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Complete Characterization of Polyhedral Self-Affine Tiles

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Abstract

A self-affine tile is a compact set $G \subset \mathbb{R}^d$ that admits a partition (tiling) by parallel shifts of the set $M^{-1}G$, where M is an expanding matrix. We find all self-affine tiles which are polyhedral sets, i.e., unions of finitely many convex polyhedra. It is shown that there exists an infinite family of such polyhedral sets, not affinely equivalent to each other. A special attention is paid to integral self-affine tiles with standard digit sets, when the matrix M and the translation vectors are integer. Applications to the approximation theory and to the functional analysis are discussed.

Keywords Self-affine tile · Polyhedron · Cone

Mathematics Subject Classification 42C40 · 39A99 · 52C22 · 12D10

1 Introduction

A *tiling* of a compact set $G \subset \mathbb{R}^d$ is its partition to finely many sets that are disjoint (up to measure zero) and are translations of one compact set T of positive Lebesgue measure. If T is similar to G by means of some affine operator with the expanding linear part M, then the tiling is *self-affine* and the set G is referred to as *a self-affine tile*.

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Self-affine tiles have been studied in an extensive literature. Most of known tiles, apart from parallelepipeds, have fractal-like properties, which is natural due to their self-similarity. An important problem is to find possibly simple sets that admit self-affine tilings. For example, disc-like sets [2], polyhedral sets [7, 12, 14, 17, 21, 22], etc. Besides the geometric interest, this question has obvious applications in the space tiling, the crystallography, the theory of functional equations, approximation theory, in constructing of orthonormal functional systems, in particular, Haar systems and wavelets, etc. For example, if G possesses a piecewise-smooth boundary, then its characteristic function χ_G has the maximal possible (among piecewise-constant functions) regularity in $L_2(\mathbb{R}^d)$: its Hölder exponent is $1/2$. Hence, the Haar system generated by χ_G has the best approximation properties, the corresponding subdivision scheme and the cascade algorithm have the fastest rate of convergence, etc., see [8, 20]. This holds, in particular, for polyhedral tiles. Not to mention, of course, that polyhedral sets are convenient in practical applications.

In this paper we classify all polyhedral self-affine tiles. The first results in this direction originated with Gröchenig and Madych [7], who studied self-affine tiles which are parallelepipeds and related Haar bases in $L_2(\mathbb{R}^d)$. A complete classification of linear operators and parallel translations generating self-affine tiles which are parallelepipeds was done in [12, Chapter 5] and in [22, Theorems 1 and 2]. Other polyhedral tilings were addressed in [14, 21, 22].

This is relatively simple to show that among convex polyhedra, only parallelepiped admits a self-affine tiling. For non-convex polyhedra, the problem is more complicated. The two-dimensional case was done in [22, Thm. 3] and the conclusion is the same: there are no self-affine polygonal tiles different from parallelograms. For higher dimensions, the corresponding conjecture was left open [22, Conjecture 1]. A big step towards the solution of this problem was done in the recent paper [21]. To formulate it, we need to recall one definition. Let a subset $G \subset \mathbb{R}^d$ be given; a point $\boldsymbol{v} \in G$ is said to be a vertex of *convex polyhedral corner* if the intersection of G with some neighbourhood of $\boldsymbol{v}$ is congruent to a neighbourhood of an apex of a convex polyhedral cone. As usual, a convex polyhedral cone is the set of solutions of a system of homogeneous linear inequalities which is pointed (does not contain straight bi-infinite lines) and possesses a nonempty interior. The main result of [21] asserts that if a self-affine tile has at least one convex polyhedral corner, then this tile is equivalent to a union of integer shifts of a unit cube. This strong result, however, does not solve the problem of characterising polyhedral self-affine tiles. Already in $\mathbb{R}^2$ there are polyhedral sets without convex corners (for example, the polyhedral set consisting of four triangles on Fig. 1). Moreover, in $\mathbb{R}^3$ there are (non-convex) polyhedra without convex corners. For example, at each vertex of a regular tetrahedron we cut off a small tetrahedron and replace it by a dimple of the same form (Fig. 2). We obtain a polyhedron with 16 vertices none of which has a convex corner.

On the other hand, there is a variety of disconnected polyhedral sets (unions of several convex polyhedra) that do admit self-affine tilings [22, Sect. 10]. However, their complete classification has been obtained only for one-dimensional case, see [11] which is based on [4] and [22, Thm. 8].

In this paper we classify all polyhedral self-affine tiles in $\mathbb{R}^d$, i.e., polyhedral sets that admit self-affine tilings (Theorem 2.1). Moreover, we also classify the integer

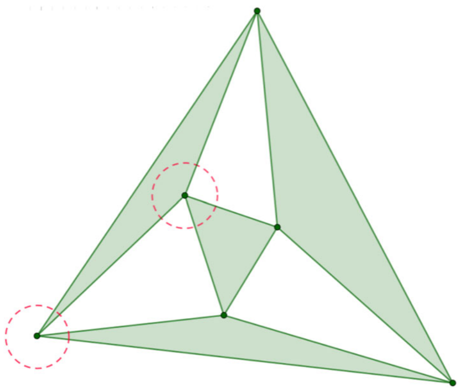

Fig. 1 2D polyhedral set without convex corners

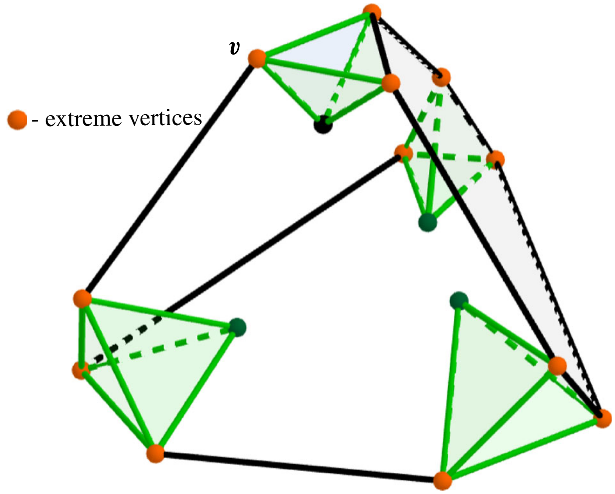

Fig. 2 3D polyhedron without convex corners

polyhedral self-affine tiles with standard digit sets (Theorem 2.2). This, in particular, gives a complete characterization of Haar bases with polyhedral structure in $L_2(\mathbb{R}^d)$. It is worth mentioning that the self-affine partitions of polyhedra have also been studied under less restrictive conditions, when not only parallel translations but also rotations are allowed [1, 3, 18, 19].

2 Main Results

Let G be a closed subset of $\mathbb{R}^d$ of positive Lebesgue measure, M be an expanding matrix, i.e., all its eigenvalues are larger than one in absolute value.

Definition 1 A *tiling* of the set G is its partition to a union of compact sets $G = \bigcup_i T_i$, such that all T_i are parallel shifts of each other and their pairwise intersections are of

Lebesgue measure zero. A tiling is *self-affine* if the sets T_i are congruent to $M^{-1}G$. In this case, G is called a self-affine tile.

We consider only the finite tilings, in which case G is compact. For example, all self-affine tilings are finite. A tiling will be denoted as $\mathcal{T} = \{T_i\}_{i=1}^N$; all the sets T_i are called *elements of the partition*. We denote $\mathcal{T} = \mathcal{T}(T, D)$, where $T_i = M^{-1}(G + d_i)$, $D = \{d_i\}$. Thus, the self-affine tile G satisfies the equation

$$G = \bigcup_{d_i \in D} M^{-1}(G + d_i). \tag{1}$$

Definition 2 A *convex polyhedron* is a compact subset of $\mathbb{R}^d$ with a nonempty interior defined by several linear inequalities. A *polyhedral set* is a union of finitely many convex polyhedra called *composing polyhedra*.

