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Equivalence Classes of Full-Dimensional 0/1-Polytopes with Many Vertices

William Y. C. Chen · Peter L. Guo

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Abstract Let Q_n denote the n-dimensional hypercube with vertex set $V_n = \{0, 1\}^n$. A $0/1$-polytope of Q_n is the convex hull of a subset of V_n. This paper is concerned with the enumeration of equivalence classes of full-dimensional $0/1$-polytopes under the symmetries of the hypercube. With the aid of a computer program, Aichholzer obtained the number of equivalence classes of full-dimensional $0/1$-polytopes of Q_4 and Q_5 with any given number of vertices and those of Q_6 up to 12 vertices. Let $F_n(k)$ denote the number of equivalence classes of full-dimensional 0/1-polytopes of Q_n with k vertices. We present a method to compute $F_n(k)$ for $k > 2^{n-2}$. Let $A_n(k)$ denote the number of equivalence classes of $0/1$-polytopes of Q_n with k vertices, and let $H_n(k)$ denote the number of equivalence classes of $0/1$-polytopes of Q_n with k vertices that are not full-dimensional. So we have $A_n(k) = F_n(k) + H_n(k)$. It is known that $A_n(k)$ can be computed by using the cycle index of the hyperoctahedral group. We show that for $k > 2^{n-2}$, $H_n(k)$ can be determined by the number of equivalence classes of $0/1$-polytopes with k vertices that are contained in every hyperplane spanned by a subset of V_n. We also find a way to compute $H_n(k)$ when k is close to 2^{n-2}. For the case of Q_6, we can compute $F_6(k)$ for $k > 12$. Together with the computation of Aichholzer, we reach a complete solution to the enumeration of equivalence classes of full-dimensional 0/1-polytopes of Q_6.

W. Y. C. Chen · P. L. Guo
Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071,
People's Republic of China
e-mail: lguo@nankai.edu.cn

W. Y. C. Chen
e-mail: chen@nankai.edu.cn

W. Y. C. Chen
Center for Applied Mathematics, Tianjin University, Tianjin 300072, People's Republic of China

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1 Introduction

Let Q_n denote the n-dimensional hypercube with vertex set $V_n = \{0, 1\}^n$. A 0/1-polytope of Q_n is defined to be the convex hull of a subset of V_n. The study of 0/1-polytopes has received much attention, see, for example [6,7,11–13,15,18,19].

In this paper, we are concerned with the problem of determining the number of equivalence classes of n-dimensional 0/1-polytopes of Q_n under the symmetries of Q_n, which has been considered as a difficult problem, see Ziegler [18]. It is also listed by Zong [19, Problem 5.1] as one of the fundamental problems concerning 0/1-polytopes.

An n-dimensional 0/1-polytope of Q_n is also called a full-dimensional 0/1-polytope of Q_n. Two 0/1-polytopes are said to be equivalent if one can be transformed to the other by a symmetry of Q_n. Such an equivalence relation is called the 0/1-equivalence relation. For example, Fig. 1 gives the representatives of 0/1-equivalence classes of Q_2, among which (d) and (e) are full-dimensional.

As the first nontrivial case, full-dimensional 0/1-equivalence classes of Q_4 were counted by Below, see Ziegler [18]. With the aid of a computer program, Aichholzer [1] completed the enumeration of full-dimensional 0/1-equivalence classes of Q_5, and those of Q_6 up to 12 vertices, see also Aichholzer [3] and Ziegler [18]. The 5-dimensional hypercube Q_5 has been considered as the last case that one can hope for a complete solution to the enumeration of full-dimensional 0/1-equivalence classes.

Let $F_n(k)$ denote the number of full-dimensional 0/1-equivalence classes of Q_n. The objective of this paper is to present a method to compute $F_n(k)$ for $k > 2^{n-2}$. We also find a way to compute $F_n(k)$ when k is close to 2^{n-2}. Using our approach, we can determine $F_6(k)$ for $k > 12$. Combining the computation of Aichholzer [1], we reach a complete solution for the case of Q_6.

