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Capturing Polytopal Symmetries by Coloring the Edge-Graph

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Abstract

A (convex) polytope $P \subset \mathbb{R}^d$ and its edge-graph G_P can have very distinct symmetry properties, in that the edge-graph can be much more symmetric than the polytope. In this article we ask whether this can be "rectified" by coloring the vertices and edges of G_P, that is, whether we can find such a coloring so that the combinatorial symmetry group of the colored edge-graph is actually isomorphic (in a natural way) to the linear or orthogonal symmetry group of the polytope. As it turns out, such colorings exist and some of them can be constructed quite naturally. However, actually proving that they "capture polytopal symmetries" involves applying rather unexpected techniques from the intersection of convex geometry and spectral graph theory.

Keywords Convex polytopes · Linear symmetries · Orthogonal symmetries · Edge-graph · Graph coloring · Graph symmetries

Mathematics Subject Classification 51M20 · 52B05 · 52B11 · 52B15 · 05C50

1 Introduction

In the context of this article, a *polytope* $P \subset \mathbb{R}^d$ will always be a *convex* polytope, that is, P is the convex hull of finitely many points. A *geometric symmetry* of P is a transformation of the ambient space from a certain pre-chosen base group that fixes the polytope set-wise. Various base groups can be considered, such as Euclidean, affine

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or projective transformations. For this article we focus on

$$\mathrm{Aut}_{\mathrm{GL}}(P) := \{T \in \mathrm{GL}(\mathbb{R}^d) \mid TP = P\} \quad \text{and}$$
$$\mathrm{Aut}_{\mathrm{O}}(P) := \{T \in \mathrm{O}(\mathbb{R}^d) \mid TP = P\},$$

the *linear* and *orthogonal symmetry group* of P, for which each symmetry also fixes the origin. This is the most convenient choice for our setting, but the reader might as well assume the polytopes to be suitably translated to have maximal symmetry.

While initially defined geometrically, one can ask whether it is possible to capture these symmetry groups combinatorially. This could mean to identify a purely combinatorial object $\mathcal{C}$ whose combinatorial symmetry group $\mathrm{Aut}(\mathcal{C})$ is isomorphic to $\mathrm{Aut}_{\mathrm{GL}}(P)$ resp. $\mathrm{Aut}_{\mathrm{O}}(P)$ in a natural way. For example, consider the edge-graph G_P of the polytope. Every, say, linear symmetry $T \in \mathrm{Aut}_{\mathrm{GL}}(P)$ induces a distinct combinatorial symmetry $\sigma_T \in \mathrm{Aut}(G_P)$ of the edge-graph (see Fig. 1). In other words, the edge-graph is at least as symmetric as the polytope. Usually however, it is strictly more symmetric and is therefore unsuited for "capturing the polytope's symmetries" in our sense.

In this article we ask whether this can be fixed by coloring the vertices and edges of the edge-graph, thereby encoding further geometric information, and hopefully creating a combinatorial object that is exactly as symmetric as P (see Fig. 2). As we shall see, this is indeed possible, but proving this requires us to involve rather unexpected techniques from the intersection of convex geometry and spectral graph theory.

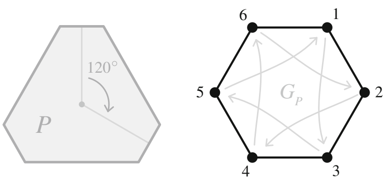

Fig. 1 The clockwise 120°-rotational symmetry of the hexagon permutes its vertices. This permutation corresponds to a combinatorial symmetry $\sigma = (135)(246)$ of the edge-graph. Not every combinatorial symmetry of G_P comes from such a geometric symmetry, e.g. $(123456) \in \mathrm{Aut}(G_P)$. The polygon is therefore strictly less symmetric than its edge-graph

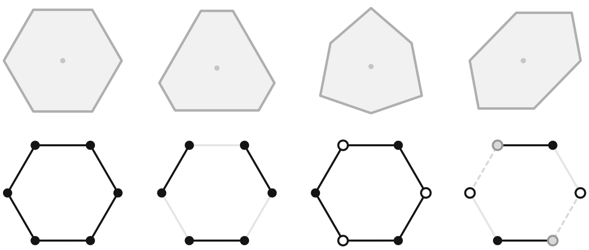

Fig. 2 Various hexagons, and to each a coloring of its edge-graph that gives it "the same symmetries" as the polygon

We believe that this finding is surprising for at least two reasons. First, it is established wisdom that the edge-graph of a general polytope in dimension $d \geq 4$ carries only little information about the full combinatorics or even the dimension of the polytope, even for polytopes with "relatively few" edges (such as hypercubes [8]). Thus, whether the geometric symmetries of P can be captured by coloring only the edges and vertices of P (instead of, say, also higher dimensional faces) should be at least controversial. Second, the analogous statement turns out to be wrong in more general geometric settings (e.g. for graph embeddings, see Example 2.6). In fact, our proof for the existence of these colorings is based on a construction by Ivan Izmestiev [7], which relies heavily on the *convexity* of P. Because of this, it is unclear whether our result generalizes to even some form of non-convex polytopes or polytopal complexes.

Our investigation is in part motivated from a result by Bremner et al. [2, 3]: given a polytope $P \subset \mathbb{R}^d$ with n vertices, the authors construct a coloring of the complete graph K_n, so that the symmetry group of the colored graph is isomorphic to $\mathrm{Aut}_{\mathrm{GL}}(P)$ (see Sect. 2.1). We can interpret this as follows: if we are allowed to color not only the vertices and edges of P, but also other pairs of vertices without a direct counterpart in the polytope's combinatorics, then "capturing the polytope's symmetries" is indeed possible. The major result of our article is then that coloring these "non-geometric edges" is not actually necessary.

Related questions for 3-dimensional polytopes have also been studied by Morozov [11]. In his article he considered *edge-length preserving symmetries* of the edge-graph. In our setting, this corresponds to preserving a coloring of G_P in which each edge is colored according to its length in P. Morozov shows that if each face of $P \subset \mathbb{R}^3$ is inscribed in a circle (e.g. if P is itself inscribed in a sphere, which means that all vertices of P lie on this sphere), then this coloring indeed captures the polytope's orthogonal symmetries.

We reiterate this introduction in a more formal manner.

1.1 Notation and Setting

Throughout the text we let $P \subset \mathbb{R}^d$ denote a convex polytope that is full-dimensional (i.e., not contained in any proper affine subspace of $\mathbb{R}^d$) and contains the origin in its interior (i.e., $0 \in \mathrm{int}(P)$). By $\mathcal{F}_\delta(P)$ we denote the set of δ-dimensional faces of P. We assume a fixed enumeration $v_1, \dots, v_n \in \mathcal{F}_0(P)$ of the polytope's vertices. In particular, n will always denote the number of the vertices.

