Guest Editors' Foreword

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(photo courtesy of Bojan Mohar)

The volume you are holding in your hand is in memory of Branko Grünbaum—one of the giants and founders of Discrete Geometry as we know it today. Throughout his 60-year career, Branko (sometimes almost single-handedly) created and revamped major subfields of this fascinating field. That list includes the theory of convex polytopes, abstract polytopes, Helly's theorem and its relatives, arrangements and spreads, patterns and tilings, and the geometric graph theory, just to name a few. The papers presented in this volume attest to the fact that these areas are very much alive and continue to thrive!

Describing Branko's mathematical influence (which includes more than 200 descendants) is a daunting task, especially because very few can claim the mathemat-

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ical depth and breadth that Branko Grünbaum had. So instead of trying to encompass the unencompassable, we will try to concentrate on a few areas that are closer to our hearts.

We start with the theory of convex polytopes, a huge, rapidly developing and vibrant area of research that would not even resemble what it is today were it not for Branko's 1967 book "Convex Polytopes". Steinitz's famous theorem from the early 1900s, or more precisely, Branko's version of that result, asserts that a simple graph is the graph of a 3-dimensional polytope if and only if it is planar and 3-connected (that is, removing any two vertices does not disconnect the graph). Despite being a fundamental result in discrete geometry and a beginning of so many research directions, the first readable account of its proof only appeared in Branko's book. In fact, according to Branko, his "*working through the Steinitz–Rademacher book in the summer of 1963, translating Steinitz's complicated process of establishing the Fundamentalsatz der konvexen Typen into the easy-to-follow proof of the graph-theoretical formulation*" is one of the things that led to the creation of the book. Another impetus were Vic Klee's groundbreaking papers on the Dehn–Sommerville equations and the Upper Bound Conjecture.

In 1964–1965, Branko gave a course on the combinatorics of convex polytopes, for which he prepared lecture notes. In just 2 years these lecture notes resulted in a book that even today remains a major reference for polytope theory. The book described both classic and completely new techniques and results, such as the theory of Gale diagrams developed by Micha A. Perles as the book was nearing completion.

As Branko wrote in the preface to the first edition, "*While the author is confident that the current surge of interest and research in the combinatorial properties of convex polytopes will continue and will render the book obsolete within a few years, he may only hope that the book itself will contribute to the revitalization of the field and act as a stimulant to further research.*" This hope has been very much realized. For proof, consider the second edition of the book, published in 2003. At the end of each chapter there is a section of notes summarizing a selection of the relevant outstanding discoveries made from 1967 to 2003. Major breakthroughs since 2003 include, for instance, Francisco Santos' disproof (in 2010) of the Hirsch conjecture on the diameter of polytopes.

This volume has several papers on polytopes. They are *Treetopes and their graphs* by David Eppstein, *Complexity yardsticks for f -vectors of polytopes and spheres* by Eran Nevo, and *Unconditional reflexive polytopes* by Florian Kohl, McCabe Olsen, and Raman Sanyal. This is just a tiny sample of what is currently happening in the polytope world. Many new breakthroughs and discoveries are constantly made. For instance, just a couple of weeks before we closed this volume, Branko's great-great-granddaughter to be—Lei Xue, a current graduate student at the University of Washington—proved the Lower Bound Conjecture for general polytopes from Branko's 1967 book. It was too late to include her preprint in this volume.

"Convex Polytopes," as well as many of Branko's other writings, combined metric and combinatorial aspects of geometry and convexity. The interface between the continuous and the discrete plays a role in several papers of this volume, for instance *On polyatomic tomography over Abelian groups: some remarks on consistency, tree packings and complexity* by Peter Gritzmann and Barbara Langfeld, and *Triangula-*

tions and a discrete Brunn–Minkowski inequality in the plane by Károly J. Böröczky, Máté Matolcsi, Imre Z. Ruzsa, Francisco Santos, and Oriol Serra.

Another extremely influential paper is a survey by Branko, in collaboration with Ludwig Danzer and Vic Klee, titled “Helly’s theorem and its relatives”, from 1963. Helly’s theorem (1913) states that if we are given $n \geq d + 1$ convex closed sets in $\mathbb{R}^d$ with the property that every $d + 1$ of them have a non-empty intersection, then all of these sets have a non-empty intersection. This result, which many consider to be the beginning of modern Discrete Geometry, has a tremendous amount of relatives, extensions, and generalizations, among them a recent counter-example (2015) by Florian Frick to the topological Tverberg conjecture. Similarly to Branko’s book on polytopes, the paper by Danzer, Grünbaum, and Klee played a catalytic role in many of these developments: it was encyclopedically comprehensive and advertised many unsolved problems.

In this volume, this area of Discrete Geometry is represented by papers *Theorems of Carathéodory, Helly, and Tverberg without dimension* by Karim Adiprasito, Imre Bárány, Nabil H. Mustafa, and Tamás Terpai; *Plus minus analogues for affine Tverberg type results* by Pavle Blagojević and Günter Ziegler; and *Intersection patterns of planar sets* by Gil Kalai and Zuzana Patáková.

