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Triangle-Free Geometric Intersection Graphs with No Large Independent Sets

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Abstract It is proved that there are triangle-free intersection graphs of line segments in the plane with arbitrarily small ratio between the maximum size of an independent set and the total number of vertices.

Keywords Intersection graph · Line segments · Triangle-free · Maximum independent set · Fractional chromatic number

1 Introduction

Pawlik et al. [7] proved that there are triangle-free intersection graphs of line segments in the plane with arbitrarily large chromatic number. The graphs they construct have independent sets containing more than $1/3$ of all the vertices. It has been left open whether there is a constant $c > 0$ such that every triangle-free intersection graph of n segments in the plane has an independent set of size at least cn. Fox and Pach [3] conjectured a much more general statement, that K_k-free intersection graphs of curves in the plane have linear-size independent sets, for every k. This would imply a well-known conjecture that k-quasi-planar graphs (graphs drawn in the plane so that no k edges cross each other) have linearly many edges [5], which is proved up to $k = 4$ [1].

In this note, I resolve the independent set problem in the negative, proving the following strengthening of the result of Pawlik et al.:

Theorem *There are triangle-free segment intersection graphs with arbitrarily small ratio between the maximum size of an independent set and the total number of vertices.*

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The constructions presented in the next two sections give rise to triangle-free intersection graphs of n segments in the plane with maximum independent set size $\Theta(n/\log\log n)$.

2 Construction

Pawlik et al. [7] construct, for $k \geqslant 1$, a triangle-free graph G_k and a family $\mathcal{P}_k$ of subsets of $V(G_k)$, called *probes*, with the following properties:

(i) $|\mathcal{P}_k| = 2^{2^{k-1}-1}$,
(ii) every member of $\mathcal{P}_k$ is an independent set of G_k,
(iii) for every proper coloring of the vertices of G_k, there is a probe $P \in \mathcal{P}_k$ such that at least k colors are used on the vertices in P.

They are built by induction on k, as follows. The graph G_1 has just one vertex v, and $\mathcal{P}_1$ has just one probe $\{v\}$. For $k \geqslant 2$, first, a copy $(G, \mathcal{P})$ of $(G_{k-1}, \mathcal{P}_{k-1})$ is taken. Then, for every probe $P \in \mathcal{P}$, another copy $(G_P, \mathcal{P}_P)$ of $(G_{k-1}, \mathcal{P}_{k-1})$ is taken. There are no edges between vertices from different copies. Finally, for every probe $P \in \mathcal{P}$ and every probe $Q \in \mathcal{P}_P$, a new vertex d_Q connected to all vertices in Q, called the *diagonal* of Q, is added. The resulting graph is G_k. The family of probes $\mathcal{P}_k$ is defined by

$$\mathcal{P}_k = \big\{P \cup Q\colon\ P \in \mathcal{P} \text{ and } Q \in \mathcal{P}_P\big\} \cup \big\{P \cup \{d_Q\}\colon\ P \in \mathcal{P} \text{ and } Q \in \mathcal{P}_P\big\}.$$

It is easy to check that the graph G_k is indeed triangle-free and the conditions (i)–(iii) are satisfied for $(G_k, \mathcal{P}_k)$—see [7] for details. It is also shown in [7] how the graph G_k is represented as a segment intersection graph.

I will show that there is an assignment w_k of positive integer weights to the vertices of G_k with the following properties:

(iv) the total weight of G_k is $\frac{k+1}{2} \cdot 2^{2^{k-1}-1}$,
(v) for every independent set I of G_k, the number of probes $P \in \mathcal{P}_k$ such that $P \cap I \neq \emptyset$ is at least the weight of I.