The edge-graph of P is the finite simple graph $G_P = (V, E)$ with vertex set $V = \{1, \dots, n\}$ and edge set $E \subseteq \binom{V}{2}$. We implicitly assume that $i \in V$ corresponds to the vertex $v_i \in \mathcal{F}_0(P)$, and that $ij \in E$ (short for $\{i, j\} \in E$) if and only if $\mathrm{conv}\{v_i, v_j\} \in \mathcal{F}_1(P)$. The (combinatorial) symmetry group of G_P is defined as[1]

$$\mathrm{Aut}(G_P) = \{\sigma \in \mathrm{Sym}(V) \mid ij \in E \Leftrightarrow \sigma(i)\sigma(j) \in E\} \subseteq \mathrm{Sym}(V),$$

where $\mathrm{Sym}(V)$ denotes the *symmetric group*, i.e., the group of permutations of the vertex set V.

[1] For convenience, notions like the symmetry group, colorings, the adjacency matrix, etc. are only introduced for the edge-graph, but it is understood that they apply to more general graphs as well.

A *coloring* of G_P is a map $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ that assign colors to both vertices and edges, where $\mathfrak{C}$ denotes an arbitrary set of colors. The pair $(G_P, \mathfrak{c})$ is then a *colored edge-graph* and will be abbreviated by $G_P^{\mathfrak{c}}$. Its combinatorial symmetry group is

$$\mathrm{Aut}(G_P^{\mathfrak{c}}) := \left\{\sigma \in \mathrm{Aut}(G_P) \;\middle|\; \begin{array}{ll} \mathfrak{c}(i) = \mathfrak{c}(\sigma(i)) & \text{for all } i \in V \\ \mathfrak{c}(ij) = \mathfrak{c}(\sigma(i)\sigma(j)) & \text{for all } ij \in E \end{array}\right\}.$$

If $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we also say that σ *preserves* the coloring $\mathfrak{c}$. The *colored adjacency matrix* of $G_P^{\mathfrak{c}}$ is the matrix $A^{\mathfrak{c}} \in (\mathfrak{C} \cup \{0\})^{n\times n}$ with entries

$$A_{ij}^{\mathfrak{c}} := \begin{cases} \mathfrak{c}(i) & \text{if } i = j, \\ \mathfrak{c}(ij) & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We explicitly allow for $0 \in \mathfrak{C}$, which causes no problems as we never try to read the graph structure from the colored adjacency matrix. Given the graph structure, the coloring however is uniquely determined by $A^{\mathfrak{c}}$.

A geometric symmetry $T \in \mathrm{Aut}_{\mathrm{GL}}(P)$ maps vertices of P onto vertices of P and therefore induces a permutation $\sigma_T \in \mathrm{Sym}(V)$ on the vertex set, that is,

$$T v_i = v_{\sigma_T(i)}, \qquad \text{for all } i \in V.$$

Since T also maps edges of P onto edges of P, we have $\sigma_T \in \mathrm{Aut}(G_P)$. Recall that P is full-dimensional, hence σ_T determines T uniquely, and we can think of $\mathrm{Aut}_{\mathrm{GL}}(P)$ as a subgroup of $\mathrm{Aut}(G_P)$. For convenience we shall simply write $\sigma \in \mathrm{Aut}_{\mathrm{GL}}(P)$ if $\sigma = \sigma_T$ for some $T \in \mathrm{Aut}_{\mathrm{GL}}(P)$.

The inclusion $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P)$ can be phrased as "the edge-graph G_P is at least as symmetric as P". In general however G_P is strictly more symmetric, i.e., $\mathrm{Aut}_{\mathrm{GL}}(P) \subsetneq \mathrm{Aut}(G_P)$. Our hope is to find a suitable coloring $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ that "destroys the additional symmetries", so that we have $\mathrm{Aut}_{\mathrm{GL}}(P) \cong \mathrm{Aut}(G_P^{\mathfrak{c}})$, where the isomorphism is the inclusion. We say that $\mathfrak{c}$ *captures* the linear symmetries of P. To establish $\mathrm{Aut}_{\mathrm{GL}}(P) \cong \mathrm{Aut}(G_P^{\mathfrak{c}})$, two inclusions need to be checked:

- $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}})$: while always true for an uncolored graph, a coloring might break this inclusion. We must make sure that the colored edge-graph is still *at least* as symmetric as the polytope.
- $\mathrm{Aut}_{\mathrm{GL}}(P) \supseteq \mathrm{Aut}(G_P^{\mathfrak{c}})$: for each $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$ we need to construct a linear symmetry $T_\sigma \in \mathrm{Aut}_{\mathrm{GL}}(P)$ with $T_\sigma v_i = v_{\sigma(i)}$ for all $i \in V$.

The discussion also applies verbatim to the orthogonal symmetry group $\mathrm{Aut}_{\mathrm{O}}(P)$. In particular, we shall write $\mathrm{Aut}_{\mathrm{O}}(P) \subseteq \mathrm{Aut}(G_P)$, and $\sigma \in \mathrm{Aut}_{\mathrm{O}}(P)$ if $\sigma = \sigma_T$ for some $T \in \mathrm{Aut}_{\mathrm{O}}(P)$. We say that a coloring $\mathfrak{c}$ *captures* the orthogonal symmetries if $\mathrm{Aut}_{\mathrm{O}}(P) \cong \mathrm{Aut}(G_P)$ via inclusion. The main results of this article are explicit constructions for

- colorings that capture linear symmetries (Theorem 4.7),
- colorings that capture orthogonal symmetries (Theorem 5.2).

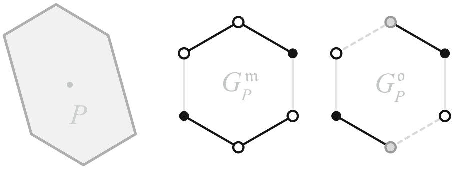

Fig. 3 A hexagon and its edge-graph colored with the metric coloring (middle, Sect. 2.1) resp. the orbit coloring (right, Sect. 2.2)

1.2 Overview

In Sect. 2 we introduce the *metric coloring* and the *orbit coloring*, two very natural candidates for capturing certain polytopal symmetries. In this section we do not show that either coloring capture linear or orthogonal symmetries, but we establish relevant properties used in the upcoming sections. In Sect. 3 we derive a sufficient condition for a coloring of the form $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathbb{R}$ (the colors are real numbers) to capture linear symmetries. The criterion will be in terms of the eigenspaces of the (colored) adjacency matrix of the edge-graph. We shall call this the "linear algebra criterion". In Sect. 4 we introduce the *Izmestiev coloring* (based on a construction by Ivan Izmestiev [7]) and we show that it satisfies the "linear algebra criterion" from Sect. 3. We thereby establish the existence of a first coloring that captures linear symmetries (Theorem 4.7). As a corollary we find that the orbit coloring captures linear symmetries as well (Theorem 4.8). In Sect. 5 we show that a combination of the Izmestiev coloring and the metric coloring captures orthogonal symmetries (Theorem 5.2).

