
Equilateral triangles of sidelengths $1, 2^{-t}, 3^{-t}, 4^{-t}, \ldots$ can be packed perfectly into an equilateral triangle, provided that $1/2 < t \le 37/72$. Moreover, for t slightly greater than 1/2, squares of sidelengths $1, 2^{-t}, 3^{-t}, 4^{-t}, \ldots$ can be packed perfectly into a square S_t in such a way that some squares have a side parallel to a diagonal of S_t and the remaining squares have a side parallel to a side of S_t.

Keywords Packing · Perfect packing · Triangle · Square

Mathematics Subject Classification 52C15

1 Introduction

Let $C, C_1, C_2, C_3, \ldots$ be planar convex bodies. We say that $C_1, C_2, \ldots$ can be *packed* into C if it is possible to apply translations and rotations to the sets C_n so that the resulting translated and rotated bodies are contained in C and have mutually disjoint interiors. If the area of C is equal to the sum of areas of the bodies, then the packing is *perfect*.

There are many results concerning packings. For example, Moon and Moser showed [12] that any collection of squares whose total area does not exceed $1/2$ can be packed into a square of sidelength 1. Richardson [15] proved that any collection of triangles

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homothetic to T, whose total area does not exceed half the area of T, can be packed in T and made the conjecture that such a result is true also for the translative packing by positive homothetic copies. This has been confirmed in [4].

In this note, we will study perfect packing.

In 1966 Moser posed the following well-known problem (see problem LM6 in [13]): Find the smallest $\varepsilon \geq 0$ such that the squares of harmonic sidelengths $1/2, 1/3, 1/4, \ldots$ can be packed into a rectangle of area $\frac{1}{6}\pi^2 - 1 + \varepsilon$ (the sum of areas of the squares equals $\frac{1}{6}\pi^2 - 1$).

The following upper bounds for ε were obtained sequentially: $1/20$ [11], $1/127$ [8], $1/198$ [1], and $1/1244918662$ (see [14], [9], and [3]).

This packing problem can be extended. Let S_n^t be a square of sidelength n^{-t} for $n = 1, 2, \ldots$. If $t > 1/2$, then the total area of the squares is equal to $\sum_{n=1}^{\infty} \frac{1}{n^{2t}} = \zeta(2t)$, where $\zeta(s)$ is the Riemann zeta function. The question is whether $S_1^t, S_2^t, \ldots$ (for $t > 1/2$) can be packed perfectly into a rectangle. Obviously, for $t = 1$, we get Moser's original question.

Some results are known for $t < 1$. Chalcraft [2] showed that $S_1^t, S_2^t, S_3^t, \ldots$ can be packed perfectly into a square for all t in the range $[0.5964, 0.6]$. Joós [10] checked that these squares can be packed perfectly for all t in the range $[\log_3 2, 2/3]$ ($\log_3 2 \approx 0.63$). Wästlund [17] proved that $S_1^t, S_2^t, S_3^t, \ldots$ can be packed into a finite collection of squares of the same area as the sum of areas of the squares, provided that $1/2 < t < 2/3$. In [5] it is shown that for all t in the range $(1/2, 2/3]$, the squares $S_1^t, S_2^t, S_3^t, \ldots$ can be packed perfectly into a single square. Recently, Tao [16] proved that for any $1/2 < t < 1$, and any n_0 that is sufficiently large depending on t, the squares $S_{n_0}^t, S_{n_0+1}^t, \ldots$ can be packed perfectly into a square.

In this note, we will give an analog of this problem for the packing of triangles.

Let T_n^t be an equilateral triangle of sidelength n^{-t} for $n = 1, 2, \ldots$. The question arises whether $T_1^t, T_2^t, \ldots$ (for $t > 1/2$) can be packed perfectly into an equilateral triangle. More precisely: whether $T_1^t, T_2^t, \ldots$ can be packed perfectly, for $0.5 < t \leq 0.761202\ldots$ (now $1 + 1/2^t \leq \sqrt{\zeta(2t)}$); whether $T_2^t, T_3^t, \ldots$ can be packed perfectly, provided that $0.761202\ldots < t \leq 0.943674\ldots$ (for such values of t the sum of sidelengths of T_2^t and T_3^t is smaller than $\sqrt{\zeta(2t) - 1}$); whether $T_3^t, T_4^t, \ldots$ can be packed perfectly, provided that $0.943674\ldots < t \leq 1.121936\ldots$ (now $1/3^t + 1/4^t \leq \sqrt{\zeta(2t) - 1 - 1/2^{2t}}$), etc.. It is only known [7] that $T_3^1, T_4^1, \ldots$ can be packed into an equilateral triangle of side of length $(\pi^2/6 - 5/4)^{1/2} + 1/270$.

In Sect. 2, we will show that equilateral triangles of sidelengths $1, 2^{-t}, 3^{-t}, 4^{-t}, \ldots$ can be packed perfectly into an equilateral triangle T_t, provided that $1/2 < t \leq 37/72$.

In Sect. 3, we will check that if $1/2 < t \leq 37/72$, then all packed triangles can be positive homothetic copies of T_t as well as all packed triangles can be negative homothetic copies of T_t.

In addition, in Sect. 4, we will consider square-packing. We will prove that, for $1/2 < t \leq (154 + 3\sqrt{2})/306$, squares of sidelengths $1, 2^{-t}, 3^{-t}, 4^{-t}, \ldots$ can be packed perfectly into a square S_t in such a way that some squares have a side parallel to a diagonal of S_t and the remaining squares have a side parallel to a side of S_t.

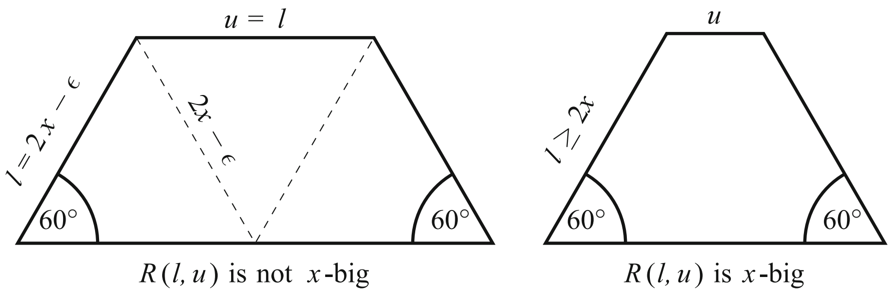

Fig. 1 Standard trapezoids $R(l, u)$

2 Perfect Packing

Let t be a fixed number from the interval $(1/2, 37/72]$ and let T_t be an equilateral triangle of area $\frac{\sqrt{3}}{4}\zeta(2t)$.

The outline of the packing method is as follows. For each $n \geq 2$, the empty space in T_t, i.e., the part of T_t not covered by packed triangles $T_1^t, \ldots, T_{n-1}^t$, will be divided into at most $3(n-1)$ trapezoids. Then, T_n^t will be packed into a corner of one of these trapezoids.