To describe our approach, let $A_n(k)$ denote the number of 0/1-equivalence classes of Q_n with k vertices, and let $H_n(k)$ denote the number of 0/1-equivalence classes of Q_n with k vertices that are not full-dimensional. So we have

$$A_n(k) = F_n(k) + H_n(k). \quad (1.1)$$

It is clear that $F_n(k) = 0$ for $1 \le k \le n$ since any full-dimensional 0/1-polytope of Q_n has at least $n + 1$ vertices. As will be seen in Sect. 2, the values $A_n(k)$ for any k

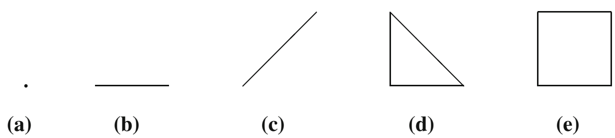

Fig. 1 Representatives of 0/1-equivalence classes of Q_2

can be computed from the cycle index of the hyperoctahedral group B_n. Hence $F_n(k)$ can be determined by $H_n(k)$.

To compute $H_n(k)$, we need a relation between the dimension of a 0/1-polytope and the number of vertices. Let P be a 0/1-polytope of Q_n, and let $\dim(P)$ denote the dimension of P. It is known that P is affinely equivalent to a full-dimensional 0/1-polytope of Q_d for some $d \leq n$, see Ziegler [18]. Thus we have the following consequence.

Theorem 1.1 *Let P be a* 0/1*-polytope of Q_n with more than 2^m vertices, where $1 \leq m < n$. Then we have*

$$\dim(P) \geq m + 1.$$

From Theorem 1.1, we see that if a 0/1-polytope P of Q_n has more than 2^{n-1} vertices, then P has dimension n. Thus, for $k > 2^{n-1}$, we have $F_n(k) = A_n(k)$.

Based on Theorem 1.1, we show that the computation of $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$ can be carried out by determining the number of equivalence classes of 0/1-polytopes with k vertices that are contained in every hyperplane spanned by vertices of Q_n. When $2^{n-2} < k \leq 2^{n-1}$, we can apply Pólya's theorem to count equivalence classes of 0/1-polytopes with k vertices that are contained in a hyperplane spanned by vertices of Q_n. In particular, when $n = 6$, we obtain $F_6(k)$ for $16 < k \leq 32$.

We also find a way to compute $H_n(k)$ when k is close to 2^{n-2}. In particular, when $n = 6$, we obtain $F_6(k)$ for $13 \leq k \leq 16$.

This paper is organized as follows. In Sect. 2, we recall a method introduced by Chen [9] to determine the cycle structure of a symmetry w in the hyperoctahedral group B_n in terms of the number of vertices of Q_n fixed by w. Sections 3–6 are devoted to the computation of $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$. In Sect. 7, we provide a way to compute $H_n(k)$ when k is close to 2^{n-2}. This enables us to determine $H_n(k)$ for $n = 6$ and $13 \leq k \leq 16$.

2 The Cycle Index of the Hyperoctahedral Group

The group of symmetries of Q_n is known as the hyperoctahedral group B_n. In this section, we give an overview of a method introduced by Chen [9] to compute the cycle index of B_n, which will be used in the determination of the cycle index of the subgroup consisting of symmetries that fix a hyperplane spanned by vertices of Q_n.

We proceed with a brief review of the cycle index of a finite group acting on a finite set, see, for example, Brualdi [8]. Let G be a finite group that acts on a finite set X. Then each element $g \in G$ induces a permutation on X. The cycle type of a permutation is defined to be a multiset $\{1^{k_1}, 2^{k_2}, \ldots\}$, where k_i is the number of cycles of length i that appear in the cycle decomposition of the permutation. For $g \in G$, let $c(g)$ denote the cycle type of the permutation on X induced by g. Let $z = (z_1, z_2, \ldots)$ be a sequence of indeterminants, and let

$$z^{c(g)} = z_1^{k_1} z_2^{k_2} \cdots .$$

The cycle index of G is defined as

$$Z_G(z) = Z_G(z_1, z_2, \ldots) = \frac{1}{|G|}\sum_{g \in G} z^{c(g)}. \tag{2.1}$$

Pólya's enumeration theorem shows that the cycle index in (2.1) can be applied to count nonisomorphic colorings of X by using a given number of colors. To be more specific, let us color the elements of X by using m colors, say $c_1, c_2, \ldots, c_m$. Let $C_G(u_1, \ldots, u_m)$ be the polynomial obtained from the cycle index $Z_G(z)$ by substituting z_i with $u_1^i + \cdots + u_m^i$. Pólya's enumeration theorem states that the number of nonisomorphic colorings of X by using the m colors $c_1, \ldots, c_m$ such that a_i elements of X receive the color c_i equals

$$\left[u_1^{a_1} \cdots u_m^{a_m}\right] C_G(u_1, \ldots, u_m),$$

where $\left[u_1^{a_1} \cdots u_m^{a_m}\right] C_G(u_1, \ldots, u_m)$ is the coefficient of $u_1^{a_1} \cdots u_m^{a_m}$ in $C_G(u_1, \ldots, u_m)$.