2 Two Useful Colorings

This section is preliminary, in that we introduce two natural colorings of the edge-graph, the *metric coloring* and the *orbit coloring*, without establishing either coloring as capturing polytopal symmetries. In fact, this is an open question for the metric coloring (see Question 6.6). The orbit coloring captures polytopal symmetries, but we are not able to show this right away. Both colorings will play a role in the upcoming sections. Figure 3 shows a polygon and its edge-graph with either coloring applied.

2.1 The Metric Coloring

It is a well-known folklore result that a combinatorial symmetry $\sigma \in \mathrm{Aut}(G_P)$ corresponds to a Euclidean symmetry of the polytope if and only if $\|v_i - v_j\| = \|v_{\sigma(i)} - v_{\sigma(j)}\|$ for all distinct $i, j \in V$. If in addition $\|v_i\| = \|v_{\sigma(i)}\|$ for all $i \in V$, then also $\sigma \in \mathrm{Aut}_{\mathrm{O}}(P)$. This can be stated using colorings:

Theorem 2.1 *Given a polytope $P \subset \mathbb{R}^d$ with vertex set $\mathcal{F}_0(P) = \{v_1, \ldots, v_n\}$. Consider the coloring $\mathfrak{c}$ on the complete graph K_n with*

$$\begin{aligned} \mathfrak{c}(i) &:= \|v_i\|^2, && \textit{for all } i \in \{1, \ldots, n\}, \\ \mathfrak{c}(ij) &:= \langle v_i, v_j \rangle, && \textit{for all distinct } i, j \in \{1, \ldots, n\}. \end{aligned}$$

Then $\mathrm{Aut}(K_n^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.[2]

For convenience reasons (see below) we used the inner product $\langle v_i, v_j \rangle$ in place of $\|v_i - v_j\|$ to color the edges, which is easily seen to give an equivalent result.

The strength of this result lies in its immediate applicability: constructing this "complete metric coloring" requires no knowledge of the edge-graph (which is usually hard to come by), but only the vertex coordinates of P.[3] In practice this is probably one of the best tools for explicitly computing the orthogonal symmetries of a polytope (or, in fact, of a general point configuration).

From a theoretical and aesthetic perspective however, this construction has the shortcoming of containing massively redundant data and stepping outside the combinatorial structure of the polytope (we assign color to vertex pairs that are not edges of the polytope). Naturally, we ask whether one can get away with coloring fewer of these "non-edges", ideally only the actual edges of the edge-graph. Based on this hope we define the following:

Definition 2.2 The *metric coloring* of G_P is the coloring $\mathfrak{m}\colon V \cup E \to \mathbb{R}$ with

$$\begin{aligned} \mathfrak{m}(i) &:= \|v_i\|^2, && \text{for all } i \in V, \\ \mathfrak{m}(ij) &:= \langle v_i, v_j \rangle, && \text{for all } ij \in E. \end{aligned}$$

Whether the metric coloring captures orthogonal symmetries is an open question (see also Question 6.6). Our reason for introducing it anyway is that in Sect. 5 the metric coloring will be one ingredient to a coloring that indeed captures orthogonal symmetries.

We close by mentioning that *linear* symmetries can be captured by quite similar ideas. Note first that the complete metric coloring of K_n in Theorem 2.1 can also be given via its colored adjacency matrix $A^{\mathfrak{c}} = \Phi^\top \Phi$, where $\Phi := (v_1, \ldots, v_n) \in \mathbb{R}^{d \times n}$ is the matrix the columns of which are the vertex coordinates of P. It was shown in [2, Thm. 2] that using $A^{\mathfrak{c}} = \Phi^\dagger \Phi$ instead captures the linear symmetries, where $\Phi^\dagger \in \mathbb{R}^{n \times d}$ is the Moore–Penrose pseudo inverse of Φ.

We recollect the adjacency matrix versions:

Theorem 2.3 *Let $K_n^{\mathfrak{c}}$ be the colored complete graph with colored adjacency matrix $A^{\mathfrak{c}}$.*

(i) *If $A^{\mathfrak{c}} = \Phi^\top \Phi$, then* $\mathrm{Aut}(K_n^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$ (*see Theorem* 2.1).

[2] A coloring whose colors are real numbers is still a *purely combinatorial object*. These numbers are just used for a concise definition and could be replaced by any other finite set of distinguishable values. The only information used from the coloring (in the form of the combinatorial symmetry group of the colored graph) is whether two vertices/edges are of the same or a different color.

[3] If P is given in $\mathcal{H}$-representation, one can apply Theorem 2.1 to compute the orthogonal symmetry group of the dual polytope P°, which is identical to $\mathrm{Aut}_{\mathrm{O}}(P)$ as a matrix group.

(ii) *If* $A^{\mathfrak{c}} = \Phi^{\dagger}\Phi$, *then* $\mathrm{Aut}(K_n^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{GL}}(P)$ (*see* [2, Thm. 2]).

The theory developed in Sect. 3 (Example 3.3) allows us to include a short proof of Theorem 2.3 (ii) (essentially equivalent to the proof given in [4, Cor. 3.2]).

2.2 The Orbit Coloring

A coloring $\mathfrak{c}$ that captures, say, linear symmetries must assign the same color to vertices and edges in the same $\mathrm{Aut}_{\mathrm{GL}}(P)$-orbit, as otherwise $\mathrm{Aut}(G_P^{\mathfrak{c}})$ cannot act transitively on the points in each orbit. This is exactly what guarantees the inclusion $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}})$. The coloring that takes this idea to the extreme is the *orbit coloring*:

Definition 2.4 The (*linear*) *orbit coloring* $\mathfrak{o}$ of G_P assigns the same color to vertices (resp. edges) of G_P if and only if the corresponding vertices (resp. edges) of P are in the same $\mathrm{Aut}_{\mathrm{GL}}(P)$-orbit.

An analogous coloring can be defined for orthogonal symmetries, which we shall call the *orthogonal orbit coloring* of G_P, still denoted by $\mathfrak{o}$. For the sake of conciseness, this section only discusses the (linear) orbit coloring, but all statements carry over to the orthogonal version in the obvious way.