Let $R(l, u)$ be an isosceles trapezoid with legs of length l, with the measure of the base angles equal to $60°$ and with the shorter base of length u. Clearly, the length of the longer base of $R(l, u)$ is equal to $l + u$. If $u = 0$, then $R(l, 0)$ is an equilateral triangle.

A trapezoid $R(l, u)$ is x*-big*, provided that $l \geq 2x$ (see Fig. 1, right).

A trapezoid $R(l, u)$ is *standard*, provided that $u \leq l$.

Obviously, each x-big trapezoid is also v-big for any $v < x$. Moreover, each standard trapezoid is x-big for sufficiently small x.

Proposition 1 *The area of any standard trapezoid that is not x-big is smaller than* $3\sqrt{3}x^2$.

Proof The area of any standard trapezoid that is not x-big is smaller than three times the area of an equilateral triangle of sidelength $2x$ (see Fig. 1, left), i.e., is smaller than $3 \cdot \frac{\sqrt{3}}{4} \cdot (2x)^2 = 3\sqrt{3}x^2$. □

Lemma 2 *Let $R(l, u)$ be an x-big trapezoid. Then $R(l, u)$ can be divided into either four or five parts: an equilateral triangle of sidelength x and at most four trapezoids which are either standard or x-big.*

Proof Let T_i be the equilateral triangle of sidelength x. We divide $R(l, u)$ into T_i (denoted by"+"in Figs. 2 and 3) and four sets A_i, B_i, C_i, D_i which are either standard trapezoids or x-big trapezoids (possibly some of them are triangles), or the empty set.

Case 1: $l \geq 4x$. The trapezoid $R(l, u)$ is divided into: $T_i \cup A_i \cup B_i \cup C_i$, where $A_i = R(x, x)$, $B_i = R(l - 2x, u)$ and $C_i = R(2x, l + u - 3x)$. The trapezoid A_i is standard. Moreover, B_i and C_i are x-big (see Fig. 2, left, when $l > 4x$). In this case $D_i = \emptyset$.

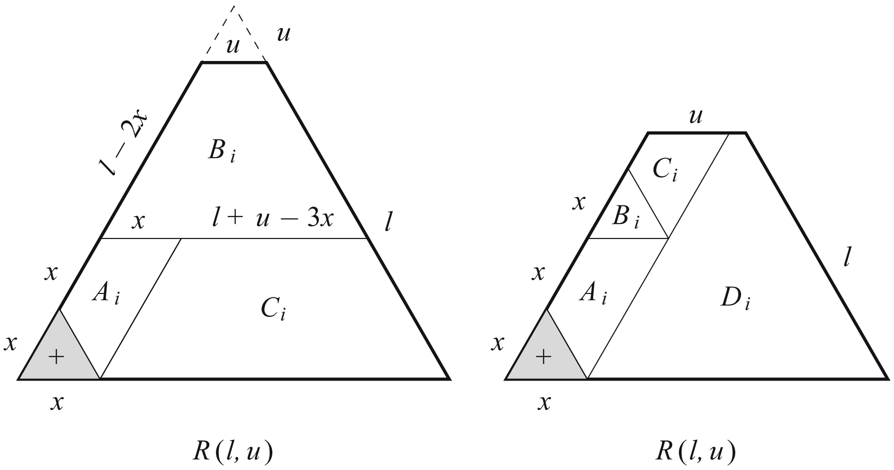

Fig. 2 Divisions of the x-big trapezoid $R(l,u)$, when $l \geq 3x$

Case 2: $3x \leq l < 4x$.

Subcase 2a: $u < x$. The trapezoid $R(l,u)$ is divided into: T_i, two standard trapezoids $A_i = R(x,x)$, $B_i = R(l-2x,u)$ and one x-big trapezoid $C_i = R(2x, l+u-3x)$ (as in Fig. 2, left, when $3x < l \leq 4x$). We take $D_i = \emptyset$.

Subcase 2b: $u \geq x$. The trapezoid $R(l,u)$ is divided into: T_i, three standard trapezoids $A_i = R(x,x)$, $B_i = R(x,0)$, $C_i = R(x, l-3x)$ and one x-big trapezoid $D_i = R(l, u-x)$ (as in Fig. 2, right).

Case 3: $2x \leq l < 3x$.

Subcase 3a: $u < x$ and $l+u \geq 3x$. The trapezoid $R(l,u)$ is divided into: T_i and three standard trapezoids $A_i = R(x,x)$, $B_i = R(x, l+u-3x)$, and $C_i = R(l-x,u)$ as in Fig. 3, left. We take $D_i = \emptyset$.

Subcase 3b: $u < x$ and $l+u < 3x$. The trapezoid $R(l,u)$ is divided into: T_i and three standard trapezoids $A_i = R(x, l+u-2x)$, $B_i = R(x,0)$, and $C_i = R(l-x,u)$ as in Fig. 3, middle. We take $D_i = \emptyset$.

Subcase 3c: $u \geq x$. The trapezoid $R(l,u)$ is divided into: T_i, three standard trapezoids $A_i = B_i = R(x,0)$, $C_i = R(x, l-2x)$ and one x-big trapezoid $D_i = R(l, u-x)$ as in Fig. 3, right. □

Since $t \leq 37/72$, it follows that the sidelength of T_t is greater or equal to $\sqrt{\zeta(37/36)} > 6$.

2.1 Packing Method $M_\triangle$

[1] The first triangle is packed into the lower left vertex of T_t. After packing T_1^t, the uncovered part of T_t is divided into $A_1 \cup B_1 \cup C_1$ as in the proof of Lemma 2 (comp. Fig. 2, left, for $u = 0$). We take $\mathcal{R}_1 = \{A_1, B_1, C_1\}$.

[2] Assume that $n > 1$, that the triangles $T_1^t, \ldots, T_{n-1}^t$ are packed into T_t and that the family $\mathcal{R}_{n-1}$ is defined. We choose one of the n^{-t}-big trapezoids from $\mathcal{R}_{n-1}$ in any

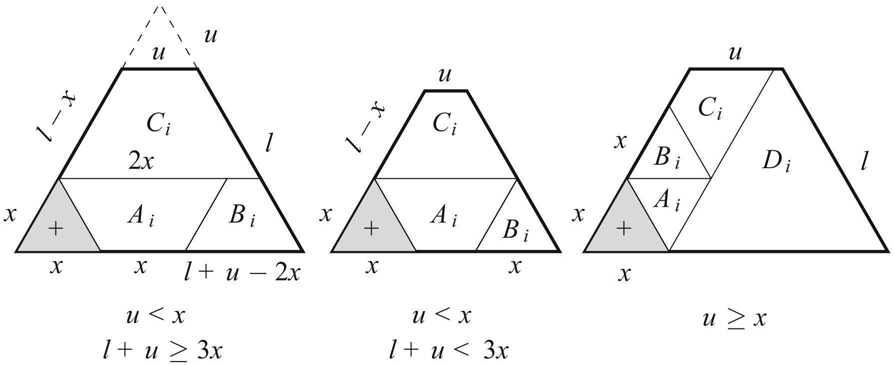

Fig. 3 Divisions of the x-big trapezoid $R(l, u)$, when $2x \leq l < 3x$

way. Denote this trapezoid by R. We pack T_n^t into R at the vertex at the longer base of R. After packing T_n^t, the uncovered part of R is divided into $A_n \cup B_n \cup C_n \cup D_n$, as in the proof of Lemma 2, and we take $\mathcal{R}_n = \big(\mathcal{R}_{n-1} \setminus \{R\}\big) \cup \{A_n, B_n, C_n, D_n\}$. It is possible that $D_n = \emptyset$.