In the one-dimensional case all polyhedral sets are unions of several segments. All such self-affine tiles were characterised in [22], it is also reduced to a result from [11]. To formulate this result we introduce some further notation. For natural numbers a, n, we denote $\mathcal{S}(a, n) = \{ka \mid k = 0, \dots, n-1\}$. This is an arithmetic progression of length n with the difference a starting at zero. For a natural r and positive vectors $\boldsymbol{a}, \boldsymbol{n} \in \mathbb{Z}^r$, let $\mathcal{S}(\boldsymbol{a}, \boldsymbol{n}) = \mathcal{S}(a_1, n_1) + \dots + \mathcal{S}(a_r, n_r)$ be the Minkowski sum of the progressions $\mathcal{S}(a_i, n_i)$. A pair of vectors $\boldsymbol{a}, \boldsymbol{n} \in \mathbb{Z}^r$ is called *admissible* if $a_1 = n_1 = 1$ and for each $i \ge 2$, we have $a_i \ge 2$, $n_i \ge 2$, and a_i is divisible by $a_{i-1}n_{i-1}$. Now we formulate the characterization of univariate polyhedral tilings [22, Thm. 8]:

A union of finitely many segments in $\mathbb{R}$ denoted by G possesses a self-affine tiling if and only if there are $r \in \mathbb{N}$ and an admissible pair of vectors $\boldsymbol{a}, \boldsymbol{n} \in \mathbb{Z}^r$ such that G is equivalent (up to normalisation) to the set

$$\{[k, k+1] \mid k \in \mathcal{S}(\boldsymbol{a}, \boldsymbol{n})\}. \tag{2}$$

Thus, a polyhedral subset of $\mathbb{R}$ that admits a self-affine tiling is equivalent to a disjoint union of integer shifts of the unit segment. The number of segments is $n_1 \dots n_r$. The case of one segment corresponds to $r = 1$.

Our first result classifies all polyhedral sets in $\mathbb{R}^d$ that possess self-affine tilings.

Theorem 2.1 *A set $G \subset \mathbb{R}^d$ is a polyhedral self-affine tile if and only if it is affinely equivalent to a disjoint union of integer translates of the unit cube which is a direct product of d one-dimensional sets of the form* (2)*. Each of those d sets corresponds to its own triple $(r, \boldsymbol{a}, \boldsymbol{n})$, where $\boldsymbol{a}, \boldsymbol{n}$ is an admissible pair from $\mathbb{Z}^r$.*

An example of a polyhedral self-affine tile in $\mathbb{R}^2$ is given on Fig. 3. The first triple (axis x) is $(3; (1, 2, 8); (1, 2, 3))$, the second one (axis y) is $(3; (1, 2, 12); (1, 3, 2))$.

The second result characterises integral self-affine polyhedral tiles with standard digit sets.

Definition 3 An integral self-affine tile with standard digit set is a self-affine tile G whose expanding matrix M has integer entries $G = \bigcup_{i=1}^m M^{-1}(G + \boldsymbol{s}_i)$, where $m = |\det M|$ and $D = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_m\}$ is the "digit set" which is a complete set of coset representatives in $\mathbb{Z}^d / M\mathbb{Z}^d$.

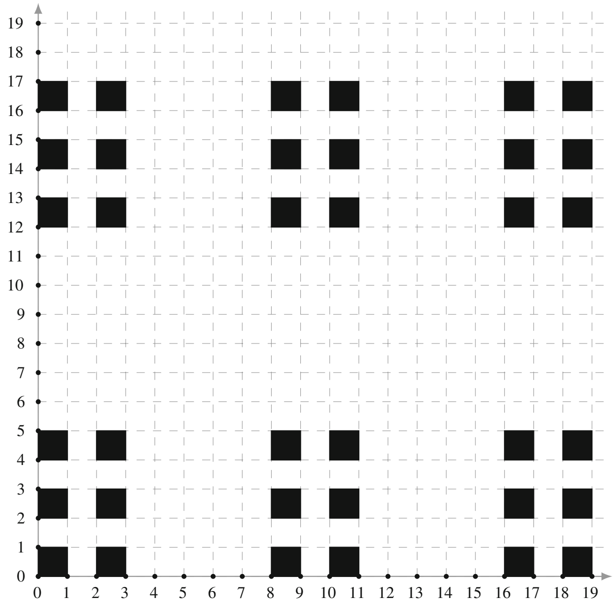

Fig. 3 Two-dimensional self-affine polyhedral tile

Thus, in the integer tiling the matrix M is integer and the shift vectors compose a complete set of digits with respect to M. This means that $\boldsymbol{s}_i - \boldsymbol{s}_j \notin M\mathbb{Z}^d$ for all $i \neq j$ and for every $\boldsymbol{s} \in \mathbb{Z}^d$ there exists j such that $\boldsymbol{s} - \boldsymbol{s}_j \in M\mathbb{Z}^d$. Integral self-affine tiles are applied in the combinatorics and number theory as well as in the construction of orthonormal Haar bases in $L_2(\mathbb{R}^d)$. By Theorem 2.1, a polyhedral integral self-affine tile with standard digit set must be equivalent to a union of integer shifts of the unit cube. However, among all the sets described in Theorem 2.1 only one case corresponds to an this integer tiling. The following theorem gives a complete classification.

Theorem 2.2 *If a polyhedral set is an integral self-affine tile with standard digit set, then it is a parallelepiped.*

Note that one integral self-affine tile can be generated by different dilation matrices and by sets of digits.

Remark 1 Note that in some cases there are no self-similar integer tiles (even not necessarily polyhedral) other than parallelepipeds. This situation emerges, for example, for tiles with an isotropic dilation matrix and with two digits [23]. This case is applied in construction of the Haar bases in $L_2(\mathbb{R}^d)$.

3 The Roadmap of the Proofs

The proof of Theorem 2.1 consists of several steps. We shall briefly describe them below and then in Sect. 5 give a detailed proof. The proof of Theorem 2.2 is much shorter (of course, after referring to Theorem 2.1); its idea is described in the end of this section. A complete proof of Theorem 2.2 is given in Sects. 5 and 6. We use the standard notation $\mathrm{co}(X)$ for the convex hull of the set X.

The Steps of the Proof of Theorem 2.1

Let us have a polyhedral set $G \subset \mathbb{R}^d$ with a self-affine tiling $\mathcal{T}$.

`Step 1.` The convex hull of the set G is a convex polyhedron. Vertices of this polyhedron are referred to as *extreme vertices* of G. For an arbitrary extreme vertex $\boldsymbol{v}$, we consider the corresponding *corner* K of G, which is the intersection of G with a small ball centered at $\boldsymbol{v}$. This is a cone with an apex $\boldsymbol{v}$, maybe non-convex. In the partial order defined by the cone $\mathrm{co}(K)$ (we keep the same notation for a corner and for the corresponding cone) the vertex $\boldsymbol{v}$ is a unique minimal element of the polyhedron $\mathrm{co}(G)$. This fact implies that there exists a unique element of partition $T \in \mathcal{T}$ that contains $\boldsymbol{v}$ (see Proposition 3). Moreover, a sufficiently small corner of G at the extreme vertex $\boldsymbol{v}$ is also a corner of T, and $\boldsymbol{v}$ is an extreme vertex for T as well (Proposition 4).

