A Finite Chiral 4-Polytope in $\mathbb{R}^4$

Javier Bracho · Isabel Hubard · Daniel Pellicer

Received: 24 October 2013 / Revised: 12 August 2014 / Accepted: 14 August 2014 /
Published online: 9 September 2014
© Springer Science+Business Media New York 2014

Abstract In this paper, we give an example of a chiral 4-polytope in projective 3-space. This example naturally yields a finite chiral 4-polytope in euclidean 4-space, giving a counterexample to a theorem previously published in the literature [Theorem 11.2 of McMullen (Discrete Comput Geom 32(1):1–35, 2004)].

Keywords Chiral polytopes · Realisations of polytopes · Projective polytopes

1 Introduction

Abstract polytopes are combinatorial structures that resemble convex polytopes. Of particular interest are those with high degree of symmetry, together with their realisations in euclidean spaces. Regular polytopes have maximum degree of symmetry, with their automorphism group being as big as possible. Chiral polytopes have maximum possible rotational symmetry, but no reflections. (See [9] for formal definitions of these concepts.)

A finite (abstract) n -polytope is said to be of full rank if it can be realised in euclidean n -space. In 1977 Grünbaum [7] gave the list of the 18 finite regular polyhedra of full rank and a few years later Dress [4,5] showed that the list was complete. Regular

J. Bracho · I. Hubard
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico city, Mexico
e-mail: isahubard@im.unam.mx

J. Bracho
e-mail: jbracho@im.unam.mx

D. Pellicer
Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Morelia, Mexico
e-mail: pellicer@matmor.unam.mx

Fig. 1 The graph $K_{4,4}$, with the edges coloured with 4 colours

polytopes of full rank have been studied by McMullen in [8]. Theorem 11.2 of the same paper claims that there are no chiral n -polytopes of full rank. Schulte [10, 11] independently proved this result for $n = 3$. In this paper we give a counterexample to the theorem, for $n = 4$.

Our approach follows [2, 3], where a vertex-transitive realisation of a finite polytope in $\mathbb{R}^{n+1}$ naturally corresponds to a projective polytope in $\mathbb{P}^n$. Hence, to give a chiral 4-polytope of full rank we construct a chiral 4-polytope in the projective space $\mathbb{P}^3$.

2 A Chiral 4-Polytope in $\mathbb{P}^3$

In this section we construct a chiral 4-polytope of Schläfli type $\{4, 3, 3\}$ [9, pp. 29, 30]. To this end, we start by considering the complete bipartite graph $K_{4,4}$. As it is shown in Fig. 1, we can colour the edges of $K_{4,4}$ with 4 colours in such a way that two edges of the same colour are not incident, so that each colour induces a perfect matching in the graph. We label the vertices of the graph $v_0, v_1, v_2, v_3, u_0, u_1, u_2, u_3$ as in the figure.

It is not difficult to see that with the colouring of $K_{4,4}$ given in Fig. 1, we can obtain a colourful 4-polytope $\mathcal{P}$ in the sense of [1] (and hence, $K_{4,4}$ is the 1-skeleton of $\mathcal{P}$). In fact, the 2-faces of $\mathcal{P}$ are the 4-cycles of $K_{4,4}$ that have exactly two colours. Hence, each of the alternating squares of two given colours is a 2-face of $\mathcal{P}$. The facets of $\mathcal{P}$ are defined by the subgraphs coloured with exactly three colours. Then, we can see that $\mathcal{P}$ has 4 facets and each of them is a cube. (In fact, we observe that the graph Q_3 of the cube is precisely $K_{4,4}$ minus a perfect matching.) The automorphisms of $\mathcal{P}$ are all the colour respecting automorphisms of $K_{4,4}$, that is, all the automorphisms of $K_{4,4}$ that induce a permutation on the colours (see [1]).