For a coloring of Q_n with two colors, say, black and white, the black vertices can be considered as vertices of a 0/1-polytope of Q_n. This establishes a one-to-one correspondence between equivalence classes of colorings and 0/1-equivalence classes of Q_n. Let $Z_n(z)$ denote the cycle index of B_n acting on the vertex set V_n, and let $C_n(u_1, u_2)$ be the polynomial obtained from $Z_n(z)$ by substituting z_i with $u_1^i + u_2^i$. By Pólya's theorem, we have

$$A_n(k) = \left[u_1^k u_2^{2^n - k}\right] C_n(u_1, u_2). \tag{2.2}$$

The computation of $Z_n(z)$ has been studied by Pólya [16] and Harrison and High [14]. Explicit expressions of $Z_n(z)$ for $n \le 6$ are given by Aguila [5], which are listed below.

$$\begin{aligned}
Z_1(z) &= z_1,\\
Z_2(z) &= \frac{1}{8}\left(z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4\right),\\
Z_3(z) &= \frac{1}{48}\left(z_1^8 + 6z_1^4 z_2^2 + 13z_2^4 + 8z_1^2 z_3^2 + 12z_4^2 + 8z_2 z_6\right),\\
Z_4(z) &= \frac{1}{384}\begin{pmatrix} z_1^{16} + 12z_1^8 z_2^4 + 12z_1^4 z_2^6 + 51z_2^8 + 48z_8^2 \\ + 48z_1^2 z_2 z_4^3 + 84z_4^4 + 96z_2^2 z_6^2 + 32z_1^4 z_3^4 \end{pmatrix},\\
Z_5(z) &= \frac{1}{3840}\begin{pmatrix} z_1^{32} + 20z_1^{16} z_2^8 + 60z_1^8 z_2^{12} + 231z_2^{16} + 80z_1^8 z_3^8 + 240z_1^4 z_2^2 z_4^6 \\ + 240z_2^4 z_4^6 + 520z_4^8 + 384z_1^2 z_5^6 + 160z_1^4 z_2^2 z_3^4 z_6^2 + 720z_2^4 z_6^4 \\ + 480z_8^4 + 384z_2 z_{10}^3 + 320z_4^2 z_{12}^2 \end{pmatrix},
\end{aligned}$$

$$Z_6(z) = \frac{1}{46080}\begin{pmatrix} z_1^{64}+30z_1^{32}z_2^{16}+180z_1^{16}z_2^{24}+120z_1^{8}z_2^{28}+1053z_2^{32}+160z_1^{16}z_3^{16} \\ +640z_1^{4}z_3^{20}+720z_1^{8}z_2^{4}z_4^{12}+1440z_1^{4}z_2^{6}z_4^{12}+2160z_2^{8}z_4^{12}+4920z_4^{16} \\ +2304z_1^{4}z_5^{12}+960z_1^{8}z_2^{4}z_3^{8}z_6^{4}+5280z_2^{8}z_6^{8}+3840z_1^{2}z_2z_3^{2}z_6^{9}+5760z_8^{8} \\ +1920z_2^{2}z_6^{10}+6912z_2^{2}z_{10}^{6}+3840z_4^{4}z_{12}^{4}+3840z_4z_{12}^{5} \end{pmatrix}.$$

For $k > 2^{n-1}$, we have shown that $F_n(k) = A_n(k)$. Thus, by (2.2) we obtain that for $k > 2^{n-1}$,

$$F_n(k) = [u_1^k u_2^{2^n - k}] C_n(u_1, u_2).$$

For $n = 4$, 5 and 6, the values of $F_n(k)$ for $k > 2^{n-1}$ are given in Tables 1, 2 and 3.