Once this is achieved, the proof of the theorem of this paper follows easily. Namely, it follows from (i) and (v) that every independent set I of G_k has weight at most $2^{2^{k-1}-1}$. We can take the representation of G_k as a segment intersection graph and replace every segment representing a vertex $v \in V(G_k)$ by $w_k(v)$ parallel segments lying very close to each other, so as to keep the property that any two segments representing vertices $u, v \in V(G_k)$ intersect if and only if $uv \in E(G_k)$. It follows from (iv) that the family of segments obtained this way has size $\frac{k+1}{2} \cdot 2^{2^{k-1}-1}$, while every independent set of its intersection graph has size at most $2^{2^{k-1}-1}$.

The assignment w_k of weights to the vertices of G_k is defined by induction on k, following the inductive construction of $(G_k, \mathcal{P}_k)$. The weight of the only vertex of G_1 is set to 1. This clearly satisfies (iv) and (v). For $k \geqslant 2$, let G, $\mathcal{P}$, G_P, $\mathcal{P}_P$ and d_Q be defined as in the inductive step of the construction of $(G_k, \mathcal{P}_k)$. Let $p = |\mathcal{P}_{k-1}| = 2^{2^{k-2}-1}$. The weights w_k of the vertices of G are their original weights w_{k-1} in G_{k-1} multiplied by p. The weights w_k of the vertices of every G_P are equal to their original weights w_{k-1} in G_{k-1}. The weight w_k of every diagonal d_Q is set to 1. It remains to prove that (iv) and (v) are satisfied for $(G_k, \mathcal{P}_k, w_k)$ assuming that they hold for $(G_{k-1}, \mathcal{P}_{k-1}, w_{k-1})$.

The proof of (iv) is straightforward:

$$w_k(G_k) = w_k(G) + \sum_{P\in\mathcal{P}} \big(w_k(G_P) + |\mathcal{P}_P|\big) = 2pw_{k-1}(G_{k-1}) + p^2 = \tfrac{k+1}{2}\cdot 2^{2^{k-1}-1}.$$

For the proof of (v), let I be an independent set in G_k. Let $\mathcal{I} = \{P \in \mathcal{P}\colon\ P\cap I \neq \emptyset\}$. For every probe $P \in \mathcal{P}$, define

$$\begin{aligned} &\mathcal{I}_P = \{Q \in \mathcal{P}_P\colon\ Q\cap I \neq \emptyset\}, && \mathcal{P}'_P = \{P\cup Q\colon\ Q\in\mathcal{P}_P\}\cup\{P\cup\{d_Q\}\colon\ Q\in\mathcal{P}_P\},\\ &D_P = \{d_Q\colon\ Q\in\mathcal{P}_P\}, && \mathcal{I}'_P = \{P'\in\mathcal{P}'_P\colon\ P'\cap I\neq\emptyset\}.\end{aligned}$$

By the induction hypothesis, we have

$$w_k(V(G)\cap I) \leqslant p|\mathcal{I}|, \qquad w_k(V(G_P)\cap I)\leqslant |\mathcal{I}_P|.$$

Suppose $P \in \mathcal{I}$. It follows that $(P\cup Q)\cap I\neq\emptyset$ and $(P\cup\{d_Q\})\cap I\neq\emptyset$ for every $Q\in\mathcal{P}_P$. Hence $|\mathcal{I}'_P| = |\mathcal{P}'_P| = 2p$. Moreover, we have $d_Q\notin I$ whenever $Q\in\mathcal{I}_P$, because d_Q is connected to all vertices in Q, one of which belongs to I. Hence

$$w_k(V(G_P)\cap I) + w_k(D_P\cap I) \leqslant |\mathcal{I}_P| + |\mathcal{P}_P\smallsetminus\mathcal{I}_P| = |\mathcal{P}_P| = p.$$

Now, suppose $P\in\mathcal{P}\smallsetminus\mathcal{I}$. If $Q\in\mathcal{I}_P$, then $(P\cup Q)\cap I\neq\emptyset$, $d_Q\notin I$ (by the same argument as above), and $(P\cup\{d_Q\})\cap I=\emptyset$. If $Q\in\mathcal{P}_P\smallsetminus\mathcal{I}_P$, then $(P\cup Q)\cap I=\emptyset$, and $(P\cup\{d_Q\})\cap I\neq\emptyset$ if and only if $d_Q\in I$. Hence

$$w_k(V(G_P)\cap I) + w_k(D_P\cap I) \leqslant |\mathcal{I}_P| + |D_P\cap I| = |\mathcal{I}'_P|.$$