Therefore, $\mathcal{P}$ is isomorphic to the hemi-hypercube $\{4, 3, 3\}/2$ shown in Fig. 2, and hence we view $\mathcal{P}$ as living in the projective 3-space. (Recall that the hemi-hypercube in $\mathbb{P}^3$ can be understood as the quotient of the hypercube $\{4, 3, 3\}$ in $\mathbb{R}^4$ by the central inversion of $\mathbb{R}^4$ that identifies antipodal points of the hypercube. In fact the vertices of $\{4, 3, 3\}/2$, in homogeneous coordinates, have all entries ± 1 , and the edges are the geodesics between two vertices that differ in one entry. Furthermore, the four colours of our embedding of $K_{4,4}$ correspond to the four coordinates, or directions.) Observe that there is an edge between opposite vertices of a given facet of $\{4, 3, 3\}/2$. In fact, the edge has precisely the colour that is missing in that cube (see Fig. 2).

Fig. 2 The regular colouring of $K_{4,4}$ in $\mathbb{P}^3$

Fig. 3 The two chiral colourings of $K_{4,4}$ in $\mathbb{P}^3$

It is well known that the hemi-hypercube $\{4, 3, 3\}/2$ is a regular 4-polytope in the projective space. Note that symmetries of the hemi-hypercube correspond not only to the colour respecting automorphisms of the graph, but also to the isometries of $\mathbb{P}^3$ that preserve the graph.

Now consider the graph $K_{4,4}$ embedded in $\mathbb{P}^3$ as fixed, and observe that it admits two colourings as in Fig. 3. We shall refer to these colourings as *chiral colourings*. Note that the chiral colourings are enantiomorphic, in the sense that any reflection on a projective plane that preserves the embedding of $K_{4,4}$ sends one colouring to the other. They are combinatorially equivalent to the colouring of Figs. 1 and 2, because all its bi-coloured cycles are squares. (In fact, they correspond to the Petrie polygons of the hemi-hypercube.)

A simple inspection shows that the two chiral colourings have the following properties: (a) each colour has an edge in each direction (is transversal to the regular colouring), and (b) each 2-face of the regular hemi-hypercube has the four colours. It is not hard to see that any of these properties defines the chiral colourings.

We now analyse these new 4-polytopes. It should be clear that what we say about one of them can be similarly said about the other, and hence we use the one on the left of Fig. 3.

As before, we regard this new 4-polytope $\mathcal{Q}$ as a colourful polytope: the 2-faces of $\mathcal{Q}$ are the 4-cycles of exactly two colours and the facets are determined by the subgraphs with exactly 3 colours. Hence, we see that the 2-faces are again 4-gons, that

Fig. 4 A 2-face and a 3-face of $\mathcal{Q}$

we see now as helices in the projective space (see Fig. 4). In fact, each of the 2-faces of $\mathcal{Q}$ corresponds to a Petrie polygon of $\mathcal{P}$ (see [9, p. 163]).

We note that the edge colouring of $K_{4,4}$ in $\mathcal{Q}$ is the same as its colouring in $\mathcal{P}$. Hence $\mathcal{P}$ and $\mathcal{Q}$ are combinatorially isomorphic. On the other hand, the 2-faces of $\mathcal{Q}$ are Petrie polygons of the hemicube $\mathcal{P}$ and vice-versa.

Theorem 1 *Let $\mathcal{Q}$ be the 4-polytope of type $\{4, 3, 3\}$ in the projective space constructed above. Then $\mathcal{Q}$ is a combinatorially regular 4-polytope isomorphic to the hemi-hypercube $\{4, 3, 3\}/2$ but is geometrically chiral, with geometrically chiral facets.*

Proof The 1-skeleton of $\mathcal{Q}$ is exactly the same as the 1-skeleton of the hemi-hypercube $\mathcal{P}$ and every isometry of $\mathbb{P}^3$ that preserves such graph is a symmetry of $\mathcal{P}$, then every symmetry of $\mathcal{Q}$ is a symmetry of $\mathcal{P}$. Recall that $\mathcal{P}$ has 192 symmetries.