`Step 2.` Thus, for every extreme vertex $\boldsymbol{v}$ of G, there exists a unique element from $\mathcal{T}$ containing $\boldsymbol{v}$, for which $\boldsymbol{v}$ is also an extreme vertex with the same corner. This defines a map from the set of extreme vertices of G to extreme vertices of T. Since the tiling is self-affine, T is affinely similar to G and hence this map defines a map of the set of extreme vertices of G to itself. The graph corresponding to this map has one outgoing edge from each vertex and hence has a cycle of some length n. Consider the nth iteration $\mathcal{T}^n$ of the tiling $\mathcal{T}(T, D)$, which is $\mathcal{T}^n = \mathcal{T}(T_n, D_n)$, where $T_n = M^{-n}G$, $D_n = \bigl\{\sum_{k=0, s_k \in D}^{n-1} M^k s_k\bigr\} = D + MD + \cdots + M^{n-1}D$. Thus,

$$G = \sum_{s \in D_n} M^{-n}(G + s).$$

Since a graph has a cycle of length n, it follows that for the tiling $\mathcal{T}(T_n, D_n)$, there exists a *stationary* vertex $\boldsymbol{v}$ corresponding to itself. To simplify the notation, we use the same symbol T for all iterations. Hence, the vertex $\boldsymbol{v}$ corresponds to the same vertex of T. Moreover, after extra iterations of the tiling, it may be assumed that the convex corner $\mathrm{co}(K)$ has the same corresponding faces in G as in T. This is proved in Proposition 5.

`Step 3.` Thus, G has at least one *stationary vertex* $\boldsymbol{v}$, which corresponds to the same extreme vertex of G and of the element of partition T. If K is the corner at the vertex $\boldsymbol{v}$, then the set $\mathrm{co}(K)$ is a corner of $\mathrm{co}(G)$; it is convex and has an apex at the vertex $\boldsymbol{v}$. Our goal is to prove that actually the corner K is convex and simple (has exactly d edges). To this end, we first show that if K contains a face of the convex corner $\mathrm{co}(K)$, then the intersection of all elements with that face define a self-affine tiling on it (Lemma 1). This will allow us to apply induction arguments in the dimension of the faces.

`Step 4.` If the corner K at a stationary vertex $\boldsymbol{v}$ contains a j-dimensional face L of $\mathrm{co}(K)$, then it contains a parallelepiped in L with the same corner at $\boldsymbol{v}$ (Lemma 2).

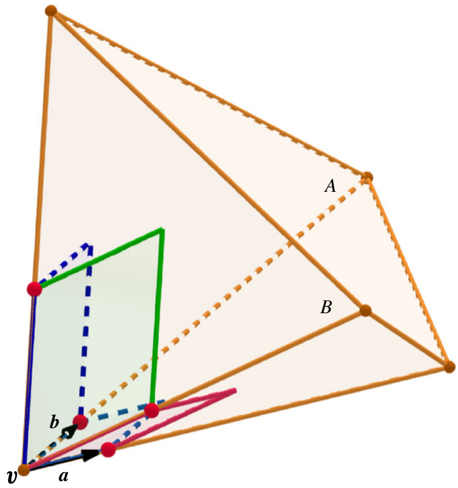

Fig. 4 The "lily": the parallelepipeds on facets spanned by vectors along the edges

Step 5. This is a keystone of the proof. We establish the following auxiliary geometrical fact, which is, probably, of some independent interest. Let us have a convex polyhedral cone $C \subset \mathbb{R}^d$. Assume all its facets (faces of co-dimension one) are simple, i.e., affinely similar to $\mathbb{R}_+^{d-1}$ (see Sect. 4.1 for detailed definitions). For arbitrary vectors going from the apex along the edges (one vector on each edge), the following holds: either C is simple, or there exist two facets A, B of C and two vectors $\boldsymbol{a}, \boldsymbol{b}$ from our family on different edges such that the shifted parallelepipeds $\boldsymbol{a} + P(A)$ and $\boldsymbol{b} + P(B)$ have a common interior point, where $P(X)$ is the parallelepiped spanned by the vectors on a facet X (see Fig. 4). In the former case (C is simple) those parallelepipeds form a boundary of a full-dimensional parallelepiped with the corner C. This is Lemma 3, which can be called "the lily lemma" since the parallelepipeds form "lily petals".

Step 6. Applying the result of the previous step we prove by induction in j the following statement: if a vertex $\boldsymbol{v}$ is stationary, then for each $j = 1, \ldots, d$, the corner K contains all faces of dimension j of the convex corner $\mathrm{co}(K)$ and all those faces are simple (Proposition 6). Taking $j = d$ we immediately obtain that the corner K is convex and simple.

Step 7. Thus, the corner of G at every stationary vertex is convex and simple. Hence, by [21, Theorem 1.9], the set G is equivalent to a union of several integer shifts of the unit cube. Then we establish Lemma 4 according to which there exists a subset of the tiling $\mathcal{T}$ that forms a tiling of a rectangular parallelepiped. Then we apply the main result of the paper [13] on discrete tiling of a cube and conclude that T is a direct product of d one-dimensional tiles (note that in case of infinite discrete tilings of the set $\mathbb{N} \times \mathbb{N}$ this does not hold, see [15]). Hence so is G. It remains to invoke the classification of one-dimensional tilings from [22, Thm. 8] formulated in Sect. 2.

3.1 The Idea of the Proof of Theorem 2.2

Applying Theorem 2.1 we obtain that G is affinely similar to a direct product of special sets of the form (2). In particular, this yields that G is equivalent to a union of disjoint integer shifts of the unit cube. If the tiling $\mathcal{T}$ is integer, then all its iterations are also integer. Taking a sufficiently big iteration we can assume that all elements of partition have diameter less than one. Hence, each element of partition is contained in a unique cube. This implies that the shifts cannot be from different quotient classes, provided G contains at least two cubes. The proof is in Sect. 7.

4 Notation and Preliminary Facts

We use convex separation theorems. Let X and Y be subsets of $\mathbb{R}^d$. The set X is separated from Y by a nonzero element $\boldsymbol{c} \in \mathbb{R}^d$ if $(\boldsymbol{c}, \boldsymbol{x}) \le (\boldsymbol{c}, \boldsymbol{y})$ for all $\boldsymbol{x} \in X$, $\boldsymbol{y} \in Y$. This separation is *strong* if the set $Y \backslash X$ is nonempty and $(\boldsymbol{c}, \boldsymbol{x}) < (\boldsymbol{c}, \boldsymbol{y})$ for all $\boldsymbol{x} \in X$, $\boldsymbol{y} \in Y \backslash X$. Let us remark that this separation is not symmetric in X, Y. By the convex separation theorem, if X, Y are both convex, the interior of Y is non-empty and does not intersect X, then X can be separated from Y. Every face of a convex polyhedron G is strongly separated from it. A point on the surface of G is strongly separated from it precisely when it is extreme.

4.1 Cones

A *cone* K with the apex at the origin is a closed subset of $\mathbb{R}^d$ such that if $\boldsymbol{x} \in K$ then for each $\lambda \ge 0$, we have $\lambda \boldsymbol{x} \in K$. A cone is nondegenerate if it possesses a nonempty interior. A cone is *pointed* if it does not contain a straight line. A cone is *convex* if for every $\boldsymbol{x}, \boldsymbol{y} \in K$, we have $\boldsymbol{x} + \boldsymbol{y} \in K$. Any ray that belongs to the boundary of a cone is called its *generatrix*. If a generatrix does not belong to a convex hull of other generatrices, it is called *an extreme edge*. In particular, a generatrix of a convex cone is an extreme edge if it does not belong to a linear span of two other generatrices. In this case we drop the word "extreme" and call it edge.

In what follows we always assume that a convex cone is nondegenerate and pointed. In this case it can be strongly separated from its apex. A convex cone $K \subset \mathbb{R}^d$ is called polyhedral if it is defined by a system of linear inequalities. It has faces of all dimensions from zero (the apex) to d (the whole cone). A *facet* is a face of dimension $d-1$; an edge is a face of dimension one. A cone is *simple* if is affinely equivalent to the positive orthant $\mathbb{R}^d_+$. So, a simple cone has exactly d facets and d edges. A cone possesses simple facets if all their facets are simple $(d-1)$-dimensional cones. Clearly, in this case its faces of all dimensions $\le d-1$ are simple.

A *corner* is an intersection of a cone with a ball centered at the apex. The cone is an *extension* of every its corner. We often identify a corner and its extension and use the same notation for them.