A coloring $\mathfrak{c}$ is said to be *finer* than a coloring $\bar{\mathfrak{c}}$ if

$$\begin{aligned} \mathfrak{c}(i) = \mathfrak{c}(\hat{\imath}) &\implies j\bar{\mathfrak{c}}(i) = \bar{\mathfrak{c}}(\hat{\imath}), && \text{for all } i, \hat{\imath} \in V, \\ \mathfrak{c}(ij) = \mathfrak{c}(\hat{\imath}\hat{\jmath}) &\implies \bar{\mathfrak{c}}(ij) = \bar{\mathfrak{c}}(\hat{\imath}\hat{\jmath}), && \text{for all } ij, \hat{\imath}\hat{\jmath} \in E. \end{aligned}$$

Conversely, $\bar{\mathfrak{c}}$ is said to be *coarser* than $\mathfrak{c}$. So the following three statements emerge as equivalent necessary criteria for a coloring $\mathfrak{c}$ to capture linear symmetries:

- $G_P^{\mathfrak{c}}$ is at least as symmetric as P.
- $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}})$.
- $\mathfrak{c}$ is coarser than the orbit coloring.

As we shall prove in Sect. 4 (Theorem 4.8), the orbit coloring indeed captures linear symmetries and is therefore the *finest* coloring with this property. However, this is surprisingly hard to show directly. Our eventual indirect proof of this fact will make use of the following:

Lemma 2.5 *If there is any coloring that captures linear symmetries, then so does the orbit coloring* $\mathfrak{o}$.

Proof Clearly $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{o}})$. For the converse assume that $\mathfrak{c}$ is a coloring which captures linear symmetries, in particular, $\mathrm{Aut}(G_P^{\mathfrak{c}}) \subseteq \mathrm{Aut}_{\mathrm{GL}}(P)$. Since $\mathfrak{o}$ is finer than $\mathfrak{c}$, and since joining color classes can only increase the amount of combinatorial symmetries, we find

$$\mathrm{Aut}(G_P^{\mathfrak{o}}) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}}) \subseteq \mathrm{Aut}_{\mathrm{GL}}(P).$$

Thus $\mathrm{Aut}_{\mathrm{GL}}(P) \cong \mathrm{Aut}(G_P^{\mathfrak{o}})$. $\square$

Note that for general graph embeddings the orbit coloring does *not* always capture the geometric symmetries of the embedding.

Example 2.6 Consider the complete bipartite graph $K_{4,4}$ with vertex set $V_1 \mathbin{\dot\cup} V_2 = \{1, 2, 3, 4\} \mathbin{\dot\cup} \{5, 6, 7, 8\}$ and an embedding into $\mathbb{R}^4$ defined as follows:

$$\begin{aligned} v_1 &= (+1, 0, 0, 0), & v_5 &= (0, 0, +1, 0),\\ v_2 &= (0, +1, 0, 0), & v_6 &= (0, 0, 0, +1),\\ v_3 &= (-1, 0, 0, 0), & v_7 &= (0, 0, -1, 0),\\ v_4 &= (0, -1, 0, 0), & v_8 &= (0, 0, 0, -1). \end{aligned}$$

One can verify that the linear symmetry group of this embedding acts transitively on the vertices as well as the edges. Thus, a coloring $\mathfrak{c}$ that is at least as symmetric as the graph embedding must assign the same color to all vertices, and like-wise, the same color to all edges. That is, $\mathrm{Aut}(K_{4,4}^{\mathfrak{c}}) = \mathrm{Aut}(K_{4,4})$. However, one can also see that the given embedding has a strictly smaller symmetry group than $\mathrm{Aut}(K_{4,4})$. For example, $\sigma := (12) \in \mathrm{Aut}(K_{4,4})$ cannot be realized as a geometric symmetry.

3 A Linear Algebra Criterion for Capturing Symmetries

For this section fix a coloring $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ with $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}})$. The goal is to derive a sufficient criterion for $\mathfrak{c}$ to ensure the opposite inclusion and thus capturing of linear symmetries. Recall that this amounts to showing that for each combinatorial symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$ we can find a linear symmetry $T_\sigma \in \mathrm{Aut}_{\mathrm{GL}}(P)$ with

$$T_\sigma v_i = v_{\sigma(i)} \quad \text{for all } i \in V. \tag{1}$$

Let us investigate the difficulties in constructing these transformations.

First, note that we can express (1) for all $i \in V$ simultaneously by rewriting it into a single matrix equation as follows:

$$T_\sigma(v_1, \dots, v_n) = (v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (v_1, \dots, v_n)\,\Pi_\sigma,$$

where $\Pi_\sigma \in \mathrm{Perm}(n)$ denotes the corresponding permutation matrix.[4] If we define $\Phi := (v_1, \dots, v_n) \in \mathbb{R}^{d\times n}$ as the matrix in which the polytope's vertices v_i appear as columns, this further compactifies to

$$T_\sigma \Phi = \Phi \Pi_\sigma. \tag{2}$$

This equation will be our benchmark: every ansatz for defining the transformations T_σ must satisfy (2), which is then also sufficient.

[4] We chose to define Π_σ so that on multiplication from *left* it permutes the *rows* as prescribed by σ. We emphasize that this, counter-intuitively, means $(\Pi_\sigma v)_i = v_{\sigma^{-1}(i)}$ for a vector $v \in \mathbb{R}^n$.

Now, if Φ were invertible, we could just solve (2) for T_σ, satisfying (2) "by force". However, $\Phi \in \mathbb{R}^{d\times n}$ is not a square matrix (since P is full-dimensional, we have $n \ge d+1$). Another hope to "solve for T_σ" is to use the *Moore–Penrose pseudo inverse* of Φ: the matrix $\Phi^\dagger \in \mathbb{R}^{n\times d}$ whose columns form a dual basis to the rows of Φ, in particular, $\Phi\Phi^\dagger = \mathrm{Id}_d$. This suggests the following ansatz:

$$T_\sigma := \Phi\Pi_\sigma\Phi^\dagger. \tag{3}$$

When does this ansatz satisfy (2)? We compute

$$T_\sigma\Phi \overset{(3)}{=} \Phi\Pi_\sigma\Phi^\dagger\Phi = \Phi\Pi_\sigma\pi_U, \tag{4}$$

where $\pi_U := \Phi^\dagger\Phi$ is the orthogonal projector onto the subspace $U := \operatorname{span}\Phi^\dagger \subseteq \mathbb{R}^n$. To arrive at (2) we would need to get rid of the projector π_U on the right side of (4). The main result of this section gives a sufficient criterion, the "linear algebra criterion", for when this is possible, expressed in terms of spectral properties of the colored adjacency matrix.

Theorem 3.1 *Let* $\mathfrak{c}\colon V \cup E \to \mathbb{R}$ *be a coloring of* G_P *with* $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}})$. *If* $U := \operatorname{span}\Phi^\dagger$ *is an eigenspace of the colored adjacency matrix* $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$*, then* $\mathfrak{c}$ *captures the linear symmetries of* P.