Table 1 $F_4(k)$ for $k > 8$

k	9	10	11	12	13	14	15	16
$F_4(k)$	56	50	27	19	6	4	1	1

Table 2 $F_5(k)$ for $k > 16$

k	17	18	19	20	21	22	23	24
$F_5(k)$	158658	133576	98804	65664	38073	19963	9013	3779
k	25	26	27	28	29	30	31	32
$F_5(k)$	1326	472	131	47	29	5	1	1

Table 3 $F_6(k)$ for $k > 32$

k	$F_6(k)$	k	$F_6(k)$
33	38580161986426	49	3492397119
34	35176482187398	50	1052201890
35	30151914536933	51	290751447
36	24289841497881	52	73500514
37	18382330104696	53	16938566
38	13061946976545	54	3561696
39	8708686182967	55	681474
40	5443544478011	56	120843
41	3186944273554	57	19735
42	1745593733454	58	3253
43	893346071377	59	497
44	426539774378	60	103
45	189678764492	61	16
46	78409442414	62	6
47	30064448972	63	1
48	10666911842	64	1

We next recall the method of Chen [9] for computing the cycle index of B_n. A symmetry of Q_n can be represented as a signed permutation on $\{1, 2, \ldots, n\}$, which is a permutation on $\{1, 2, \ldots, n\}$ with a plus or a minus sign attached to each element. Following the notation in Chen and Stanley [10] or Chen [9], we may write a signed permutation as the form of the cycle decomposition and ignore the plus signs. For example, $(\bar{2}4\bar{5})(3)(1\bar{6})$ represents a signed permutation, where (245)(3)(16) is its underlying permutation. The action of a signed permutation $w \in B_n$ on the vertices of Q_n is defined as follows. For a vertex $(x_1, x_2, \ldots, x_n)$ of Q_n, we define $w(x_1, x_2, \ldots, x_n)$ to be the vertex $(y_1, y_2, \ldots, y_n)$ of Q_n as given by

$$y_i = \begin{cases} x_{\pi(i)} & \text{if } i \text{ is associated with a plus sign,} \\ 1 - x_{\pi(i)} & \text{if } i \text{ is associated with a minus sign,} \end{cases} \tag{2.3}$$

where π is the underlying permutation of w.

We end this section with the following formula of Chen [9], which will be used in Sect. 5 to compute the cycle structure of a symmetry that fixes a hyperplane spanned by vertices of Q_n.

Let n be a positive integer, and let $p_1^{n_1} \ldots p_r^{n_r}$ be the prime factorization of n. Let $\mu(n)$ be the classical number-theoretic Möbius function, that is, $\mu(n) = (-1)^r$ if $n_1 = \cdots = n_r = 1$, and $\mu(n) = 0$ otherwise.

Theorem 2.1 *Let G be a group that acts on a finite set X. For any $g \in G$, the number of i-cycles of the permutation on X induced by g is given by*

$$\frac{1}{i} \sum_{j|i} \mu(i/j) \psi(g^j),$$

where $\psi(g^j)$ is the number of fixed points of g^j acting on X.

3 $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$

Recall that $H_n(k)$ is the number of 0/1-equivalence classes of Q_n with k vertices that are not full-dimensional. In this section, we show that for $2^{n-2} < k \leq 2^{n-1}$, the number $H_n(k)$ is determined by the number of equivalence classes of 0/1-polytopes with k vertices that are contained in every hyperplane spanned by vertices of Q_n. For this reason, it is necessary to consider all possible hyperplanes spanned by vertices of Q_n.

A hyperplane spanned by vertices of Q_n is also called a spanned hyperplane of Q_n. In other words, a spanned hyperplane of Q_n is a hyperplane in $\mathbb{R}^n$ such that the affine space spanned by the vertices of Q_n contained in this hyperplane is of dimension $n-1$. Let

$$H : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

be a spanned hyperplane of Q_n, where $a_1, \ldots, a_n$ and b are integers. For $n \leq 8$, all spanned hyperplanes of Q_n have been found by Aichholzer and Aurenhammer [4].

As will be seen, in order to compute $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$, we need to consider equivalence classes of spanned hyperplanes of Q_n under the symmetries of Q_n. Note that the symmetries of Q_n can be expressed by permuting the coordinates and changing x_i to $1 - x_i$ for some indices i. Therefore, for each equivalence class of spanned hyperplanes of Q_n, we can choose a representative of the form

$$a_1x_1 + a_2x_2 + \cdots + a_tx_t = b, \tag{3.1}$$

where $t \leq n$ and $0 < a_1 \leq a_2 \leq \cdots \leq a_t$.

