

On the Densest Packing of Polycylinders in Any Dimension

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Abstract Using transversality and a dimension reduction argument, a result of Bezdek and Kuperberg is applied to polycylinders, showing that the optimal packing density of $\mathbb{D}^2 \times \mathbb{R}^n$ equals $\pi/\sqrt{12}$ for all natural numbers n .

Keywords Polycylinders · Packing · Density · Slicing

1 Introduction

Open and closed Euclidean unit n -balls will be denoted by \mathbb{B}^n and \mathbb{D}^n respectively. The closed unit interval is denoted by \mathbb{I} . A general polycylinder C is a set congruent to $\prod_{i=1}^m \lambda_i \mathbb{D}^{k_i}$ in $\mathbb{R}^{k_1+\dots+k_m}$, where λ_i is in $[0, \infty]$. For this article, the term polycylinder refers to the special case of an infinite polycylinder over a two-dimensional disk of unit radius. A *polycylinder* is a set congruent to $\mathbb{D}^2 \times \mathbb{R}^n$ in \mathbb{R}^{n+2} . A *polycylinder packing of \mathbb{R}^{n+2}* is a family $\mathcal{C} = \{C_i\}_{i \in I}$ of polycylinders $C_i \subset \mathbb{R}^{n+2}$ with mutually disjoint interiors. The *upper density* $\delta^+(\mathcal{C})$ of a packing \mathcal{C} of \mathbb{R}^n is defined to be

$$\delta^+(\mathcal{C}) = \limsup_{r \rightarrow \infty} \frac{\text{Vol}(\mathcal{C} \cap r\mathbb{B}^n)}{\text{Vol}(r\mathbb{B}^n)}.$$

The *upper packing density* $\delta^+(C)$ of an object C is the supremum of $\delta^+(\mathcal{C})$ over all packings \mathcal{C} of \mathbb{R}^n by C .

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This article proves the following sharp bound for the packing density of infinite polycylinders:

Theorem 1 $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) = \pi/\sqrt{12}$ for all natural numbers n .

Theorem 1 generalizes a result of Bezdek and Kuperberg [1] and improves on results that may be computed using a method of Fejes Tóth and Kuperberg [3], cf. [2,5]; it gives some of the first sharp upper bounds for packing density in high dimensions.

2 Transversality

This section introduces the required transversality arguments from affine geometry. A d -flat is a d -dimensional affine subspace of \mathbb{R}^n . The *parallel dimension* $\dim_{\parallel}\{F, \dots, G\}$ of a collection of flats $\{F, \dots, G\}$ is the dimension of their maximal parallel sub-flats. The notion of parallel dimension can be interpreted in several ways, allowing a modest abuse of notation.

- For a collection of flats $\{F, \dots, G\}$, consider their tangent cones at infinity $\{F_{\infty}, \dots, G_{\infty}\}$. The parallel dimension of $\{F, \dots, G\}$ is the dimension of the intersection of these tangent cones. This may be viewed as the limit of a rescaling process $\mathbb{R}^n \rightarrow r\mathbb{R}^n$ as r tends to 0, leaving only the scale-invariant information.
- For a collection of flats $\{F, \dots, G\}$, consider each flat as a system of linear equations. The corresponding homogeneous equations determine a collection of linear subspaces $\{F_{\infty}, \dots, G_{\infty}\}$. The parallel dimension is the dimension of their intersection $F_{\infty} \cap \dots \cap G_{\infty}$.

Two disjoint d -flats are *parallel* if their parallel dimension is d , that is, if every line in one is parallel to a line in the other.

Lemma 1 *A pair of disjoint n -flats in \mathbb{R}^{n+k} with $n \geq k$, has parallel dimension strictly greater than $n - k$.*

Proof Let F and G be such a pair. By homogeneity of \mathbb{R}^{n+k} , let $F = F_{\infty}$. As F_{∞} and G are disjoint, G contains a non-trivial vector \mathbf{v} such that $G = G_{\infty} + \mathbf{v}$ and \mathbf{v} is not in $F_{\infty} + G_{\infty}$. It follows that

$$\begin{aligned} \dim(\mathbb{R}^{n+k}) &\geq \dim(F_{\infty} + G_{\infty} + \text{span}(\mathbf{v})) > \dim(F_{\infty} + G_{\infty}) \\ &= \dim(F_{\infty}) + \dim(G_{\infty}) - \dim(F_{\infty} \cap G_{\infty}). \end{aligned}$$

Count dimensions to find $n + k > n + n - \dim_{\parallel}(F_{\infty}, G_{\infty})$. □

Corollary 1 *A pair of disjoint n -flats in \mathbb{R}^{n+2} has parallel dimension at least $n - 1$.*

3 Dimension Reduction

3.1 Pairwise Foliations

The *core* a_i of a polycylinder C_i congruent to $\mathbb{D}^2 \times \mathbb{R}^n$ in \mathbb{R}^{n+2} is the distinguished n -flat defining C_i as the set of points at most distance 1 from a_i . In a packing \mathcal{C} of