Proof Fix a combinatorial symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$. Recall the following well-known property of the colored adjacency matrix $A^{\mathfrak{c}}$: if $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, then

$$\Pi_\sigma A^{\mathfrak{c}} = A^{\mathfrak{c}}\Pi_\sigma.$$

Now, if $A^{\mathfrak{c}}$ and Π_σ commute then the eigenspaces of $A^{\mathfrak{c}}$ (including U) are *invariant subspaces* of Π_σ, i.e., $\Pi_\sigma U = U$. Equivalently, Π_σ commutes with the projector π_U. This suffices to show that the map $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger$ satisfies (2):

$$T_\sigma\Phi = \Phi\Pi_\sigma\Phi^\dagger\Phi = \Phi\Pi_\sigma\pi_U = \Phi\pi_U\Pi_\sigma = \Phi(\Phi^\dagger\Phi)\Pi_\sigma = \overbrace{(\Phi\Phi^\dagger)}^{\mathrm{Id}_d}\Phi\Pi_\sigma = \Phi\Pi_\sigma.$$

Hence, the map $\sigma \mapsto T_\sigma$ defines the desired inclusion $\mathrm{Aut}(G_P^{\mathfrak{c}}) \subseteq \mathrm{Aut}_{\mathrm{GL}}(P)$. □

It might not be immediately obvious how Theorem 3.1 is a helpful reformulation of the problem. To apply it we need to construct a matrix $A^{\mathfrak{c}}$ with two very special properties: first, $A^{\mathfrak{c}}$ must be a (colored) adjacency matrix of the edge-graph G_P, that is, it must have non-zero entries only where G_P has edges. Second, we need to ensure that $A^{\mathfrak{c}}$ has U as an eigenspace. It is not even clear that these two conditions are compatible. We come back to this in the next section.

Remark 3.2 While we are mainly interested in the polytopal case, Theorem 3.1 could have been stated in much greater generality: the proof does not use that G_P is the edge-graph of a polytope or that the columns of Φ are in convex position. We address this again in the outlook (see Remark 6.5) and the following example.

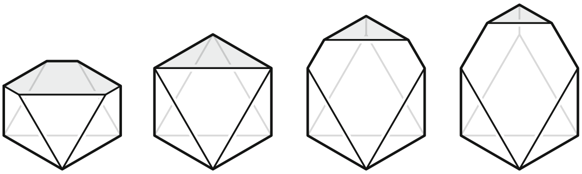

Fig. 4 Several instances of the generalized dual $P^\circ(c)$ of the cube (the usual polar dual of the cube is the regular octahedron; the second from the left). The polytopes differ by a single facet-defining plane being shifted along its normal vector

Example 3.3 Consider the "obvious" matrix $A^{\mathfrak{c}}$ with eigenspace $U := \operatorname{span} \Phi^\dagger$:

$$A^{\mathfrak{c}} := \Phi^\dagger \Phi.$$

This matrix has most likely no zero-entries and is therefore not a colored adjacency matrix of G_P (except if G_P is the complete graph). However, it is exactly the colored adjacency matrix of the complete metric coloring as discussed in Theorem 2.3 (ii). As noted in Remark 3.2, our "linear algebra criterion" can be applied nevertheless: consider the colored complete graph $K_n^{\mathfrak{c}}$ with colored adjacency matrix $A^{\mathfrak{c}}$. Then Theorem 3.1 directly implies Theorem 2.3 (ii) (this short proof is essentially equivalent to the one given in [4, Cor. 3.2]).

4 The Izmestiev Coloring

In this section we introduce a coloring of G_P which satisfies the "linear algebra condition" Theorem 3.1. This coloring is based on a construction by Izmestiev [7] and we shall call it the *Izmestiev coloring*. The coloring is built in a quite unintuitive way. First, we need to recall that for a polytope P with $0 \in \operatorname{int}(P)$ the *polar dual* P° is defined as

$$P^\circ := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \le 1 \text{ for all } i \in V\}.$$

We generalize this notion: for a vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ let

$$P^\circ(c) := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \le c_i \text{ for all } i \in V\}.$$

Then $P^\circ(1, \dots, 1) = P^\circ$ and $P^\circ(c)$ is obtained from P° by shifting facets along their normal vectors (see Fig. 4).

In the following, $\operatorname{vol}(C)$ denotes the *relative volume* (relative to the affine hull of C) of a compact convex set $C \subset \mathbb{R}^d$.

Theorem 4.1 [Izmestiev [7, Thm. 2.4]] *For a polytope $P \subset \mathbb{R}^d$ with $0 \in \operatorname{int}(P)$ consider the matrix $M \in \mathbb{R}^{n \times n}$ (which we shall call the **Izmestiev matrix** of P) with components*

$$M_{ij} := \frac{\partial^2 \operatorname{vol}(P^\circ(c))}{\partial c_i \partial c_j}\Big|_{c=(1,\dots,1)}.$$

(In particular, $\mathrm{vol}(P^\circ(c))$ *is two times continuously differentiable at* c.*)* M *then has the following properties:*

(i) $M_{ij} < 0$ *whenever* $ij \in E$.
(ii) $M_{ij} = 0$ *whenever* $ij \notin E$ *and* $i \neq j$.
(iii) M *has a unique negative eigenvalue of multiplicity one.*
(iv) $M\Phi^\top = 0$, *where* $\Phi = (v_1, \ldots, v_n) \in \mathbb{R}^{d\times n}$ *is the matrix introduced in* (2).
(v) $\dim \ker M = d$.

Remark 4.2 In the words of [7], the matrix M constructed in Theorem 4.1 is a *Colin de Verdière matrix* of the edge-graph, that is, a matrix satisfying a certain list of properties, including (i), (ii), and (iii), and the so-called *strong Arnold property* (for details, see e.g. [6]). Among the Colin de Verdière matrices, one usually cares about the ones with the largest possible kernel. The dimension of this largest kernel is known as the *Colin de Verdière graph invariant* $\mu(G_P)$ [6], and Theorem 4.1 (v) then shows that $\mu(G_P) \geq d$. This is not too surprising and was known before. However, the result of Izmestiev is remarkable for a different reason: it shows that there is a Colin de Verdière matrix whose kernel has dimension *exactly* d (property (v)) and that is compatible with the geometry of P (property (iv)).