Observe that $\mathcal{R}_{n-1}$ contains at most $3(n-1)$ trapezoids with mutually disjoint interiors, for any $n \geq 2$. Each trapezoid from $\mathcal{R}_{n-1}$ is either n^{-t}-big or standard. It is possible that a trapezoid from $\mathcal{R}_{n-1}$ is both n^{-t}-big and standard.

Example 1 Figure 4 illustrates the initial stage of the packing process for $t = 0.51$. The area of $T_{0.51}$ is equal to $\zeta(1.02)\sqrt{3}/4 \approx 21.9$, thus the sidelength σ of $T_{0.51}$ is greater than 7.11. After packing $T_1^{0.51}$, the uncovered part of $T_{0.51}$ is divided into three trapezoids: $A_1 = R(1, 1)$, $B_1 = R(\sigma - 2, 0)$ and $C_1 = R(2, \sigma - 3)$. Both trapezoids B_1 and C_1 are 2-big and therefore we have two possibilities to pack $T_2^{0.51}$. For instance, we will pack each triangle into a trapezoid of maximum leg length. After packing $T_2^{0.51}$ into B_1, the uncovered part of $T_{0.51}$ is partitioned into A_1, C_1, A_2, B_2 and C_2 ($D_2 = \emptyset$).

Theorem 3 *For each t in the range* $1/2 < t \leq 37/72 \approx 0.5138$*, the triangles* T_n^t *can be packed perfectly into the triangle* T_t *by the method* $M_\triangle$*.*

Proof The proof is similar to that presented in [6]. Let t be a fixed number from the interval $(1/2, 37/72]$. The area of T_t is equal to $\frac{\sqrt{3}}{4}\sum_{i=1}^{\infty}\frac{1}{i^{2t}} = \frac{\sqrt{3}}{4}\zeta(2t) \geq \frac{\sqrt{3}}{4}\zeta(37/36) = \frac{\sqrt{3}}{4} \cdot 36.579\ldots$. Consequently, T_t is 1-big (its leg length is greater than 2). We pack $T_1^t, T_2^t, \ldots$ into T_t by the method $M_\triangle$. To prove Theorem 3 it suffices to show that, for any n, there is at least one n^{-t}-big trapezoid in $\mathcal{R}_{n-1}$ (into which T_n^t will be packed).

First, we estimate the sum of areas of trapezoids in $\mathcal{R}_{n-1}$, i.e., the area of the uncovered part of T_t after packing T_{n-1}^t. This value is equal to the sum of areas of unpacked triangles $T_n^t, T_{n+1}^t, \ldots$, i.e., is equal to

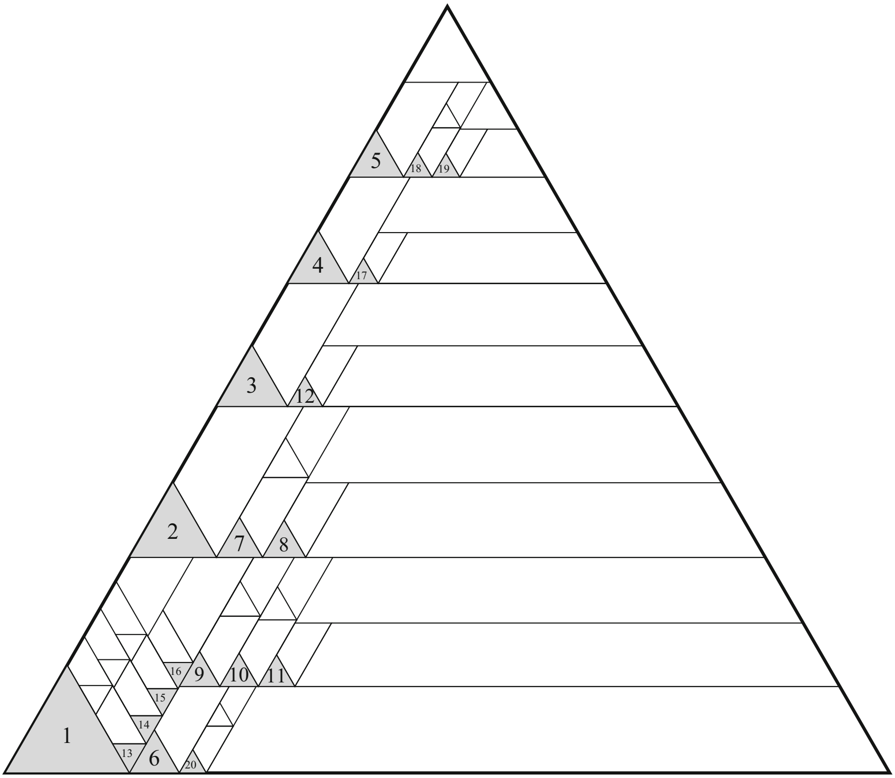

Fig. 4 Packing method $M_\triangle$ for $t = 0.51$

$$\frac{\sqrt{3}}{4}\Big(\frac{1}{n^{2t}} + \frac{1}{(n+1)^{2t}} + \dots\Big) > \frac{\sqrt{3}}{4} \cdot \int_n^{+\infty} \frac{1}{x^{2t}} \mathrm{d}x = \frac{\sqrt{3}}{4} \cdot \frac{1}{2t-1} n^{1-2t}$$
$$\geq \frac{\sqrt{3}}{4} \cdot \frac{1}{2 \cdot \frac{37}{72} - 1} n^{1-2t} = 9\sqrt{3}n^{1-2t}.$$

Assume that there is an integer n such that the triangle T_n^t cannot be packed into T_t by the method $M_\triangle$, i.e., that there is no n^{-t}-big trapezoid in $\mathcal{R}_{n-1}$. This means that all trapezoids in $\mathcal{R}_{n-1}$ are standard and that the length of the leg of each such trapezoid is smaller than $2n^{-t}$ (if $l \geq 2n^{-t}$, then $R(l, u)$ is n^{-t}-big). By Proposition 1, the area of each such trapezoid is smaller than $3\sqrt{3}n^{-2t}$. Since there are at most $3(n-1)$ trapezoids in $\mathcal{R}_{n-1}$, it follows that the total area of trapezoids in $\mathcal{R}_{n-1}$ is smaller than $(3n-3) \cdot 3\sqrt{3}n^{-2t} < 9\sqrt{3}n^{1-2t}$, which is a contradiction.