\mathbb{R}^{n+2} by polycylinders, Corollary 1 shows that, for every pair of polycylinders C_i and C_j , one can choose parallel $(n - 1)$ -dimensional subflats $b_i \subset a_i$ and $b_j \subset a_j$ and define a product foliation

$$\mathcal{F}^{b_i, b_j} : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^3$$

with \mathbb{R}^3 leaves that are orthogonal to b_i and to b_j . Given a point x in a_i , there is a distinguished \mathbb{R}^3 leaf $F_x^{b_i, b_j}$ that contains the point x . The foliation \mathcal{F}^{b_i, b_j} restricts to foliations of C_i and C_j with right-circular-cylinder leaves.

3.2 The Dirichlet Slice

In a packing \mathcal{C} of \mathbb{R}^{n+2} by polycylinders, the *Dirichlet cell* D_i associated with a polycylinder C_i is the set of points in \mathbb{R}^{n+2} which lie no further from C_i than from any other polycylinder in \mathcal{C} . The Dirichlet cells of a packing partition \mathbb{R}^{n+2} , as $C_i \subset D_i$ for all polycylinders C_i . To bound the density $\delta^+(\mathcal{C})$, it is enough to fix an i in I and consider the density of C_i in D_i .

Consider the following slicing of the Dirichlet cell D_i . Given a fixed polycylinder C_i in a packing \mathcal{C} of \mathbb{R}^{n+2} by polycylinders and a point x on the core a_i , the plane p_x is the 2-flat orthogonal to a_i and containing the point x . The *Dirichlet slice* d_x is the intersection of D_i and p_x .

Note that p_x is a sub-flat of $F_x^{b_i, b_j}$ for all j in I .

3.3 Bezdek–Kuperberg Bound

For any point x on the core a_i of a polycylinder C_i , the results of Bezdek and Kuperberg [1] apply to the Dirichlet slice d_x .

Lemma 2 *A Dirichlet slice is convex and, if bounded, a parabola-sided polygon.*

Proof Construct the Dirichlet slice d_x as an intersection. Define d^j to be the set of points in p_x which lie no further from C_i than from C_j . Then the Dirichlet slice d_x is realized as

$$d_x = \left\{ \bigcap_{j \in I} d^j \right\}.$$

Each arc of the boundary of d_x in p_x is given by an arc of the boundary of some d^j in p_x . The boundary of d^j in p_x is the set of points in p_x equidistant from C_i and C_j . Since the foliation \mathcal{F}^{b_i, b_j} is a product foliation, the arc of the boundary of d^j in p_x is also the set of points in p_x equidistant from the leaf $C_i \cap F_x^{b_i, b_j}$ of $\mathcal{F}^{b_i, b_j}|_{C_i}$ and the leaf $C_j \cap F_x^{b_i, b_j}$ of $\mathcal{F}^{b_i, b_j}|_{C_j}$. This reduces the analysis to the case of a pair of cylinders in \mathbb{R}^3 . From [1], it follows that d^j is convex and the boundary of d_j in p_x is a parabola; the intersection of such sets d^j in p_x is convex, and a parabola-sided polygon if bounded. □

Let $S_x(r)$ be the circle of radius r in p_x centered at x .

Lemma 3 *The vertices of d_x are not closer to $S_x(1)$ than the vertices of a regular hexagon circumscribed about $S_x(1)$.*

Proof A vertex of d_x occurs where three or more polycylinders are equidistant, so the vertex is the center of a $(n + 2)$ -ball B tangent to three polycylinders. Thus B is tangent to three disjoint unit $(n + 2)$ -balls B_1, B_2, B_3 . By projecting into the affine hull of the centers of B_1, B_2, B_3 , it is immediate that the radius of B is no less than $2/\sqrt{3} - 1$. \square

Lemma 4 *Let y and z be points on the circle $S_x(2/\sqrt{3})$. If each of y and z is equidistant from C_i and C_j , then the angle yxz is smaller than or equal to $2 \arccos(\sqrt{3} - 1) = 85.8828 \dots^\circ$.*

Proof Following [1, 4], the existence of a supporting hyperplane of C_i that separates $\text{int}(C_i)$ from $\text{int}(C_j)$ suffices. \square

In [1], it is shown that planar objects satisfying Lemmas 2, 3 and 4 have area no less than $\sqrt{12}$. As the bound holds for all Dirichlet slices, it follows that $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ in \mathbb{R}^{n+2} . The product of the dense disk packing in the plane with \mathbb{R}^n gives a polycylinder packing in \mathbb{R}^{n+2} that achieves this density. Combining this with the result of Thue [6] for $n = 0$ and the result of Bezdek and Kuperberg [1] for $n = 1$, Theorem 1 follows.

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