Remark 4.3 Izmestiev also shows that the matrix M can be expressed in terms of simple geometric properties of the polytope: for $ij \in E$ let $f_{ij} \in \mathcal{F}_{d-2}(P^\circ)$ be the *dual face* to the edge $\mathrm{conv}\{v_i, v_j\} \in \mathcal{F}_1(P)$. Then

$$M_{ij} = -\frac{\mathrm{vol}(f_{ij})}{\|v_i\| \cdot \|v_j\| \cdot \sin\angle(v_i, v_j)}.$$

Definition 4.4 The *Izmestiev coloring* $\mathfrak{I}\colon V \cup E \to \mathbb{R}$ of G_P is defined by

$$\begin{aligned}\mathfrak{I}(i) &:= M_{ii}, \quad \text{for all } i \in V,\\ \mathfrak{I}(ij) &:= M_{ij}, \quad \text{for all } ij \in E,\end{aligned}$$

where $M \in \mathbb{R}^{n\times n}$ is the Izmestiev matrix of P.

Observation 4.5 *Since* $M_{ij} = 0$ *whenever* $ij \notin E$ *and* $i \neq j$ *(by Theorem* 4.1 (ii)*), the colored adjacency matrix* $A^{\mathfrak{I}}$ *of* $G_P^{\mathfrak{I}}$ *is exactly the Izmestiev matrix* M.

To apply the "linear algebra criterion" Theorem 3.1 we first need to show that $G_P^{\mathfrak{I}}$ is at least as symmetric as P. This is straightforward if we use that the Izmestiev matrix is a linear invariant of P. We include a proof for completeness:

Proposition 4.6 $\mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{I}})$.

Proof Fix a linear symmetry $T \in \mathrm{Aut}_{\mathrm{GL}}(P)$ and let $\sigma_T \in \mathrm{Aut}(G_P)$ be the induced combinatorial symmetry of the edge-graph. We need to show that σ_T preserves the Izmestiev coloring, that is, $\sigma_T \in \mathrm{Aut}(G_P^{\mathfrak{I}})$. This requires two ingredients. For the first one, one checks that the generalized polar dual $P^\circ(c)$ (like the usual polar dual) satisfies

$$(TP)^\circ(c) = T^{-\top}P^\circ(c),$$

which yields

$$\mathrm{vol}\,((TP)^\circ(c)) = |\det(T^{-\top})| \cdot \mathrm{vol}\,(P^\circ(c)) = \mathrm{vol}\,(P^\circ(c)), \tag{5}$$

where we use that $\det(T^{-\top}) = \det(T) = \pm 1$ holds for all linear transformations in a finite matrix group such as $\mathrm{Aut}_{\mathrm{GL}}(P)$. The second ingredient is the following:

$$\begin{aligned}(TP)^\circ(c) &= \{x \in \mathbb{R}^d \mid \langle x, Tv_i\rangle \le c_i \text{ for all } i \in V\}\\ &= \{x \in \mathbb{R}^d \mid \langle x, v_{\sigma_T(i)}\rangle \le c_i \text{ for all } i \in V\}\\ &= \Big\{x \in \mathbb{R}^d \mid \langle x, v_i\rangle \le c_{\sigma_T^{-1}(i)} \text{ for all } i \in V\Big\} = P^\circ(\Pi_{\sigma_T} c).^5\end{aligned} \tag{6}$$

Putting everything together, we can show $\mathfrak{I}(i) = \mathfrak{I}(\sigma_T(i))$ for all $i \in V$, and equivalently for edges. We show both at the same time by proving $M_{ij} = M_{\sigma_T(i)\sigma_T(j)}$ for all $i, j \in \{1, \dots, n\}$:

$$\begin{aligned}M_{ij} = \frac{\partial^2 \mathrm{vol}\,(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=c_0} &= \frac{\partial^2 \mathrm{vol}\,(P^\circ(\Pi_\sigma c))}{\partial c_{\sigma_T(i)}\, \partial c_{\sigma_T(j)}}\Big|_{c=c_0} \overset{(6)}{=} \frac{\partial^2 \mathrm{vol}\,((TP)^\circ(c))}{\partial c_{\sigma_T(i)}\, \partial c_{\sigma_T(j)}}\Big|_{c=c_0}\\ &\overset{(5)}{=} \frac{\partial^2 \mathrm{vol}\,(P^\circ(c))}{\partial c_{\sigma_T(i)}\, \partial c_{\sigma_T(j)}}\Big|_{c=c_0} = M_{\sigma_T(i)\sigma_T(j)},\end{aligned}$$

where we set $c_0 := (1, \dots, 1) \in \mathbb{R}^n$. □

Theorem 4.7 *The Izmestiev coloring captures the linear symmetries of P.*

Proof By Theorem 4.6, the Izmestiev coloring $\mathfrak{I}$ is at least as symmetric as P, and so we can try to apply the "linear algebra criterion" (Theorem 3.1) to show that $\mathfrak{I}$ captures linear symmetries. That is, we need to show that $U := \mathrm{span}\,\Phi^\dagger$ is an eigenspace of the colored adjacency matrix $A^{\mathfrak{I}}$ of $G_P^{\mathfrak{I}}$. Recall that $A^{\mathfrak{I}}$ is exactly the Izmestiev matrix (Theorem 4.5), and so we can try to use the various properties of this matrix established in Theorem 4.1.

First, $U = \mathrm{span}\,\Phi^\dagger = \mathrm{span}\,\Phi^\top$ (since the columns of $\Phi^\top$ and $\Phi^\dagger$ are dual bases of U), and so Theorem 4.1 (iv) can be read as $U \subseteq \ker A^{\mathfrak{I}}$. Second, we have both $\dim U = \mathrm{rank}\,\Phi = d$ (since P is full-dimensional) and $\dim\ker A^{\mathfrak{I}} = d$ (by Theorem 4.1 (v)). Comparing dimensions, we thus have $U = \ker A^{\mathfrak{I}}$. We conclude that U is an eigenspace of $A^{\mathfrak{I}}$ (namely, the eigenspace to eigenvalue 0). The "linear algebra criterion" Theorem 3.1 then asserts that $\mathfrak{I}$ captures the linear symmetries of P. □

[5] Recall that Π_σ was defined so that $(\Pi_\sigma v)_i = v_{\sigma^{-1}(i)}$ for a vector $v \in \mathbb{R}^n$.

By Theorem 2.5, if there is any coloring that captures linear symmetries, then the orbit coloring does so as well:

Corollary 4.8 *The orbit coloring captures the linear symmetries of* P.

Remark 4.9 While the orbit coloring was quickly established as the finest coloring that captures linear symmetries, determining a coarsest coloring with this property (which might not be unique) seems like a challenging task. The Izmestiev coloring is in general neither the finest nor a coarsest coloring of this kind.

5 Capturing Orthogonal Symmetries

For this section we consider the orthogonal symmetry group $\mathrm{Aut}_{\mathrm{O}}(P)$ and all notations without an explicit hint to the kind of symmetry (such as the orbit coloring $\mathfrak{o}$) implicitly refer to their orthogonal versions.