We first note that all 4-fold rotations of the central cube in the left part of Fig. 3 preserve the colouring by cyclically permuting the four colours. It is also true that the rotation around an edge is a 3-fold rotation that preserves the colouring by cyclically permuting the three other colours. In particular, the symmetry group of $\mathcal{Q}$ contains the rotation subgroup of the central cube and acts transitively on the four cubes of $\mathcal{P}$. Consequently, all 96 orientation preserving elements of the symmetry group of $\mathcal{P}$ belong also to the symmetry group of $\mathcal{Q}$. Observe that the reflection with respect to the plane perpendicular to the four edges with a given colour in the regular colouring (Fig. 2) does not preserve colours in the chiral colouring and hence, it is not a symmetry of $\mathcal{Q}$. This implies that $\mathcal{Q}$ has precisely 96 symmetries.

It only remains to verify that these 96 elements are precisely the combinatorial rotations of the colourful polytope $\mathcal{Q}$. We observe that the stabilisers of a pair consisting of incident 2-face and 3-face are generated by the twist along an axis of the (helical) 2-face (note that this twist preserves the colouring). For example, the stabiliser of the 2-face in Fig. 4, as permutation of the vertices of $\mathcal{Q}$, is generated by $(v_0 u_0 v_1 u_1)(u_2 v_2 u_3 v_3)$. Furthermore, the stabiliser of a pair consisting of incident vertex and 3-face is generated by the 3-fold rotation around the edge of $\mathcal{Q}$ containing the vertex, but not contained on the 3-face. Finally, the edge pointwise stabiliser is generated by the 3-fold rotation around that edge. Following [12, p. 495], this implies that $\mathcal{Q}$ is indeed chiral.

Fig. 5 The colourful polytope $\hat{Q}$

The arguments on the previous two paragraphs also imply that the facets are geometrically chiral, although they are combinatorially regular. $\square$

As it was kindly pointed out to us by Peter McMullen, the facet of Q is interesting in its own right. It is a chiral projective realization of the cube $\{4, 3\}$ with helical faces which is not rigid, in the sense that it belongs to a continuous family of such realizations. To see this, consider the facet in Fig. 4. Its geometric symmetry group sends the edges of the deleted colour (the thin black ones) among themselves. Thus, the vertices may slide simultaneously along the lines of their corresponding deleted edges to give a family of realizations of the cube parametrized by the projective line and sharing the same geometric symmetry group. Four of these realizations are outstanding. Two become vertex unfaithful when the two vertices of each deleted edge (corresponding to a combinatorially antipodal pair in the cube) coincide in its midpoint or in its polar point in the corresponding line (which is a vertex of the hemi-crosspolytope). And two realizations become geometrically regular. They arise when a vertex and its three neighbours lie in a plane, so that the (combinatorial) reflections fixing a vertex are then realized by the 2-fold rotations along its edges.

We end this section by pointing out that the two enantiomorphic forms of Q are indeed the two polytopes arising from the diagrams in Fig. 3.

3 Finite Chiral 4-Polytopes in $\mathbb{R}^4$

In the previous section we gave an example of a chiral 4-polytope Q in $\mathbb{P}^3$. We now take the double cover $\hat{Q}$ of Q , in the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$.

Each of the vertices and edges of Q lift to two copies of them, and $K_{4,4}$ lifts into the graph G , the 1-skeleton of the hypercube $\{4, 3, 3\}$. We label the vertices of G as $\tilde{v}$ and $-\tilde{v}$, where v is a vertex of Q in such a way that the sets of vertices $\{\tilde{v} \mid v \in V(Q)\}$ and $\{-\tilde{v} \mid v \in V(Q)\}$ are the vertex sets of two disjoint cubes of $\{4, 3, 3\}$. The 4-gons of Q lift into 8-gons of $\hat{Q}$, implying that $\hat{Q}$ has Schläfli type $\{8, 3, 3\}$. Figure 6 shows a 2-face of $\hat{Q}$ and its two incident facets. Each of the 4 facets is of type $\{8, 3\}$, has 16 vertices, 24 edges and 6 faces. We note that $\hat{Q}$ is again a colourful polytope and hence

Fig. 6 A face of $\hat{Q}$ and its two incident facets

every symmetry of $\hat{Q}$ induces a permutation of the colours of the graph G . Using Fig. 5 it is straightforward to see that $\hat{Q}$ is not regular.