Every convex cone defines a partial order in $\mathbb{R}^d$ as follows: $\boldsymbol{x} \ge \boldsymbol{y}$ if $\boldsymbol{x} - \boldsymbol{y} \in K$. In particular, $\boldsymbol{x} \ge 0$ if $\boldsymbol{x} \in K$. Since K is pointed, it follows that if $\boldsymbol{x} \ge \boldsymbol{y}$, then $\boldsymbol{y} \ge \boldsymbol{x}$

is impossible, unless $\boldsymbol{x} = \boldsymbol{y}$. A point $\boldsymbol{x}$ is called *minimal* for a set $\Omega \in \mathbb{R}^d$ if $\boldsymbol{x} \le \boldsymbol{y}$ for all $\boldsymbol{y} \in \Omega$. Not every set possesses a minimal element, but if it exists, it is unique. Indeed, if there are two minimal elements $\boldsymbol{x}$ and $\boldsymbol{y}$, then $\boldsymbol{x} \le \boldsymbol{y}$ and $\boldsymbol{y} \le \boldsymbol{x}$, hence $\boldsymbol{x} = \boldsymbol{y}$.

Let B be a subset of a hyperplane L with a nonempty (in L) interior and $\boldsymbol{v} \notin L$ be a point. Then all segments connecting $\boldsymbol{v}$ with points from B form a *bounded cone* with the apex $\boldsymbol{v}$ and a base B.

4.2 Polyhedral Sets

A *corner* of a polyhedral set is its intersection with a small ball centered at a vertex $\boldsymbol{v}$ of some composing polyhedron. We always assume that the ball is small enough and intersects only those faces of the composing polyhedra adjacent to $\boldsymbol{v}$. Clearly, the extension of a corner is nondegenerate, but possibly non-convex and not pointed.

A point of a polyhedral set G is said to be *an extreme vertex* if it is a vertex of $\mathrm{co}(G)$. For example, the plane polyhedral set on Fig. 1 has three extreme vertices (the vertices of the big triangle), the set on Fig. 2 has 12 extreme vertices painted red. A point $\boldsymbol{v} \in G$ is an extreme vertex if and only if it can be strongly separated from G. That vertex is called *convex* if it is a vertex of a convex corner of G (as usual, a convex corner is assumed to be non-degenerate and pointed).

Similarly we define a *composite extreme face* L of a polyhedral set $G \subset \mathbb{R}^d$. This is the intersection of G with a separating hyperplane of L. For the sake of simplicity we usually call L *extreme face*. It is a union of several convex sets that are faces of polyhedra that form G. The maximal dimension of those sets is the *dimension of the extreme face*. An extreme face of dimension $d-1$ is an *extreme facet*. Let $G' \subset G$ be an extreme facet and S be a convex subset of G'; then a *layer* of S is a bounded cone with the base S that is contained in G. Not every convex subset of a facet has a layer. For example, if ABC is a triangle and A', B', C' are the midpoints of its sides, then the union of the triangles $AB'C'$ and $BC'A'$ is a polyhedral set with the facet AB. This facet, however, does not have a layer.

As we have already mentioned, there are polyhedra without convex vertices. However, extreme vertices always exist.

Proposition 1 *If K is a corner of a polyhedral set at its extreme vertex, then* $\mathrm{co}(K)$ *is a nondegenerate convex pointed cone.*

Proof The convexity and nondegeneracy are obvious. The pointedness of $\mathrm{co}(K)$ follows from the pointedness of the corner of $\mathrm{co}(G)$, which contains $\mathrm{co}(K)$. □

Proposition 2 *Every polyhedral set is contained in the convex hull of its extreme vertices.*

Proof Extreme vertices of a polyhedral set G are vertices of the convex polytope $\mathrm{co}(G)$. By the Minkowski theorem, a convex polyhedron is a convex hull of its vertices. □

Corollary 1 *Every polyhedral set in $\mathbb{R}^d$ possesses at least $d+1$ extreme vertices.*

5 Proof of Theorem 2.1. Part 1: Auxiliary Results

We begin with several basic facts, which hold for all tilings, not necessarily self-affine.

Proposition 3 *Suppose $\mathcal{T}$ is a tiling of a polyhedral set G; then every extreme vertex of G is contained in a unique element of $\mathcal{T}$.*

Proof Let an element $T \in \mathcal{T}$ contain an extreme vertex $\boldsymbol{v}$ of G. Denote by $\tilde{K}$ the extension of the corner of $\mathrm{co}(G)$ at the vertex $\boldsymbol{v}$. Since $\mathrm{co}(G) \subset \tilde{K}$ and $T \subset G$, it follows that $T \subset \tilde{K}$. Hence, $\boldsymbol{v}$ is the minimal element of T in the order defined by the cone $\tilde{K}$. If another element $T' \in \mathcal{T}$ also contains $\boldsymbol{v}$, then $\boldsymbol{v}$ is the minimal element of T'. From the uniqueness of the minimal element and from the fact that T' is a parallel shift of T by some nonzero vector $\boldsymbol{a}$, we see that $\boldsymbol{v} + \boldsymbol{a} = \boldsymbol{v}$; therefore, $\boldsymbol{a} = 0$ and $T' = T$. □

Proposition 4 *If an element $T \in \mathcal{T}$ contains an extreme vertex $\boldsymbol{v}$ of G, then it contains a sufficiently small corner at $\boldsymbol{v}$. In particular, if the tiling is polyhedral, then T and G have the same corner at $\boldsymbol{v}$ and this vertex is extreme for T.*

Proof By Proposition 3, T is a unique element of partition containing $\boldsymbol{v}$, hence all other elements of $\mathcal{T}$ are located on positive distances from $\boldsymbol{v}$. Choosing the radius of the ball smaller than all those distances, we obtain a corner that does not intersect other elements of $\mathcal{T}$. On the other hand, the tiling covers the whole set G, therefore the intersection of the small ball with G coincides with its intersection with T. Thus, T and G have the same corner at $\boldsymbol{v}$. Furthermore, since $\boldsymbol{v}$ can be strongly separated from G by a hyperplane, the same hyperplane separates $\boldsymbol{v}$ from T, because $T \subset G$. Consequently, $\boldsymbol{v}$ is an extreme vertex of T. □

Thus, every extreme vertex of G is associated to an extreme vertex of T, which is the minimal element of T in the order defined by the corner of $\mathrm{co}(G)$ at that vertex. The inverse correspondence may not be well-defined: an extreme vertex of T can have no corresponding vertices from G or can have several ones.

Now we turn to self-affine tilings. Each element T is similar to G by means of some affine transform $A\colon T \to G$. It maps each extreme vertex of T to a corresponding vertex of G. For all elements, those transforms have the same linear part defined by the dilation matrix M.

Definition 4 Let a polyhedral set G possess a self-affine tiling $\mathcal{T}$. Then its extreme vertex $\boldsymbol{v}$ is called *stationary* if a unique element $T \in \mathcal{T}$ containing $\boldsymbol{v}$ possesses the following properties:

- $\boldsymbol{v}$ is a fixed point of the affine transform $A\colon T \to G$;
- the transform A respects the corner K of G at the vertex $\boldsymbol{v}$ and respects all faces of the cone $\mathrm{co}(K)$;
- the extension of K contains G.