Branko’s works on finding generalizations of regular polytopes, e.g., his notion of polystroma and his construction of the 11-cell, and in particular his paper “Regularity of graphs, complexes and designs” from 1978, along with works by Coxeter and Tits, laid the groundwork for what we know today as a very rich and beautiful theory of abstract polytopes. This volume features three papers on the topic: *On regular polytopes of 2-power order* by Dong-Dong Hou, Yan-Quan Feng, and Dimitri Leemans; *The assembly problem for alternating semiregular polytopes* by Barry Monson and Egon Schulte; and *An exploration of locally spherical regular hypertopes* by Maria Elisa Fernandes, Dimitri Leemans, and Asia Ivić Weiss.

In fact, exploring symmetry played a huge role in Branko’s research throughout his whole career, see his papers “The search for symmetric Venn diagrams,” “Levels of orderliness: global and local symmetry,” “Hypersymmetric tiles” (with G. C. Shephard), “Can all tiles of a tiling have five-fold symmetry?” (with L. Danzer and G. C. Shephard). In addition to the papers on abstract polytopes mentioned above, this volume features two more papers that involve symmetry: *Pairing symmetries for Euclidean and spherical frameworks* by Katie Clinch, Anthony Nixon, Bernd Schulze, and Walter Whiteley on rigidity of frameworks and *The Schläfli fan* by Michael Joswig, Marta Panizzut, and Bernd Sturmfels on tropical geometry.

Branko’s influential monograph on arrangements of hyperplanes and other geometric objects, his monograph “Lectures on Lost Mathematics,” and his survey articles, each opened fruitful areas of research. One of these areas are Configurations. This volume contains a contribution *Chiral astral realizations of cyclic 3-configurations* by Leah Berman, Philip Deorsey, Jill R. Faudree, Tomáš Pisanski, and Arjana Žitnik which shows some of the new developments in this area.

Let us also mention a paper by Branko (in collaboration with L. Danzer) that simultaneously solved a problem due to Erdős and a problem due to Klee. The main result of their paper asserted that every collection of more than 2^d points in $\mathbb{R}^d$ determines at least one obtuse angle. The proof used an idea going back to Minkowski and was

so elegant that it made a chapter in Martin Aigner and Günter Ziegler's "Proofs from THE BOOK." Our volume features *A one-page solution of a problem of Erdős and Purdy* by Rom Pinchasi and Alexandr Polyanskii, which the authors of the paper hope is a very good candidate for the "book proof" of this conjecture.

Branko contributed greatly to the development of geometric and topological graph theory with his work on planar graphs and graphs of higher-dimensional polytopes. He was among the first to realize that the classical coloring problems exemplified with the Four Color Conjecture can be asked in a more general way. This led to notions of "Grünbaum colorings" (geometric dual notion to nowhere-zero 4-flows) and acyclic colorings. His conjectures, some of which are still open today, motivated and continue to motivate research in graph theory. In addition to some of the aforementioned papers on polytopal graphs (e.g., the paper by David Eppstein), this volume also contains the following two contributions: *Ordered and convex geometric trees with linear extremal function* by Zoltán Füredi, Alexandr Kostochka, Dhruv Mubayi, and Jacques Verstraëte, and *Embeddability of arrangements of pseudocircles and graphs on surfaces* by Éric Colin de Verdière, Carolina Medina, Edgardo Roldán-Pensado, and Gelasio Salazar.

The results and directions we have mentioned so far form a small sample of Branko's interests. There are tons of conjectures and problems that bear Branko's name, and quite a few directions of research that grew out of his papers and books. In the pre-Internet era, researchers in discrete geometry who wanted to get a head start on a question would often start by writing Branko to see what was known about the problem. He kept cabinets full of index cards in his office with notes on articles, and he always seemed to know just the right references to get one off and rolling.

Throughout his career, Branko put forth a clear (perhaps even radical) argument against the dominance of certain approaches and trends in mathematics. He criticized trends for abstraction at the expense of concrete examples and problems, and the trend of sectarianism; he criticized the loss of beautiful mathematics that does not fit the mainstream. He wrote, for example: "*it has become acceptable and even fashionable to limit one's total knowledge and interests to the same narrow area, a development I consider disastrous.*"

Let us finish with a few sentences that János Pach shared with us recently. "*As usual, when I came across a question related to polytopes or tilings, or when I needed a quick opinion about a paper in the subject that was submitted to Discrete and Computational Geometry, I wrote to Branko. He had a great memory and he was incredibly fast and sharp. I completely trusted his judgment and instincts until the very end.*"

We believe this special issue of the Discrete and Computational Geometry is a humble contribution to the celebration of life and work of our mentor, colleague, and friend. We hope the readers will enjoy it.

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