Finally, the polytope $\hat{Q}$ is chiral. This can be seen with arguments analogous to those in the previous section, using the symmetry group of the 4-cube $\{4, 3, 3\}$ instead of that of the hemi-cube. Furthermore, it is not difficult to see that the 2-faces of $\hat{Q}$ are helices in $\mathbb{R}^4$ and that the stabiliser of the pair consisting of the 2-face and any of the 3-faces in Fig. 6, is generated by the permutation of the vertices of $\hat{Q}$ given by

$$(\tilde{v}_0, \tilde{u}_0, -\tilde{v}_1, -\tilde{u}_1, -\tilde{v}_0, -\tilde{u}_0, \tilde{v}_1, \tilde{u}_1)(\tilde{u}_2, \tilde{v}_2, -\tilde{u}_3, \tilde{v}_3, -\tilde{u}_2, -\tilde{v}_2, \tilde{u}_3, -\tilde{v}_3),$$

implying that the 1-step rotation in that 2-face is also the 3-step rotation in the other red-green 2-face. Hence, the two components of the above isometry are a 1-step 8-fold rotation followed by a perpendicular 3-step 8-fold rotation. We have established the following theorem.

Theorem 2 *The polytope $\hat{Q}$ is a chiral 4-polytope of full rank.*

Of course, the facet of $\hat{Q}$ is also of interest. Its underlying graph is the Generalized Petersen graph $GP(8, 3)$, and, as it follows from our discussion in the previous section,

its Wythoff space of chiral realizations is of dimension 2 with two of its realizations being geometrically regular.

Acknowledgments The work was partially supported by PAPIIT-UNAM under the project IN112511 and by CONACyT under project 166951.

References

1. Araujo-Prado, G., Hubbard, I., Oliveros, D., Schulte, E.: Colorful polytopes and graphs. *Israel J. Math.* **195**(2), 647–675 (2013)
2. Arocha, J.L., Bracho, J., Montejano, L.: Regular projective polyhedra with planar faces I. *Aequationes Math.* **59**, 55–73 (2000)
3. Bracho, J.: Regular projective polyhedra with planar faces II. *Aequationes Math.* **59**, 160–176 (2002)
4. Dress, A.W.M.: A combinatorial theory of Grünbaum's new regular polyhedra, I: Grünbaum's new polyhedra and their automorphism group. *Aequationes Math.* **23**, 252–265 (1981)
5. Dress, A.W.M.: A combinatorial theory of Grünbaum's new regular polyhedra, II: complete enumeration. *Aequationes Math.* **29**, 222–243 (1985)
6. Grove, L.C., Benson, C.T.: *Finite Reflection Groups*. Graduate Text in Math., vol. 99, 2nd edn. Springer, New York (1985)
7. Grünbaum, B.: Regular polyhedra—old and new. *Aequationes Math.* **16**, 1–20 (1977)
8. McMullen, P.: Regular polytopes of full rank. *Discrete Comput. Geom.* **32**(1), 1–35 (2004)
9. McMullen, P., Schulte, E.: *Abstract Regular Polytopes*. Cambridge University Press, Cambridge (2002)
10. Schulte, E.: Chiral polyhedra in ordinary space. I. *Discrete Comput. Geom.* **32**(1), 55–99 (2004)
11. Schulte, E.: Chiral polyhedra in ordinary space. II. *Discrete Comput. Geom.* **34**(2), 181–229 (2005)
12. Schulte, E., Weiss, A.I.: Chiral polytopes. In: Gritzmann, P., Sturmfels, B. (eds.) *Applied Geometry and Discrete Mathematics (The Victor Klee Festschrift)*. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 4, pp. 493–516. AMS, Providence, RI (1991)