Recall the metric coloring $\mathfrak{m}\colon V \mathbin{\dot\cup} E \to \mathbb{R}$ (Definition 2.2) with

$$\begin{aligned} \mathfrak{m}(i) &= \|v_i\|^2, && \text{for all } i \in V,\\ \mathfrak{m}(ij) &= \langle v_i, v_j\rangle, && \text{for all } ij \in E. \end{aligned}$$

The inclusion $\mathrm{Aut}_{\mathrm{O}}(P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{m}})$ is easy to see. And while we consider $\mathfrak{m}$ a promising candidate for actually capturing orthogonal symmetries, we are yet unable to prove the inclusion in the other direction (see Question 6.6). Nevertheless, we can show that combining the metric coloring with a coloring that captures linear symmetries suffices to capture orthogonal symmetries.

Definition 5.1 Given two colorings $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ and $\bar{\mathfrak{c}}\colon V \mathbin{\dot\cup} E \to \overline{\mathfrak{C}}$, the *product coloring* $\mathfrak{c}\times\bar{\mathfrak{c}}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}\times\overline{\mathfrak{C}}$ is defined by

$$\begin{aligned} (\mathfrak{c}\times\bar{\mathfrak{c}})(i) &:= (\mathfrak{c}(i), \bar{\mathfrak{c}}(i)), && \text{for all } i \in V,\\ (\mathfrak{c}\times\bar{\mathfrak{c}})(ij) &:= (\mathfrak{c}(ij), \bar{\mathfrak{c}}(ij)), && \text{for all } ij \in E. \end{aligned}$$

The essential (and straightforward to verify) property of the product coloring is

$$\mathrm{Aut}(G_P^{\mathfrak{c}\times\bar{\mathfrak{c}}}) = \mathrm{Aut}(G_P^{\mathfrak{c}}) \cap \mathrm{Aut}(G_P^{\bar{\mathfrak{c}}}), \tag{7}$$

from which we immediately see that if both $G_P^{\mathfrak{c}}$ and $G_P^{\bar{\mathfrak{c}}}$ are at least as symmetric as P, then the same holds for $G_P^{\mathfrak{c}\times\bar{\mathfrak{c}}}$.

Theorem 5.2 *If a coloring* $\mathfrak{c}$ *captures linear symmetries, then the product coloring* $\mathfrak{c}\times\mathfrak{m}$ *captures orthogonal symmetries.*

Proof The inclusion $\mathrm{Aut}_{\mathrm{O}}(G_P) \subseteq \mathrm{Aut}(G_P^{\mathfrak{c}\times\mathfrak{m}})$ follows immediately from (7) and the fact that respective inclusions hold for $\mathfrak{c}$ and $\mathfrak{m}$. It remains to verify the inclusion in the other direction. For that, fix $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}\times\mathfrak{m}})$. Since $\mathfrak{c}$ captures linear symmetries, and by (7) we have $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}}) \subseteq \mathrm{Aut}_{\mathrm{GL}}(P)$, that is, σ corresponds to a linear symmetry $T_\sigma \in \mathrm{Aut}_{\mathrm{GL}}(P)$. It remains to show $T_\sigma \in \mathrm{O}(\mathbb{R}^d)$.

It suffices to verify $\langle T_\sigma v_i, T_\sigma v_j\rangle = \langle v_i, v_j\rangle$ for every two $i, j \in V$ (since P is full-dimensional and therefore its vertices contain a basis). Thus, let us fix $i, j \in V$ and define $B := \{k \in V \mid i = k \text{ or } ik \in E\}$. Note that $\{v_k \mid k \in B\}$ already contains a basis of $\mathbb{R}^d$ and so v_j can be written as

$$v_j = \sum_{k \in B} \alpha_k v_k$$

for some coefficients $\alpha_k \in \mathbb{R}$. Then

$$\begin{aligned}\langle T_\sigma v_i, T_\sigma v_j\rangle &= \sum_{k\in B} \alpha_k \langle T_\sigma v_i, T_\sigma v_k\rangle \\ &= \sum_{k\in B} \alpha_k \langle v_{\sigma(i)}, v_{\sigma(k)}\rangle \overset{(*)}{=} \sum_{k\in B} \alpha_k \langle v_i, v_k\rangle = \langle v_i, v_j\rangle\end{aligned}$$

follows, where in $(*)$ we used that for $k \in B$ either $i = k$ or $ik \in E$, and that σ preserves the metric coloring. Thus, T_σ is orthogonal. □

By (the orthogonal version of) Theorem 2.5, if there is any coloring that captures orthogonal symmetries, then so does the orthogonal orbit coloring (and it is the finest coloring with this property).

Corollary 5.3 *The orthogonal orbit coloring captures orthogonal symmetries.*

Remark 5.4 Similarly to the linear case we can give an explicit formula to translate a combinatorial symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, where $\mathfrak{c}$ captures orthogonal symmetries, into an actual orthogonal symmetry:

$$T_\sigma := \Phi \Pi_\sigma \Phi^\top,$$

where $\Phi := (v_1, \ldots, v_n) \in \mathbb{R}^{d\times n}$ is the matrix with the polytope's vertices as columns, and Π_σ is the permutation matrix to σ.

6 Outlook, Open Questions, and Further Notes

In this article we have shown that the edge-graph of a convex polytope, while generally a very weak representative of the polytope's geometric nature, still has sufficient structure to let us encode two important types of geometric symmetries: linear and orthogonal symmetries. We achieved this by coloring the vertices and edges of the edge-graph.

The first coloring for which we established that it "captures the polytope's linear symmetries" was the *Izmestiev coloring* (Theorem 4.7), based on an ingenious construction by Izmestiev. But we also found that the *orbit coloring*, a conceptually very easy coloring, does the job as well (Theorem 4.8). Analogous colorings exist for the orthogonal symmetries as well (Theorems 5.2 and 5.3).

In the following we briefly discuss various potential generalizations and follow up questions concerning these results. This further highlights the very special structure of convex polytopes that went into our theorems, emphasizing again that these results are non-trivial to achieve and to generalize. We also want to mention the following neat consequence for "very symmetric" polytopes:

Corollary 6.1 *If $P \subset \mathbb{R}^d$ is vertex- and edge-transitive (i.e., its linear resp. orthogonal symmetry group has a single orbit on vertices and edges), then P is exactly as symmetric as its edge-graph.*

This observation has previously been made in [5, Thm. 5.2]. No classification of simultaneously vertex- and edge-transitive polytopes is known so far, and so this fact might help in the study of this class.