Consequently, $T_1^t, T_2^t, \dots$ can be packed into T_t. □

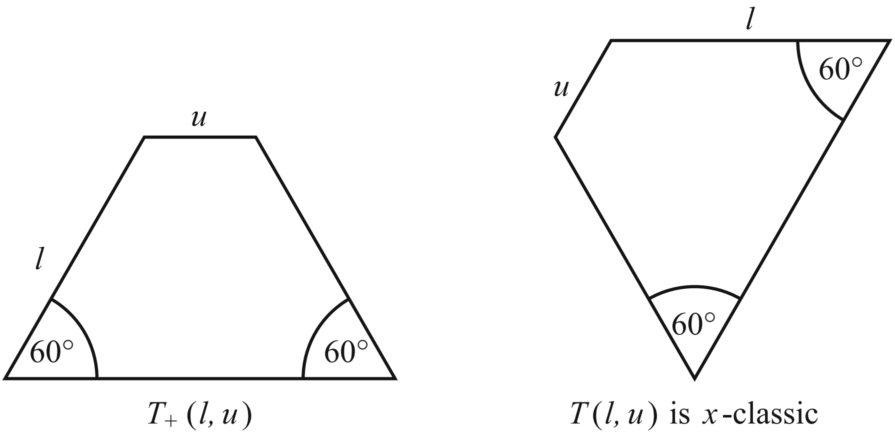

Fig. 5 Standard trapezoids $T(l, u)$

3 Positive or Negative Copies

One side of T_r is called the *bottom*. In our drawings, the bottom is the horizontal side of the triangle. By $T_+(l, u)$ we mean the trapezoid $T(l, u)$ lying such that its bases are parallel to the bottom of T_r and that the longer base is lower than the shortest one (see Fig. 5, left).

We call a trapezoid x-*classic* if it arises from rotating a trapezoid $T_+(l, u)$ by a multiple of 60°, where the respective $T(l, u)$ is x-big or standard (see Fig. 5, left and right).

The image of any positive homothetic copy of T_r in rotation of 60° is a negative homothetic copy of T_r as well as the image of any negative homothetic copy of T_r in rotation of 60° is a positive homothetic copy of T_r. The image of a triangle $T(x, 0)$ in rotation of 180° is denoted by $-T(x, 0)$.

Lemma 4 *Let R be an x-big classic trapezoid. Then R can be divided into: $T_+(x, 0)$ and at most four x-classic trapezoids. Moreover, R can be divided into: $-T_+(x, 0)$ and at most four x-classic trapezoids.*

Proof Let R be the x-big classic trapezoid.

Since R is the image of $T_+(l, u)$ in rotation by a multiple of 60°, it follows that R can be divided into the same shapes as $T_+(l, u)$, with possibly switching roles of $T_+(x, 0)$ and $-T_+(x, 0)$ (see Fig. 6). Consequently, to prove Lemma 4 it suffices to check that $T_+(l, u)$ can be divided into $T_+(x, 0)$ (denoted by "+") and at most four x-classic trapezoids as well as that $T_+(l, u)$ can be divided into $-T_+(x, 0)$ (denoted by"-"in Figs. 6, 7 and 8) and at most four x-classic trapezoids.

The first option was discussed in the proof of Lemma 2. Now consider negative copies of the triangle.

Case 1: $l \geq 4x$. The trapezoid R is divided into: $-T_+(x, 0)$, one standard trapezoid and two x-big trapezoids (as in Fig. 7, left; if $l \geq 4x$, then the upper trapezoid is x-big).

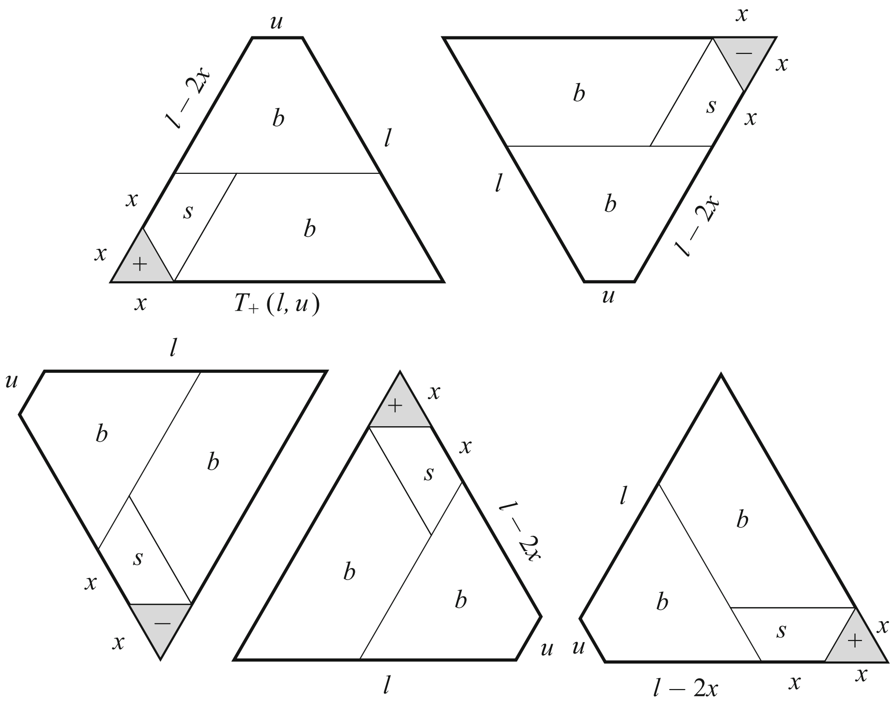

Fig. 6 Divisions of the x-big classic trapezoid R, when $l \geq 4x$

Case 2: $3x \leq l < 4x$. If $u < x$, then $T_+(l,u)$ is divided as in Fig. 7, left (now the upper trapezoid is standard). If $u \geq x$, then $T_+(l,u)$ is divided as in Fig. 7, right.

Case 3: $2x \leq l < 3x$. If $u < x$ and $l+u \geq 3x$, then $T_+(l,u)$ is divided as in Fig. 8, left. If $u < x$ and $l+u < 3x$, then $T_+(l,u)$ is divided as in Fig 8, middle. If $u \geq x$, then $T_+(l,u)$ is divided as in Fig. 8, right. □

3.1 Packing Method $M_{\Delta\pm}$

Let $\mathbb{N} = \mathbb{A} \cup \mathbb{B}$, where $\mathbb{A} \cap \mathbb{B} = \emptyset$. Since $t \leq 37/72$, it follows that the sidelength of T_t is greater or equal to $\sqrt{\zeta(37/36)} > 6$.