Extreme vertices depend only on the polyhedral set G, while the stationary vertices depend also on its self-affine tiling. As we know, G has at least $d + 1$ extreme vertices. However, for some tilings, none of those vertices are stationary. Nevertheless, at least one of them does become stationary after certain iteration of the tiling. This is guaranteed by the following

Proposition 5 *For every self-affine tiling $\mathcal{T}$ of a polyhedral set G, there is $n \in \mathbb{N}$ such that G with the tiling $\mathcal{T}^n$ possesses at least one stationary vertex.*

Proof Proposition 4 implies that every extreme vertex $\boldsymbol{u}$ of G corresponds to a unique extreme vertex $\boldsymbol{u}'$ of T. The latter, in turn, corresponds to a unique extreme vertex $A\boldsymbol{u}'$ of G. Thus, we have a map $\boldsymbol{u} \mapsto A\boldsymbol{u}'$ defined on the set of extreme vertices of G. Denote this map by φ. In general, φ may not be injective. Iterating φ, we obtain an extreme vertex $\boldsymbol{v}$ of G and a number k such that $\varphi^k(\boldsymbol{v}) = \boldsymbol{v}$. This means that, for the tiling $\mathcal{T}^k$, the vertex $\boldsymbol{v}$ is covered by the corresponding (in the sense of similarity) vertex of the element of the tiling. Thus, after taking some power of the tiling we assume (keeping the previous notation for the new tiling) that G and T have the same corresponding vertex $\boldsymbol{v}$, i.e., $A\boldsymbol{v} = \boldsymbol{v}$. By Proposition 4, the corners of G and of T at $\boldsymbol{v}$ coincide. Hence, the transform A preserves this corner K and therefore, defines a permutation of its extreme edges. Some power of this permutation is identical. This means that passing again to a power of the tiling we may assume that A maps each extreme edge of the cone K to itself. Hence, it respects all faces of $\mathrm{co}(K)$ and so, $\boldsymbol{v}$ is a stationary vertex.

With possible further iteration of the tiling it may be assumed that the element T is small enough and is contained in a small ball defining the corner K. Hence, T is contained in the extension of K. Since A maps T to G and respects K it follows that G is also contained in the extension of K. $\square$

In what follows we simplify the notation and assume that we are already given the nth power of the tiling. Thus, for a tiling $\mathcal{T}$ of the set G, there exists a stationary vertex $\boldsymbol{v}$.

Until now we dealt with elements covering the extreme vertices. Now we go further and look at extreme faces. Let us stress that we consider faces of convex cones only. Let K be a corner at the stationary vertex $\boldsymbol{v}$ of a polyhedral set G and L be a j-dimensional face of the convex cone $\mathrm{co}(K)$, $j \geq 1$. We denote $G_L = G \cap L$. If G admits a self-affine tiling, then G lies in the extension of K. Hence, in this case $\mathrm{co}(G_L)$ is a j-dimensional face of $\mathrm{co}(G)$. The following lemma reduces the dimension in the proof of Theorem 2.1.

Lemma 1 *Let $\mathcal{T}$ be a self-affine tiling of G, $\boldsymbol{v} \in G$ be its stationary vertex, and $T \in \mathcal{T}$ be the element containing $\boldsymbol{v}$. Suppose the corner K at $\boldsymbol{v}$ contains a face L of $\mathrm{co}(K)$; then all elements of $\mathcal{T}$ intersecting G_L are translations of T by vectors parallel to L. The sets of intersection form a self-affine tiling of G_L. If $T \cap L$ is a composite facet of T, then every convex subset of it has a layer in T.*

Proof Let z be an arbitrary point of G_L and an element $T' = T + \boldsymbol{a}$ contain this point. Let a hyperplane $\{\boldsymbol{x} \mid (\boldsymbol{c}, \boldsymbol{x}) = 0\}$ strongly separate L from K. Then it strongly separates G_L from K. This means that $(\boldsymbol{c}, z) = (\boldsymbol{c}, \boldsymbol{v}) = 0$, while $(\boldsymbol{c}, \boldsymbol{x}) < 0$ for all points $\boldsymbol{x} \in G \setminus G_L$. We have $z - \boldsymbol{a} \in T$, hence $(\boldsymbol{c}, z) - (\boldsymbol{c}, \boldsymbol{a}) \leq 0$. Thus, $(\boldsymbol{c}, \boldsymbol{a}) \geq (\boldsymbol{c}, z) = 0$, which implies $(\boldsymbol{c}, \boldsymbol{v} + \boldsymbol{a}) = (\boldsymbol{c}, \boldsymbol{a}) \geq 0$. However, $\boldsymbol{v} + \boldsymbol{a} \in T'$ and therefore, $\boldsymbol{v} + \boldsymbol{a} \in G$. This yields $(\boldsymbol{c}, \boldsymbol{v} + \boldsymbol{a}) \leq 0$ and consequently, $(\boldsymbol{c}, \boldsymbol{v} + \boldsymbol{a}) = 0$.

Thus, $\boldsymbol{v} + \boldsymbol{a} \in G_L$ and hence, the translation vector $\boldsymbol{a}$ is parallel to L. This is true for all elements of the partition intersecting G_L.

Thus, the sets $T' \cap L$, where T' is a translation of T parallel to L, are translations of $T \cap L$. Clearly, they cover G_L. Since the affine similarity transform A respects L it follows that all the sets $T_i \cap G_L$ which are nonempty, are similar to G_L by the transform A. To prove that they form a self-affine tiling of G_L it remains to show that the interiors (in L) of those sets are disjoint. This will be done by induction in $j = \dim L$ in the inverse order, i.e., starting with $j = d$.

Suppose $j \le d - 1$ and assume that the statement is true for every $(j+1)$-dimensional face L': the sets $T_i \cap G_{L'}$ do not have common interior points in L'. Take an arbitrary j-dimensional face L and choose a $(j+1)$-dimensional face L' containing L. By the inductive assumption, the intersections of T_i with L' (denote them by T_i') form a tiling of $G_{L'}$. If we show that the intersections of T_i' with L have disjoint interiors (in L), then they form a self-affine tiling of G_L, and the proof will be completed. Indeed, for all possible $(j+1)$-dimensional faces containing L, this tiling of G_L will be the same. To see this we note that this tiling is uniquely defined by the affine similarity transform of L and by the system of translations of L. Neither the similarity transform nor the system of translations depend of L'. Indeed, the translations of L are those translations of T parallel to L and the similarity transform is the restriction $A|_L$ of the similarity operator A to its invariant subspace L. Thus, we need to show that the intersections of T_i' with L have disjoint interiors in L. For the sake of simplicity we realize the proof for $j = d-1$, the proof for other j is literally the same. Thus, $L' = \mathrm{co}(K)$ and L is a hyperface (facet) of L'. Denote $\tilde{T} = T \cap L$ and assume the contrary: there are elements $\tilde{T} + \boldsymbol{a}$ and $\tilde{T} + \boldsymbol{b}$ that share a common interior point $\boldsymbol{x} \in L$. This point belongs to the set $P + \boldsymbol{a}$, where P is one of the convex polyhedra that form the polyhedral set G. Then $\tilde{P} + \boldsymbol{a}$, which is the intersection of $P + \boldsymbol{a}$ with L, is a facet of the convex polyhedron $P + \boldsymbol{a}$. If $\boldsymbol{x}$ is an interior point of this facet, then some its neighbourhood $U(\boldsymbol{x})$ lies in $P + \boldsymbol{a}$ (more precisely, the "half" of this neighbourhood, which is the intersection of $U(\boldsymbol{x})$ with the half-space containing G). Therefore, $U(\boldsymbol{x})$ lies in $T + \boldsymbol{a}$. However, in this case the set $T + \boldsymbol{b}$ cannot intersect $U(\boldsymbol{x})$, otherwise it will have common interior points with $T + \boldsymbol{a}$. If, on the other hand, $\boldsymbol{x}$ belongs to the boundary of $\tilde{P} + \boldsymbol{a}$, then we use our assumption that $\boldsymbol{x}$ is a common interior point of $\tilde{T} + \boldsymbol{a}$ and $\tilde{T} + \boldsymbol{b}$. Each sufficiently small shift of $\boldsymbol{x}$ along L is also an interior point of both $\tilde{T} + \boldsymbol{a}$ and $\tilde{T} + \boldsymbol{b}$. However, there is a small shift which sends $\boldsymbol{x}$ to some point $\tilde{\boldsymbol{x}}$ in the interior of $\tilde{P} + \boldsymbol{a}$. Indeed, every boundary point of a convex polyhedron can be shifted to its interior. Now we just repeat the proof above for the point $\tilde{\boldsymbol{x}}$, which belongs to the interior of $\tilde{P} + \boldsymbol{a}$. This completes the proof for the self-affine tiling of G_L.