6.1 Capturing Other Types of Symmetries

Besides linear and orthogonal symmetries, there are at least two further common groups of symmetries associated with a polytope: the *projective symmetries* and the *combinatorial symmetries* (that is, the symmetries of the face lattice). We can ask whether those too can be captured by a colored edge-graph:

Question 6.2 Is there a coloring $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ that captures projective resp. combinatorial symmetries:

$$\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{PGL}}(P) \quad \text{resp.} \quad \mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{Comb}}(P)\ ?$$

There might be a general strategy derived from the following inclusion chain of the symmetry groups:

$$\mathrm{Aut}_{\mathrm{O}}(P) \subseteq \mathrm{Aut}_{\mathrm{GL}}(P) \subseteq \mathrm{Aut}_{\mathrm{PGL}}(P) \subseteq \mathrm{Aut}_{\mathrm{Comb}(P)}.$$

As it turns out, having solved the coloring problem further to the left in the chain can help to solve the problem further to the right—at least to some degree.

For example, note that every polytope P can be linearly transformed via a transformation $T \in \mathrm{GL}(\mathbb{R}^d)$ so that $\mathrm{Aut}_{\mathrm{GL}}(P) = \mathrm{Aut}_{\mathrm{O}}(TP)$. That is, a coloring of G_P that captures the orthogonal symmetries of TP (which has the same edge-graph) also captures the linear symmetries of P. In still other words, we solved the problem of capturing linear symmetries by making use of our ability to capture orthogonal symmetries.

In our approach, we have not made use of this because we needed to solve the linear case before the orthogonal one. However, this can be of use for capturing projective symmetries. More explicitly, the question is as follows: for every polytope P, is there a projective transformation $T \in \mathrm{PGL}(\mathbb{R}^d)$ so that $\mathrm{Aut}_{\mathrm{PGL}}(P) = \mathrm{Aut}_{\mathrm{GL}}(TP)$?

The same approach seems doomed for capturing combinatorial symmetries: there are polytopes with combinatorial symmetries that cannot be realized geometrically ([1] discusses the case of a combinatorial symmetry that cannot be made linear; to our knowledge, realizing them as projective symmetries remains to be discussed).

6.2 Edge-Only Coloring

For capturing the symmetries of certain 2-dimensional polytopes it is necessary to color both vertices *and* edges (cf. Fig. 2). But it is unclear whether this is still necessary in higher dimensions.

Question 6.3 Is it sufficient to color *only the edges* if $d \geq 3$? That is, is there an edge-only coloring $\mathfrak{c}\colon E \to \mathfrak{C}$ that captures (for example) linear symmetries?

Morozov [11] showed that the answer is affirmative in the special case of 3-dimensional polytopes with inscribed 2-faces.

A vertex-only coloring is not always sufficient. For example, in even dimensions exist vertex-transitive neighborly polytopes other than the simplex: e.g. for $n \geq 6$ we have the following cyclic 4-polytope with n vertices that is not a simplex:

$$P := \operatorname{conv}\left\{ \begin{pmatrix} \cos(2\pi i/n) \\ \sin(2\pi i/n) \\ \cos(4\pi i/n) \\ \sin(4\pi i/n) \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; i \in \{1, \dots, n\} \right\}.$$

The edge-graph of P is the complete graph K_n, and P has a single orbit of vertices. Thus, if $\mathfrak{c}\colon V \to \mathfrak{C}$ is a vertex-only coloring that captures the symmetries of P, then all vertices of K_n must receive the same color. But if the edges receive no color, then $\mathrm{Aut}(K_n^{\mathfrak{c}}) = \mathrm{Sym}(V)$. However, it is known that the linear symmetry group of the cyclic polytope P other than a simplex is strictly smaller than $\mathrm{Sym}(V)$ [9].

6.3 Non-Convex Polytopes and General Graph Embeddings

Our approach suggests no immediate generalization to non-convex polytopes or various forms of polytopal complexes.

Question 6.4 What is the most general geometric setting in which the symmetries can be "captured" by coloring the edge-graph? Does it work for non-convex and/or self-intersecting polytopes? What about more general polytopal complexes?

A vaster generalization are *graph embeddings*, that is, maps $v\colon V(G) \to \mathbb{R}^d$. We already mentioned in Example 2.6 that the orbit coloring does not generally capture the linear symmetries in this setting (and then no coloring can).

One setting in which capturing symmetries of such embeddings is possible is already suggested by Theorem 3.1, whose proof nowhere uses the concept of convexity or polytopes.

Remark 6.5 (*spectral graph embeddings*) Given some graph $G = (V, E)$, let $M \in \mathbb{R}^{n\times n}$ be a weighted adjacency matrix of G (that is, M is symmetric and $ij \notin E$ imply $M_{ij} = 0$), θ an eigenvalue of M and $u_1, \dots, u_d \in \mathbb{R}^n$ a basis of the θ-eigenspace. Then the rows of the matrix $(u_1, \dots, u_d) \in \mathbb{R}^{n\times d}$ provide d-dimensional coordinates, one per vertex of G. This is called a *spectral graph embedding*. The proof of Theorem 3.1 goes through unchanged to show that the symmetries of such embeddings can

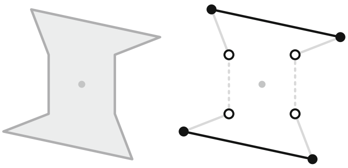

Fig. 5 A non-convex shape and two drawings of its edge-graph with metric coloring. The colored edge-graph has more symmetries than the polygon

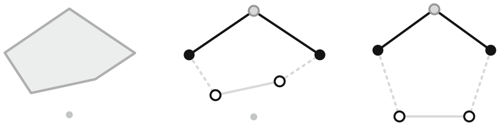

Fig. 6 A convex polygon P with $0 \notin \operatorname{int}(P)$ (the gray dot indicates the origin) and two drawings of its edge-graph with metric coloring. The colored edge-graph has more symmetries than the polygon

be captured by coloring the edges and vertices using the entries of M. Examples of such embeddings are ubiquitous, including Colin de Verdière embeddings [6], spectral graph drawings [10], and, according to Izmestiev's Theorem, polytope skeleta.

It would be interesting to determine other tangible geometric criteria under which "capturing symmetries" of graph embeddings is possible.

6.4 The Metric Coloring

It is yet unknown whether the metric coloring alone can capture orthogonal symmetries (cf. Sects. 2.1 and 5).

Question 6.6 Can the metric coloring $\mathfrak{m}$ capture orthogonal symmetries?

Again, [11] provides an affirmative answer for the special case of 3-polytopes with inscribed 2-faces. Any potential affirmative answer to Question 6.6 will need to make use of similar assumptions as the construction of the Izmestiev coloring, namely, convexity and $0 \in \operatorname{int}(P)$, as there are known counterexamples for the other cases (see Figs. 5 and 6). An interesting special case is the following:

Question 6.7 If P is inscribed (i.e., it has all its vertices on a common sphere around the origin) and has all edges of the same length, then is it true that P is as symmetric as its edge-graph, that is, $\operatorname{Aut}_{\mathrm{O}}(P) \cong \operatorname{Aut}(G_P)$?

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