[1] If $1 \in \mathbb{A}$, then the first triangle is packed into T_t in the place "+" described in the proof of Lemma 2 (see Fig. 2, left, if $u = 0$). If $1 \in \mathbb{B}$, then the first triangle is packed in the place "-" described in the proof of Lemma 4 (see Fig. 7, left, if $l > 4x$ and $u = 0$). After packing T_1^t, the uncovered part of T_t is divided into three 1-classic trapezoids as in the proof of Lemma 4. We take as $\mathcal{R}_1^{\pm}$ the family of these three trapezoids.

[2] Assume that $n > 1$, that the triangles $T_1^t, \ldots, T_{n-1}^t$ are packed into T_t and that the family $\mathcal{R}^{\pm}{}_{n-1}$ is defined. We choose one of the n^{-t}-big trapezoids from $\mathcal{R}^{\pm}{}_{n-1}$ in any way. Denote this trapezoid by R. If $n \in \mathbb{A}$, then T_n^t is packed in the place

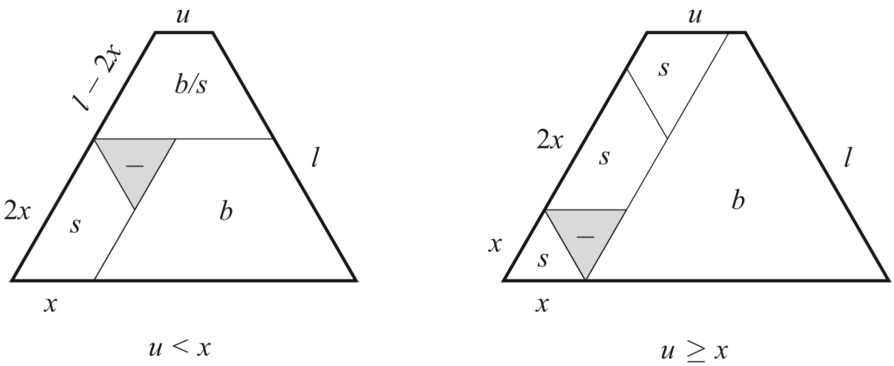

Fig. 7 Divisions of the x-big trapezoid $T_+(l, u)$, when $l \geq 3x$

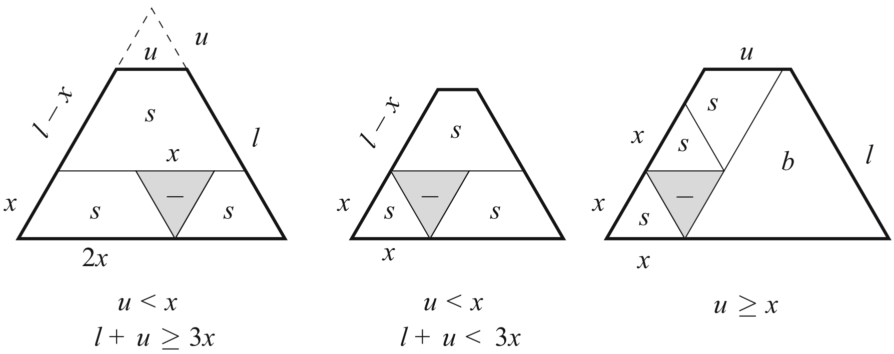

Fig. 8 Divisions of the x-big trapezoid $T_+(l, u)$, when $2x \leq l < 3x$

"+" in R; if $n \in \mathbb{B}$, then T_n^t is packed in the place "-" in R (see Figs. 2, 3, 6, 7 or 8). After packing T_n^t, the uncovered part of R is divided into at most four n^{-t}-classic trapezoids as in the proof of Lemma 4. We take as $\mathcal{R}^{\pm}{}_n$ the union of the family of these trapezoids and $\mathcal{R}^{\pm}{}_{n-1} \setminus \{R\}$.

Clearly, $\mathcal{R}^{\pm}{}_{n-1}$ contains at most $3(n-1)$ trapezoids with mutually disjoint interiors, for any $n \geq 2$. Each trapezoid from $\mathcal{R}^{\pm}{}_{n-1}$ is either n^{-t}-big or standard.

Figure 9 illustrates the initial stage of the packing process in the case when $\mathbb{A} = \mathbb{N}$ and $\mathbb{B} = \emptyset$; note that the first twelve triangles are packed in the same places using the algorithms $M_\triangle$ and $M_{\triangle^\pm}$. On the other hand, $\mathbb{B} = \mathbb{N}$ and $\mathbb{A} = \emptyset$ in Fig. 10.

Theorem 5 *For each t in the range* $1/2 < t \leq 37/72$, *the triangles* T_n^t *can be packed perfectly into the triangle* T_t *so that each packed triangle* T_n *is a positive homothetic copy of* T_n^t, *provided that* $n \in \mathbb{A}$ *and that each packed triangle* T_n^t *is a negative homothetic copy of* T_t, *provided that* $n \in \mathbb{B}$.

Proof We pack $T_1^t, T_2^t, \ldots$ into T_t by the method $M_{\triangle^\pm}$. As in the proof of Theorem 3, the sum of areas of trapezoids in $\mathcal{R}^{\pm}{}_{n-1}$ is greater than $9\sqrt{3}n^{1-2t}$. If there is an

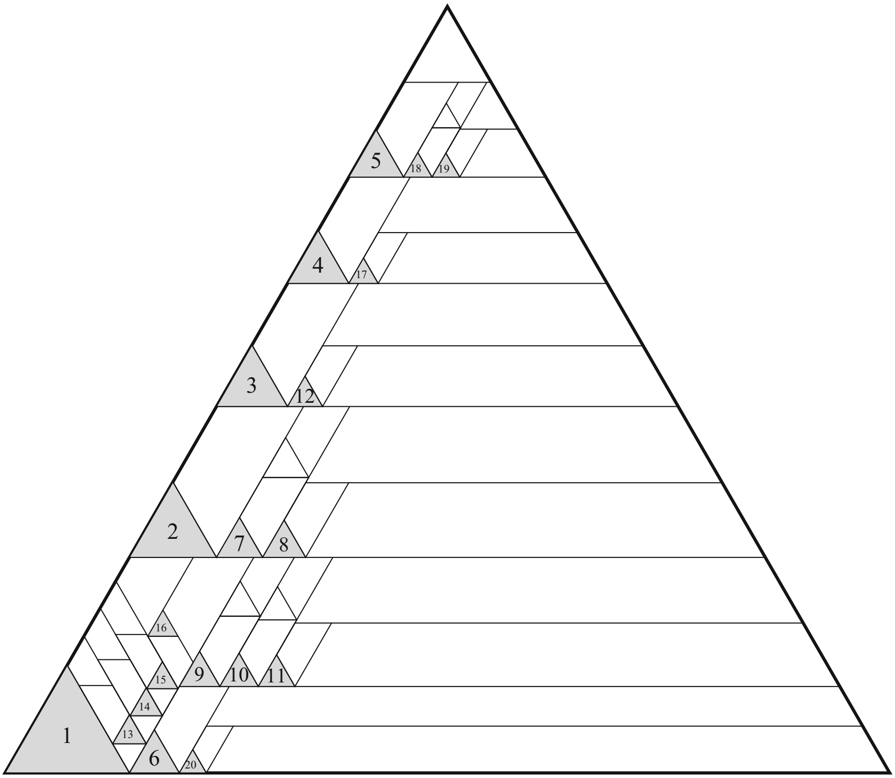

Fig. 9 Packing method for $t = 0.51$ and $\mathbb{B} = \emptyset$

integer n such that the triangle T_n^t cannot be packed into T_t by our method, then the total area of trapezoids in $\mathcal{R}^{\pm}{}_{n-1}$ is smaller than $(3n-3)\cdot 3\sqrt{3}n^{-2t} < 9\sqrt{3}n^{1-2t}$, which is a contradiction. $\square$