Finally, for every $T' \in \mathcal{T}$, $T' \neq T$, there is a bounded cone in G with the base $T \cap L$ that does not intersect T'. Since the set $\mathcal{T}$ is finite, the intersection of those cones contains a bounded cone with the base $T \cap L$. This cone does not intersect other elements from $\mathcal{T}$, hence, it lies in T. Therefore, this is a layer of the facet $T \cap L$ in T. □

Lemma 1 applied to a one-dimensional face (an edge) of $\mathrm{co}(K)$, implies the following

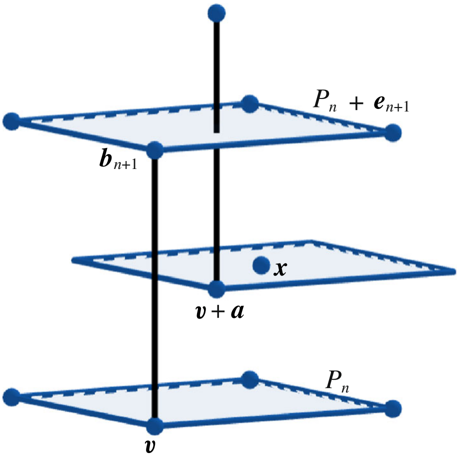

Fig. 5 Illustration to the proof of Lemma 2

Corollary 2 *Let ℓ be an extreme edge of G going from a stationary vertex $\boldsymbol{v} \in G$. Let $\mathcal{T}$ be a self-affine tiling of G and T be its element containing $\boldsymbol{v}$. Then all elements of $\mathcal{T}$ intersecting ℓ are translations of T by vectors parallel to ℓ. The sets of intersection form a tiling of ℓ.*

Proof As we know, all edges of $\mathrm{co}(K)$ (faces of dimension one) are extreme edges of K and they lie in K. Hence, the assumptions of Lemma 1 for ℓ are fulfilled. Now applying Lemma 1 in the case $j = 1$ we conclude the proof. □

Now we show that if a corner of K of a stationary vertex contains a face of the corner $\mathrm{co}(K)$, then the set $T \cap L$ contains a special parallelepiped.

Lemma 2 *Under the assumptions of Lemma 1, suppose that L is a simple j-dimensional cone with edges ℓ_s, $s = 1, \ldots, j$. Let $\boldsymbol{b}_s$ be the most distant point from $\boldsymbol{v}$ of the edge ℓ_s for which the segment $[\boldsymbol{v}, \boldsymbol{b}_s]$ is contained in T. Then T contains the j-dimensional parallelepiped spanned by the segments $[\boldsymbol{v}, \boldsymbol{b}_s]$, $s = 1, \ldots, j$.*

Proof The proof is by the induction in the dimension j. For $j = 1$, the statement is obvious. Assume it holds for some dimension $n < j$. Consider the face L_{n+1} spanned by the edges $\ell_1, \ldots, \ell_{n+1}$ and its face L_n spanned by the first n edges. Since L_{n+1} is a simple cone, we can identify it with $\mathbb{R}_+^{n+1}$ and assume that all the segments $[\boldsymbol{v}, \boldsymbol{b}_s]$ are of length one. Denote $\boldsymbol{b}_s - \boldsymbol{v} = \boldsymbol{e}_s$. We use the same notation for the elements from $\mathcal{T}$ and for their intersections with the face L_{n+1}. We need to show that the unit cube $P_{n+1} = \{\boldsymbol{x} \in \mathbb{R}_+^{n+1} \mid x_i \le 1,\ i = 1, \ldots, n+1\}$ is contained in T. By the inductive assumption, T contains its face $P_n = \{\boldsymbol{x} \in P_{n+1} \mid x_{n+1} = 0\}$. Since by Corollary 2, the shifts of the set $T \cap \ell_{n+1}$ form a tiling of ℓ_{n+1} it follows that the set $T' = T + \boldsymbol{e}_{n+1}$ is also an element of $\mathcal{T}$.

If P_{n+1} does not lie in T, then there is a point $\boldsymbol{x} \in \mathrm{int}\, P_{n+1}$ which is not in T (see Fig. 5). On the other hand, it must belong to some element $T'' = T + \boldsymbol{a}$, $\boldsymbol{a} \in \mathrm{int}\, P_{n+1}$. For every $\boldsymbol{y} \in T'$, we have $y_{n+1} \ge 1$, hence none of the points $\boldsymbol{x}, \boldsymbol{v} + \boldsymbol{a}$ belongs

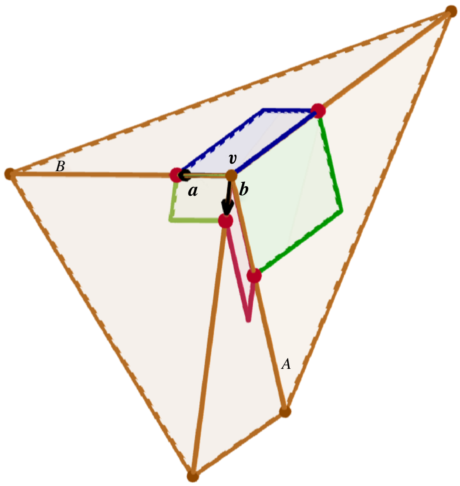

Fig. 6 Parallelepipeds on facets of the lily spanned by the vectors on the edges

to T', and so $T' \neq T''$. On the other hand, T'' must contain the parallel shift of the segment $[\boldsymbol{v}, \boldsymbol{b}_{n+1}]$ by the vector $\boldsymbol{a}$. This segment $[\boldsymbol{v} + \boldsymbol{a}, \boldsymbol{b}_{n+1} + \boldsymbol{a}]$ is of length one and it intersects the face $P_n + \boldsymbol{e}_{n+1}$ of P_{n+1}. However, this face lies in T' with some layer (Lemma 1). Hence, T and T' have a common interior point. The contradiction proves that P_{n+1} is in T, which completes the inductive step. □

6 Proof of Theorem 2.1. Part 2: Conclusion

Now we are going to prove the following main proposition, from which Theorem 2.1 simply follows:

Proposition 6 *If a polyhedral set admits a self-affine tiling, then its corners at all stationary vertices are convex and simple.*

We need the following geometrical lemma, which is, probably, of some independent interest. Let C be a convex polyhedral cone with simple facets and with the apex at the origin O. On every edge of C one chooses an arbitrary point $\boldsymbol{c} \neq 0$, which is referred to as a *directing point* and the segment $[O, \boldsymbol{c}]$ is a *directing* segment of that edge. For a given facet H of C, we denote by $P(H)$ its *directing parallelepiped*, which is spanned by the directing segments of that facet. Thus, $P(H)$ is a $(d-1)$-dimensional parallelepiped contained in H. The family of all such parallelepipeds form a "lily" based on the cone C.