To conclude, we gather all the inequalities and obtain

$$\begin{aligned} w_k(I) &= w_k(V(G)\cap I) + \sum_{P\in\mathcal{P}}\big(w_k(V(G_P)\cap I) + w_k(D_P\cap I)\big)\\ &\leqslant p|\mathcal{I}| + \sum_{P\in\mathcal{I}} p + \sum_{P\in\mathcal{P}\smallsetminus\mathcal{I}} |\mathcal{I}'_P| = \sum_{P\in\mathcal{I}}|\mathcal{I}'_P| + \sum_{P\in\mathcal{P}\smallsetminus\mathcal{I}}|\mathcal{I}'_P| = \sum_{P\in\mathcal{P}}|\mathcal{I}'_P|.\end{aligned}$$

3 Improved Construction

Pawlik et al. [7] define also a graph $\tilde{G}_k$, which arises from $(G_k, \mathcal{P}_k)$ by adding, for every probe $P\in\mathcal{P}_k$, a diagonal d_P connected to all vertices in P. This is the smallest triangle-free segment intersection graph known to have chromatic number greater than k. Define the assignment $\tilde{w}_k$ of weights to the vertices of $\tilde{G}_k$ so that $\tilde{w}_k$ is equal to w_k on the vertices of G_k and $\tilde{w}_k(d_P) = 1$ for every $P\in\mathcal{P}_k$. Let I be an independent set in $\tilde{G}_k$. Let $\mathcal{I} = \{P\in\mathcal{P}_k\colon\ P\cap I\neq\emptyset\}$. Hence $d_P\notin I$ for $P\in\mathcal{I}$. It follows that

$$\begin{aligned}\tilde{w}_k(I) &= w_k(V(G_k) \cap I) + |\{d_P : P \in \mathcal{P}_k\} \cap I| \\ &\leqslant |\mathcal{I}| + |\mathcal{P}_k \setminus \mathcal{I}| = |\mathcal{P}_k| = 2^{2^{k-1}-1}, \\ \tilde{w}_k(\tilde{G}_k) &= w_k(G_k) + |\mathcal{P}_k| = \tfrac{k+3}{2} \cdot 2^{2^{k-1}-1}.\end{aligned}$$

The graph $\tilde{G}_k$ is the smallest one for which I can prove that it has a weight assignment such that the ratio between the maximum weight of an independent set and the total weight is at most $\frac{2}{k+3}$. It is not difficult to prove (e.g. using weak LP duality) that the assignment of weights $\tilde{w}_k$ to the vertices of $\tilde{G}_k$ is optimal (gives the least ratio) for this particular graph.

Both constructions give rise to triangle-free intersection graphs of n segments in the plane with maximum independent set size $\Theta(n/\log\log n)$. On the other hand, it follows from the result of McGuinness [4] that every triangle-free intersection graph of n segments has chromatic number $O(\log n)$ and maximum independent set size $\Omega(n/\log n)$.

4 Other Geometric Shapes

It is known that the graphs G_k and $\tilde{G}_k$ have intersection models by many other geometric shapes, for example, L-shapes, axis-parallel ellipses, circles, axis-parallel square boundaries [6] or axis-parallel boxes in $\mathbb{R}^3$ [2]. The result of this paper can be extended to those models for which every geometric object X representing a vertex of the intersection graph can be replaced by many pairwise disjoint objects approximating X. This is possible, for example, for intersection graphs of L-shapes, circles or axis-parallel square boundaries, but not for intersection graphs of axis-parallel ellipses or axis-parallel boxes in $\mathbb{R}^3$. The problem whether triangle-free intersection graphs of the latter kind of shapes have linear-size independent sets remains open.

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