4 Squares

Denote by S_t the square of area $\zeta(2t)$, where $1/2 < t \le (154 + 3\sqrt{2})/306 \approx 0.517$. Let $P(h, a)$ be a right trapezoid with height h and with bases of length a and $a - h$. In this section, a trapezoid $P(h, a)$ is x-*big*, provided that $h \ge \frac{3\sqrt{2}}{2}x$. A trapezoid $P(h, a)$ is *standard*, if $a \le \frac{3\sqrt{2}}{2}h$. A trapezoid $P(h, a)$ is x-*classic*, provided that it is either standard or x-big and provided that its bases are parallel either to a side of S_t or to a diagonal of S_t. Observe that the area of each standard trapezoid that is not x-big is smaller than $\left(\frac{3\sqrt{2}}{2} - \frac{1}{2}\right)\left(\frac{3\sqrt{2}}{2}x\right)^2 = \frac{27\sqrt{2}-9}{4}\cdot x^2$.

Lemma 6 *Let P be an x-big classic trapezoid. Then P can be divided into: a square of sidelength x with a side parallel to the bases of P and at most four x-classic trapezoids.*

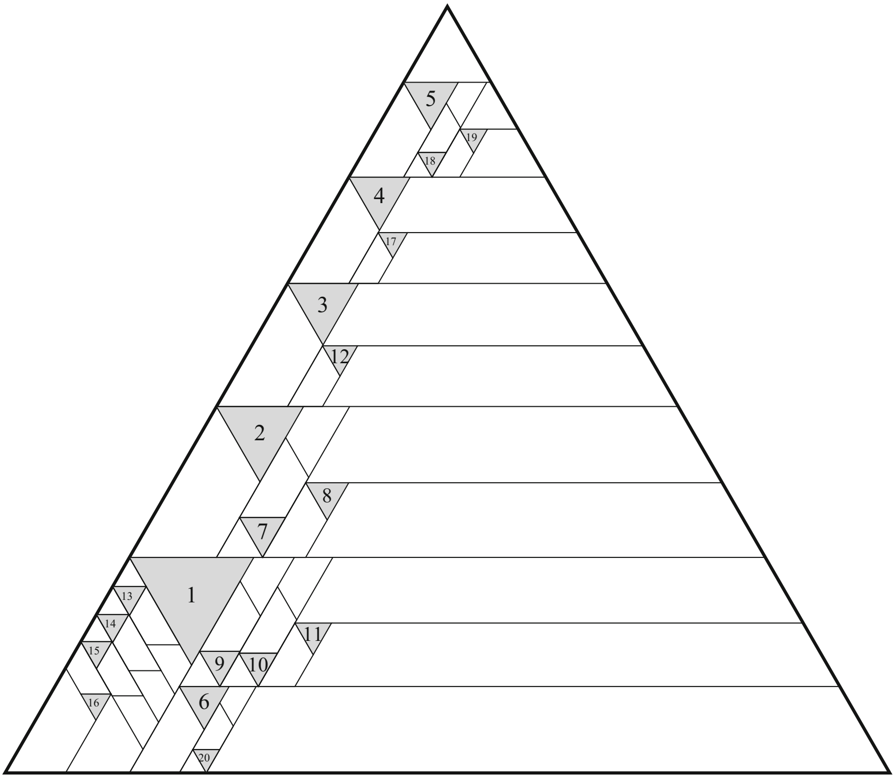

Fig. 10 Packing method for $t = 0.51$ and $\mathbb{A} = \emptyset$

Moreover, P can be divided into: a square of sidelength x with a diagonal parallel to the bases of P and at most five x-classic trapezoids.

Proof Observe that $P(h, a)$ can be divided (see Fig. 11) into a square of sidelength x with a side parallel to bases of $P(h, a)$ and into:

Case 1: two standard trapezoids $P(x, x)$ and $P(x, (3\sqrt{2}/2 - 1)x)$ and two big trapezoids: $P(3\sqrt{2}x/2, a - x)$ and $P(h - 3\sqrt{2}x/2, a - 3\sqrt{2}x/2)$, provided that $h \geq 3\sqrt{2}x$;

Case 2: two standard trapezoids $P(x, h/2)$ and one big trapezoid $P(h, a - x)$, provided that $h < 3\sqrt{2}x$ and $a - x \geq h$; clearly $h/2 < 3\sqrt{2}x/2$;

Case 3: four standard trapezoids: $P(h - x, a - x)$, $P(x, x)$ and two trapezoids $P(x, (a - x)/2)$, provided that $a - x < h < 3\sqrt{2}x$ and $a > 3x$; clearly $(a - x)/2 < 3\sqrt{2}x/2$ as well as $a - x < h = 2(h - x) - h + 2x \leq 2(h - x) - 3\sqrt{2}x/2 + 2x < 2(h - x)$;

Case 4: two standard trapezoids: $P(h - x, a - x)$ and $P(x, a - x)$, provided that $a - x < h < 3\sqrt{2}x$ and $a \leq 3x$; clearly $a - x \leq 3x - x = 2x$ as well as $a - x < h < 2(h - x)$.