Lemma 3 (lily lemma) *For a convex polyhedral cone C with simple facets and for arbitrary directing points on its edges, the following holds: either C is simple, or there exist two facets A, B and two directing points $\boldsymbol{a}, \boldsymbol{b}$ on different edges of C such that the shifted parallelepipeds $\boldsymbol{a} + P(A)$ and $\boldsymbol{b} + P(B)$ have a common interior point (see Figs. 6 and 7).*

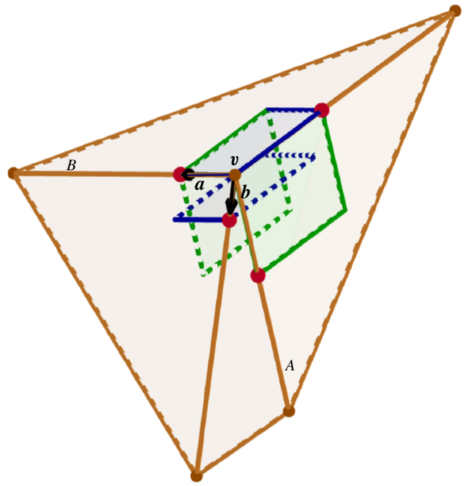

Fig. 7 Intersection of shifted parallelepipeds

Proof If C is not simple, then there are two of its facets A, B without a common $(d-2)$-dimensional face. Then there are two edges of the cone C: $a \subset A$ and $b \subset B$ such that the $(d-1)$-dimensional cones $C_a = \mathrm{co}\,\{a, B\}$ and $C_b = \mathrm{co}\,\{b, A\}$ have a common interior point. Clearly, a is not in B, otherwise $C_a = B$ and it cannot have common interior points with C_b since A and B have no common facets. Similarly, b is not in A. Since a and b are both in $C_a \cap C_b$ it follows that the set $C_a \cap C_b$ has interior points arbitrary close to $\mathrm{co}\,\{a, b\}$. Now denote by $\boldsymbol{a}_i$, $i = 1, \dots, d-1$, the directing points on the edges of the face A, $\boldsymbol{a}_1 \in a$. Analogously, $\boldsymbol{b}_i$, $i = 1, \dots, d-1$, are the directing points on the edges of B, $\boldsymbol{b}_1 \in b$. A common interior point $\boldsymbol{x}$ of C_a and C_b is expressed as follows:

$$\boldsymbol{x} = \alpha \boldsymbol{a}_1 + \sum_{i=1}^{d-1} t_i \boldsymbol{b}_i = \beta \boldsymbol{b}_1 + \sum_{i=1}^{d-1} s_i \boldsymbol{a}_i, \tag{3}$$

where all the coefficients α, β, t_i, s_i are strictly positive. After multiplication of this equality by a positive constant it can be assumed that $\alpha < 1$ and $\beta < 1$. Moreover, choosing $\boldsymbol{x}$ sufficiently close to $\mathrm{co}\,\{a, b\}$ we may assume that all t_i and s_i are small, in particular, all of them are less than 1 and $t_1 < \beta$, $s_1 < \alpha$. Rewriting (3) we obtain

$$-(\beta - t_1)\boldsymbol{b}_1 + \sum_{i=2}^{d-1} t_i \boldsymbol{b}_i = -(\alpha - s_1)\boldsymbol{a}_1 + \sum_{i=2}^{d-1} s_i \boldsymbol{a}_i.$$

Now we add $\boldsymbol{a}_1 + \boldsymbol{b}_1$ to both sides of this equality and get

$$\boldsymbol{a}_1 + (1 - (\beta - t_1))\boldsymbol{b}_1 + \sum_{i=2}^{d-1} t_i \boldsymbol{b}_i = \boldsymbol{b}_1 + (1 - (\alpha - s_1))\boldsymbol{a}_1 + \sum_{i=2}^{d-1} s_i \boldsymbol{a}_i. \tag{4}$$

Note that the point $(1-(\beta-t_1))\boldsymbol{b}_1+\sum_{i=2}^{d-1} t_i\boldsymbol{b}_i$ belongs to the interior of the parallelepiped $P(B)$. Indeed, this is a linear combination of the vectors $\boldsymbol{b}_1,\dots,\boldsymbol{b}_{d-1}$ and all coefficients of this combination are from the interval $(0,1)$. Hence, the left-hand side of equality (4) is an interior point of the parallelepiped $\boldsymbol{a}_1+P(B)$. Analogously, the left-hand side is an interior point of $\boldsymbol{b}_1+P(A)$. Thus, the parallelepipeds $\boldsymbol{a}_1+P(B)$ and $\boldsymbol{b}_1+P(A)$ possess a common interior point. It remains to denote $\boldsymbol{a}=\boldsymbol{a}_1, \boldsymbol{b}=\boldsymbol{b}_1$, which completes the proof. □

Proof of Proposition 6 As usual, G denotes a polyhedral set, $\mathcal{T}$ is a self-affine tiling of G, K is a corner of G at a stationary vertex $\boldsymbol{v}$, and $T\in\mathcal{T}$ is the element covering $\boldsymbol{v}$. After possible iteration of the tiling it can be assumed that the size of T is as small as needed. We prove the following statement which immediately implies Proposition 6:

Claim 1 *For each $j=1,\dots,d$, the corner K contains all faces of dimension j of the convex corner* $\mathrm{co}(K)$ *and all those faces are simple.*

Applying this statement for $j=d$ we obtain that K contains the whole convex cone $\mathrm{co}(K)$, which is, moreover, simple. So $K=\mathrm{co}(K)$, and hence K is convex and simple. The proof of this statement is by induction in the dimension j.

$\underline{j=1}$. In this case the faces are the extreme edges of $\mathrm{co}(K)$, which, as we know, coincide with the extreme edges of K. So, in this case the statement is true.

$\underline{j\to j+1}$. Assume the statement holds for some $j\le d-1$. Suppose the converse: the corner $\mathrm{co}(K)$ possesses a face L of dimension $j+1$ that does not lie in K. Then the set $S=L\setminus K$ is nonempty. Since $\mathrm{co}(K)$ is small enough, it follows that S is a corner with apex $\boldsymbol{v}$.

Let H be an arbitrary j-dimensional face of L. It is also a face of $\mathrm{co}(G)$ and by the inductive assumption, H is simple and is contained in K. For an arbitrary edge of L, we take its directing point $\boldsymbol{a}$, for which the segment $[\boldsymbol{v},\boldsymbol{a}]$ (the directing segment) is the biggest by inclusion segment of that edge contained in T. Invoking now Lemma 2 we conclude that the j-dimensional parallelepiped $P(H)$ generated by j directing segments of the face H is contained in T. By Corollary 2, for every directing segment $[\boldsymbol{v},\boldsymbol{a}]$ of L, the set $T+(\boldsymbol{a}-\boldsymbol{v})$ is an element of $\mathcal{T}$ and hence, this set lies in K. Therefore, $P(H)+(\boldsymbol{a}-\boldsymbol{v})\subset K$. Thus, for every facet H of L and for every directing point $\boldsymbol{a}$ of L, the shifted parallelepiped $P(H)+(\boldsymbol{a}-\boldsymbol{v})$ lies in K. Now we consider two cases.

1) L is a simple cone. Then the $j+1$ shifted parallelepipeds $P(H)+(\boldsymbol{a}-\boldsymbol{v})$, where H is a facet of L and $\boldsymbol{a}\notin H$ is a directing point, form the boundary of the $(j+1)$-dimensional parallelepiped spanned by the directing segments of L. This boundary lies in K and must intersect the open cone S. Hence $K\cap S\neq\emptyset$, which contradicts the definition of S. Thus, $S=\emptyset$ and so $L\subset K$.

2) L is not simple, i.e., it has at least $j+2$ edges. Let us show that this case is impossible. Applying Lemma 3 to the cone L, we find two of its j-dimensional faces A, B and two of its different edges a, b with the directing points $\boldsymbol{a}\in a, \boldsymbol{b}\in b$ such that the parallelepipeds $P(A)+(\boldsymbol{a}-\boldsymbol{v})$ and $P(B)+(\boldsymbol{b}-\boldsymbol{v})$ have a common interior point. The first one is contained in the element $T+(\boldsymbol{a}-\boldsymbol{v})$, the second one is in the

element $T + (\boldsymbol{b} - \boldsymbol{v})$. Moreover, since $P(A)$ has a layer in T (Lemma 1), it follows that those two elements have a common interior point. Hence, they coincide, which is impossible, because $\boldsymbol{a} - \boldsymbol{v} \neq \boldsymbol{b} - \boldsymbol{v}$, since those vectors are in different edges of L.

This completes the proof of the inductive step, which proves the proposition. □

Thus, we have proved that for every stationary vertex of G, the corresponding corner of G is convex and simple. Since G has at least one convex corner, it follows that it possesses a simple convex corner. Now we can apply the main result of [21, Theorem 1.9]: if a set admits a self-affine tiling and has at least one convex polyhedral corner, then it is affinely similar to a union of integer shifts of the unit cube. The next step is to show that a self-affine tiling of this set contains a tiling of a parallelepiped.