Moreover, $P(h, a)$ can be divided (see Fig. 12) into a square of sidelength x with a diagonal parallel to bases of $P(h, a)$ and into:

Case 5: two standard trapezoids $P(x,x)$ and $P(x,2x)$ and two big trapezoids: $P(3\sqrt{2}x/2, a-\sqrt{2}x)$ and $P(h-3\sqrt{2}x/2, a-3\sqrt{2}x/2)$, provided that $h \geq 3\sqrt{2}x$;

Case 6: five standard trapezoids: $P(z,\sqrt{2}x)$, $P(h-z,h-z)$, where $z=\sqrt{2}x-(a-h)$, $P(\sqrt{2}x,\sqrt{2}x)$ and two trapezoids $P(x,(h-z)\sqrt{2}/2-x/2)$, provided that $h<3\sqrt{2}x$ and $z>a-h$; since $z+a-h=\sqrt{2}x$ and $z>a-h$, it follows that $z>\sqrt{2}x/2$; clearly, $\frac{\sqrt{2}}{2}(h-z)-\frac{1}{2}x<\frac{\sqrt{2}}{2}\Big(3\sqrt{2}x-\frac{\sqrt{2}}{2}x\Big)-\frac{1}{2}x=2x$;

Case 7: five standard trapezoids: $P(h-z,h-z)$, $P(a-h,\sqrt{2}x)$, $P(\sqrt{2}x,\sqrt{2}x)$ and two trapezoids $P(x,h\sqrt{2}/2-x)$, provided that $5\sqrt{2}x/2\leq h<3\sqrt{2}x$ and $z\leq a-h<\sqrt{2}x$; clearly, $h\frac{\sqrt{2}}{2}-x<3\sqrt{2}x\cdot\frac{\sqrt{2}}{2}-x=2x$ as well as $\sqrt{2}x=z+a-h\leq a-h+a-h=2(a-h)$;

Case 8: four standard trapezoids: $P(h-z,h-z)$, $P(a-h,\sqrt{2}x)$ and two trapezoids $P(x,h\sqrt{2}/2-x/2)$, provided that $h<5\sqrt{2}x/2$ and $z\leq a-h<\sqrt{2}x$; clearly, $h\sqrt{2}/2-x/2<(5\sqrt{2}x/2)\cdot(\sqrt{2}/2)-x/2=2x$ as well as $\sqrt{2}x=z+a-h\leq a-h+a-h=2(a-h)$;

Case 9: one big trapezoid $P(h,a-\sqrt{2}x)$ and four standard trapezoids $P(x,(h\sqrt{2}+x)/4)$, provided that $h<3\sqrt{2}x$ and $a-h\geq\sqrt{2}x$; clearly, $(h\sqrt{2}+x)/4<(3\sqrt{2}x\cdot\sqrt{2}+x)/4<\frac{3\sqrt{2}}{2}x$. □

Let $\mathbb{N}=\mathbb{A}\cup\mathbb{B}$, where $\mathbb{A}\cap\mathbb{B}=\emptyset$. Since $t\leq(154+3\sqrt{2})/306$, it follows that the sidelength of S_t is greater or equal to $\sqrt{\zeta((154+3\sqrt{2})/153)}>5.4$.

4.1 Packing Method $M_\square$

[1] The square S_t is partitioned into two right isosceles triangles: A_1 containing the lower left vertex of S_t and B_1 containing the upper right vertex of S_t. We choose one of them, say A_1, and pack the first square into it in the following way. If $1\in\mathbb{A}$, then the first square is packed in the place marked in Fig. 11 (the upper left picture). If $1\in\mathbb{B}$, then the first square is packed in the place marked in Fig. 12 (the upper left picture). After packing S_1^t, the uncovered part of A_1 is divided into four 1-classic trapezoids as in Figs. 11 and 12. We take as $\mathcal{P}_1$ the union of the family of these four trapezoids and $\{B_1\}$.

[2] Assume that $n>1$, that the squares $S_1^t,\dots,S_{n-1}^t$ are packed into S_t and that the family $\mathcal{S}_{n-1}$ is defined. We choose one of the n^{-t}-big trapezoids from $\mathcal{P}_{n-1}$ in any way. Denote this trapezoid by P. The square S_n^t is packed into P in the place marked in Figs. 11 and 12. After packing S_n^t, the uncovered part of P is divided into at most five trapezoids. We take as $\mathcal{P}_n$ the union of the family of these trapezoids and $\mathcal{P}_{n-1}\setminus\{P\}$.

Figure 13 illustrates the initial stage of the square-packing process when $\mathbb{A}$ is the set of even numbers and $\mathbb{B}$ is the set of odd numbers.

Theorem 7 *For each t in the range $1/2<t\leq(154+3\sqrt{2})/306\approx0.517$, the squares S_n^t can be packed perfectly into the square S_t so that a side of each packed*

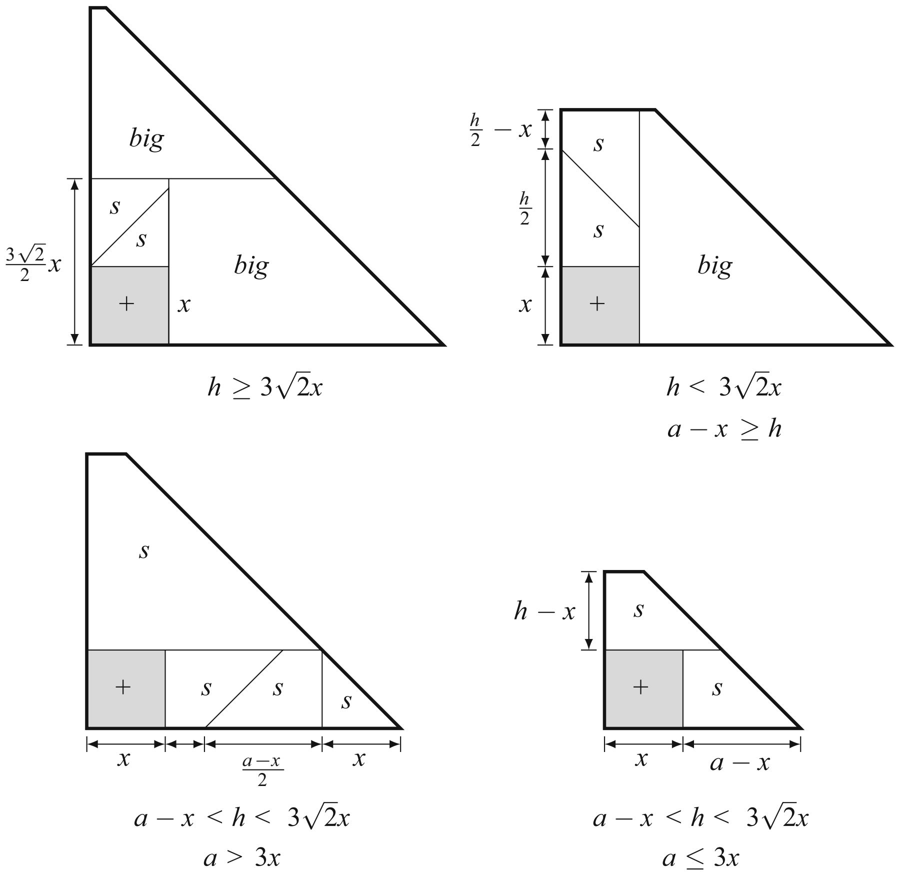

Fig. 11 Divisions of the x-big classic trapezoid $P(h, a)$, when a side of the square is parallel to the bases of $P(h, a)$

square S_i is parallel to a side of S_t for $i \in \mathbb{A}$ while a side of each packed square S_i is parallel to a diagonal of S_t for $i \in \mathbb{B}$.

Proof Let t be a fixed number from the interval $(1/2, (154 + 3\sqrt{2})/306]$. We place $S_1^t, S_2^t, \ldots$ by the method $M_\square$. The sum of areas of trapezoids in $\mathcal{P}_{n-1}$ is greater than

$$\int_n^{+\infty} \frac{1}{x^{2t}} \mathrm{d}x \geq \frac{1}{2 \cdot \frac{154+3\sqrt{2}}{306} - 1} \cdot n^{1-2t} = (27\sqrt{2} - 9)n^{1-2t}.$$

Assume that there is an integer n such that the square S_n^t cannot be packed into S_t by our method, i.e., that there is no n^{-t}-big trapezoid in $\mathcal{P}_{n-1}$. Since there are at most $4n$ trapezoids in $\mathcal{P}_{n-1}$, it follows that the total area of trapezoids in $\mathcal{P}_{n-1}$ is smaller than $4n \cdot \frac{27\sqrt{2}-9}{4} \cdot n^{-2t} = (27\sqrt{2} - 9)n^{1-2t}$, which is a contradiction. $\square$

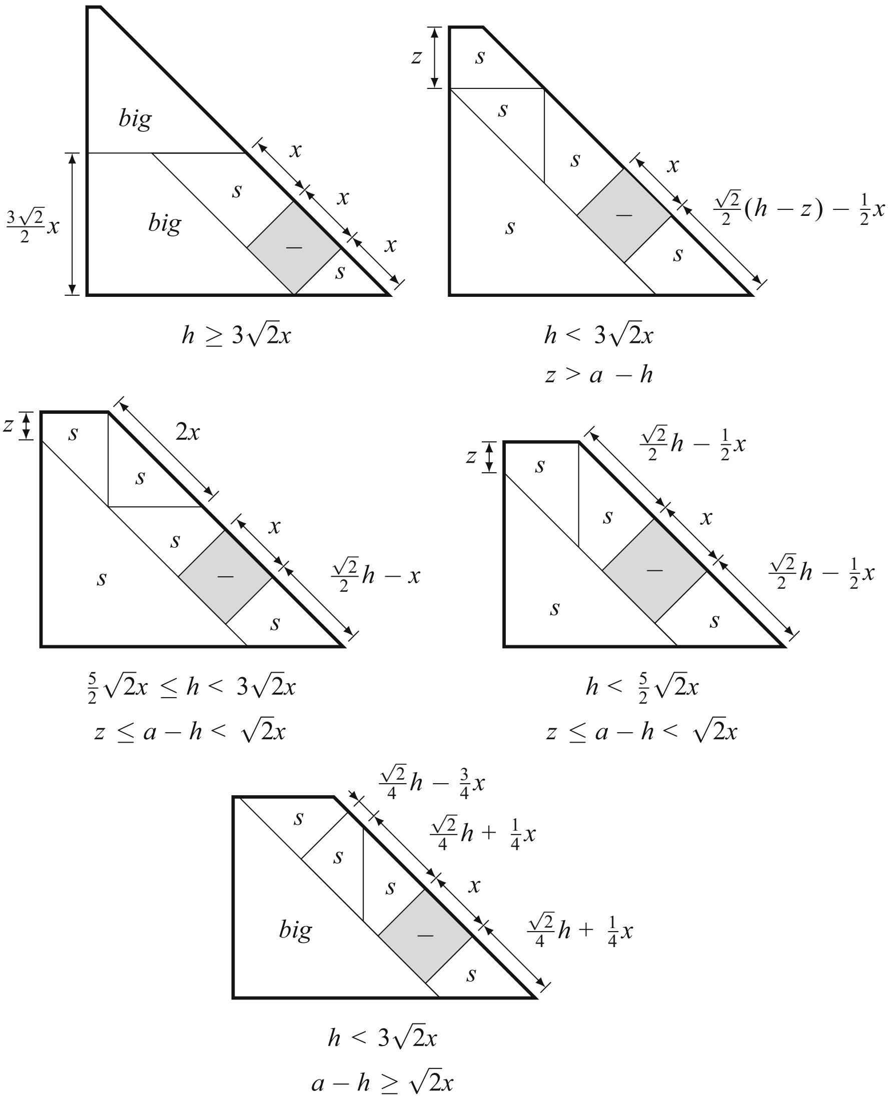

Fig. 12 Divisions of the x-big classic trapezoid $P(h, a)$, when a diagonal of the square is parallel to the bases of $P(h, a)$

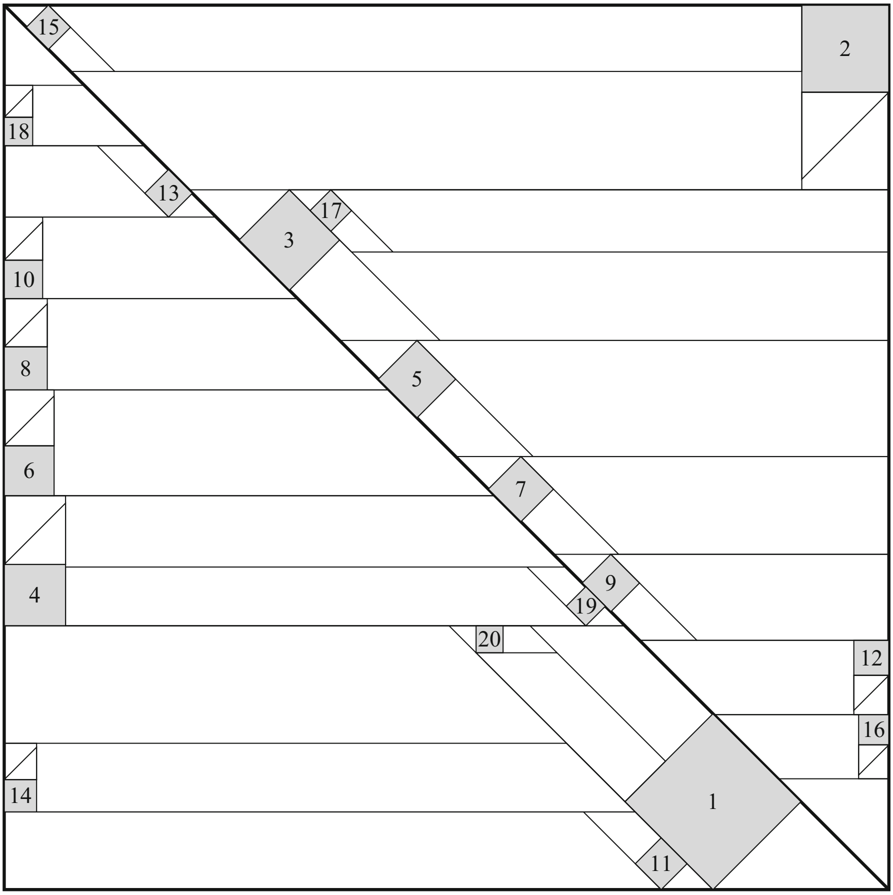

Fig. 13 Square-packing method for $t = 0.51$

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Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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