Lemma 4 *Let a set G be a union of integer shifts of a unit cube. Suppose G has a convex simple corner K at its extreme vertex and has a self-affine tiling $\mathcal{T}$; then there is an iteration of the tiling $\mathcal{T}$ whose subset forms a tiling of some parallelepiped with the same corner K.*

Proof By iterating the tiling, we may assume that M is a diagonal matrix and the diameter of T is less than one. Let P_0 be the parallelepiped spanned by the edges $[\boldsymbol{v}, \boldsymbol{b}_s]$, $s = 1, \dots, d$, from Lemma 2. Let us show that the parallelepiped $P = MP_0$ is one we are looking for. Since K is simple, we identify it with $\mathbb{R}^d_+$. The image of an integer shift of the unit cube under the action of M^{-1} will be called *brick*. Since G consists of shifts of the unit cube, it follows that T consists of bricks.

We have $T + \boldsymbol{b}_i \in \mathcal{T}$, therefore, the interior of the parallelepiped $P_0 + \boldsymbol{b}_i$, being a part of the element $T + \boldsymbol{b}_i$, does not intersect T. Denote by $\{\boldsymbol{e}_j\}_{j=1}^d$ the canonical basis of $\mathbb{R}^d$ and by h_j the lengths of the edges of P. Applying the similarity of G and T we conclude that the interior of the parallelepiped $P + h_1\boldsymbol{e}_i$ does not intersect G. Among all unit cubes forming G we choose the "highest" ones with respect to the ith coordinate (whose center has the largest ith coordinate among all cubes). There are two possible cases.

Suppose none of the highest cubes intersect the axis Ox_i; then the element $T' \in \mathcal{T}$ containing the vertex $h_i\boldsymbol{e}_i$ of P possesses a brick higher than that vertex, i.e., the ith coordinate of the center of that brick exceeds h_i. Since the diameter of T' is smaller than one, that brick is contained in $P + h_i\boldsymbol{e}_i$, which is impossible. Therefore, for every i, there exists the highest with respect to the ith coordinate cube in G intersecting the axis Ox_i. Hence, T also has the highest brick (denote it by B_i) which intersects the axis Ox_i.

If some element $T + \boldsymbol{a} \in \mathcal{T}$ intersecting the interior of P has a point whose ith coordinate exceeds h_i, then the brick $B_i + \boldsymbol{a}$ is above the level $x_i = h_i$. On the other hand, all other coordinates of that brick are on the segments $[0, h_k]$, $k \neq i$. Hence, $B_i + \boldsymbol{a} \subset P + h_i\boldsymbol{e}_i$. Therefore, the brick $B_i + \boldsymbol{a}$ is out of G, which is impossible, since this is a part of the element $T + \boldsymbol{a}$. Thus, all points of each element intersecting P have the ith coordinate at most h_i. Applying this argument for all i, we see that every element intersecting P lies in P. Hence, those elements form a tiling of P. □

Remark 2 If among the unit cubes composing the set G there is at least one separated from others, then Lemma 4 is obvious. Indeed, if all elements of some iteration of $\mathcal{T}$

are of diameter smaller than one, then all the elements intersecting the separated cube do not intersect the others. Hence, they form a tiling of that cube. However, if there are no separated cubes in G, then Lemma 4 is less obvious.

Proof of Theorem 2.1 Let G be a polyhedral set and $\mathcal{T}$ be its self-affine tiling. By Proposition 5, there is some power of this tiling for which G has a stationary vertex $\boldsymbol{v}$. Proposition 6 asserts that the corner K at $\boldsymbol{v}$ is convex and simple. Hence, by [21, Thm. 1.9], the set G is equivalent to a union of several integer shifts of the unit cube. All corners of this set are rectangular, i.e., equal to the cone $\mathbb{R}^d_+$. Now we invoke Lemma 4 and conclude that there is a power of the tiling $\mathcal{T}$ whose subset forms a tiling of a parallelepiped P with the corner K. Hence, P is a rectangular parallelepiped. Since the element T covering the vertex $\boldsymbol{v}$ is affinely similar to G and has the same corner K, it follows that after an affine transform with a diagonal matrix, T becomes a union of integer shifts of a unit cube, and several integer shifts of T cover a parallelepiped. Replacing each unit cube by its center we obtain a discrete tiling of a parallelepiped in $\mathbb{Z}^d$. According to the main result of [13], every discrete tiling is a direct product of univariate discrete tilings of a segment of integer numbers. Therefore, $\mathcal{T}$ is a direct product of d tilings of a segment. Applying the classification of univariate tilings of a segment [22, Thm. 8] completes the proof. □

7 Proof of Theorem 2.2

Let a polyhedral set G admit a self-affine tiling $\mathcal{T}$. If G is an integral self-affine tile with standard digit set, then $\mathcal{T} = \{M^{-1}(G + \boldsymbol{s}) \mid \boldsymbol{s} \in D\}$, M is an integer expanding matrix, D is a set of digits, i.e., complete set of coset representatives $\mathbb{Z}^d/M\mathbb{Z}^d$.

Proof of Theorem 2.2 Applying Theorem 2.1 we obtain that G is a direct product of d sets of the form (2). Hence, G is equivalent to a union of disjoint integer shifts of the unit cube. For every $n \geq 2$, recall that the nth iteration $\mathcal{T}^n$ of the tiling $\mathcal{T}$ is defined by the matrix M^n and by the set of digits $D_n = D + MD + \cdots + M^{n-1}D$. Clearly, all the elements of D_n are from different quotient classes of $\mathbb{Z}^d/M^n\mathbb{Z}^d$. If n is large enough, then the diameter of the set $M^{-n}G$ is smaller than one. Hence, each element is contained in one of the unit cubes composing G.

If G contains more than one unit cube, then we take two of them C and C'. We have $C' = C + \boldsymbol{a}$, where $\boldsymbol{a} \in \mathbb{Z}^d$. Let $\boldsymbol{u}$ be an arbitrary vertex of C and $T \in \mathcal{T}^n$ be an element of partition containing $\boldsymbol{u}$. In the partial order defined by the corner of the cube C at $\boldsymbol{u}$ the point $\boldsymbol{u}$ is the minimal point of C. Since $T \subset C$ it follows that $\boldsymbol{u}$ is the minimal point of T. The vertex $\boldsymbol{u} + \boldsymbol{a}$ of the cube C' is covered by another element $T' \in \mathcal{T}^n$. Let $T' = T + \boldsymbol{b}$, $\boldsymbol{b} \in \mathbb{Z}^d$. By the same argument we show that $\boldsymbol{u} + \boldsymbol{a}$ is the minimal point of T' with respect to the same order. Since the parallel translation respects minimal points, we have $\boldsymbol{u} + \boldsymbol{a} = \boldsymbol{u} + \boldsymbol{b}$ and so $\boldsymbol{a} = \boldsymbol{b}$. Thus, $T' = T + \boldsymbol{a}$. Since $\mathcal{T}^n = \{M^{-n}(G + \boldsymbol{s}) \mid \boldsymbol{s} \in D_n\}$, we have $T = M^{-n}(G + \boldsymbol{s}_1)$, $T' = M^{-n}(G + \boldsymbol{s}_2)$, where $\boldsymbol{s}_1, \boldsymbol{s}_2 \in D_n$. Thus, $T' = T + \boldsymbol{a} = T + M^{-n}(\boldsymbol{s}_2 - \boldsymbol{s}_1)$. This means that $\boldsymbol{a} = M^{-n}(\boldsymbol{s}_2 - \boldsymbol{s}_1)$ and so $\boldsymbol{s}_2 - \boldsymbol{s}_1 = M^n\boldsymbol{a}$, which is impossible since $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are from different quotient classes of $\mathbb{Z}^d/M^n\mathbb{Z}^d$. □

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