A complete list of spanned hyperplanes of Q_n for $n \leq 6$ can be found in Aichholzer [2]. The following hyperplanes are representatives of equivalence classes of spanned hyperplanes of Q_4:

$$\begin{aligned}
&x_1 = 0,\\
&x_1 + x_2 = 1,\\
&x_1 + x_2 + x_3 = 1,\\
&x_1 + x_2 + x_3 + x_4 = 1 \text{ or } 2,\\
&x_1 + x_2 + x_3 + 2x_4 = 2.
\end{aligned}$$

In addition to the above hyperplanes, which can also be viewed as spanned hyperplanes of Q_5, we have the following representatives of equivalence classes of spanned hyperplanes of Q_5:

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 = 1 \text{ or } 2,\\
&x_1 + x_2 + x_3 + x_4 + 2x_5 = 2 \text{ or } 3,\\
&x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 2 \text{ or } 3,\\
&x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 = 3 \text{ or } 4,\\
&x_1 + x_2 + x_3 + x_4 + 3x_5 = 3,\\
&x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 3,\\
&x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 = 4.
\end{aligned}$$

When $n = 6$, for the purpose of computing $F_6(k)$ for $16 < k \leq 32$, we need the representatives of equivalence classes of spanned hyperplanes of Q_6 containing more than 16 vertices. There are 6 such representatives:

$$\begin{aligned}
&x_1 = 0,\\
&x_1 + x_2 = 1,\\
&x_1 + x_2 + x_3 = 1,\\
&x_1 + x_2 + x_3 + x_4 = 2,\\
&x_1 + x_2 + x_3 + x_4 + x_5 = 2,\\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3.
\end{aligned}$$

Note that two equivalent spanned hyperplanes of Q_n contain the same number of vertices of Q_n because the symmetry of Q_n preserves the number of vertices. So we may say that an equivalence class of spanned hyperplanes of Q_n contains k vertices, by which we mean that every spanned hyperplane in this class contains k vertices of Q_n.

To state the main result of this section, we need to define an equivalence relation on $0/1$-polytopes contained in a set of points in $\mathbb{R}^n$. Given a set $S \subset \mathbb{R}^n$, consider the set of $0/1$-polytopes of Q_n that are contained in S. Restricting the $0/1$-equivalence relation to this set induces an equivalence relation. More precisely, two $0/1$-polytopes in the set of $0/1$-polytopes of Q_n contained in S are equivalent if one can be transformed to the other by a symmetry of Q_n. Such an equivalence class is called a partial $0/1$-equivalence class of S. Denote by $\mathcal{P}(S, k)$ the set of partial $0/1$-equivalence classes of S with k vertices. The cardinality of $\mathcal{P}(S, k)$ is denoted by $N_S(k)$.

Let $h(n, k)$ denote the number of equivalence classes of spanned hyperplanes of Q_n that contain at least k vertices. Assume that $H_1, H_1, \ldots, H_{h(n,k)}$ are the representatives of equivalence classes of spanned hyperplanes of Q_n containing at least k vertices. We use $\mathcal{H}_n(k)$ to denote the set of $0/1$-equivalence classes of Q_n with k vertices that are not full-dimensional. We shall define a map, denoted Φ, from the (disjoint) union of $\mathcal{P}(H_i, k)$, where $1 \leq i \leq h(n, k)$, to $\mathcal{H}_n(k)$. Given a partial $0/1$-equivalence class $\mathcal{P} \in \mathcal{P}(H_i, k)$, we define $\Phi(\mathcal{P})$ to be the unique $0/1$-equivalence class in $\mathcal{H}_n(k)$ containing $\mathcal{P}$. Then we have the following theorem.

Theorem 3.1 *For $2^{n-2} < k \leq 2^{n-1}$, the map Φ is a bijection.*

Proof We first show that Φ is injective. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two distinct partial $0/1$-equivalence classes with k vertices, which are contained in the spanned hyperplanes H_i and H_j of Q_n, respectively. Let P_1 be a $0/1$-polytope in $\mathcal{P}_1$, and P_2 be a $0/1$-polytope in $\mathcal{P}_2$. To prove that Φ is an injection, it suffices to show that P_1 and P_2 are not equivalent. This is clear when $i = j$. We now consider the case $i \neq j$. Suppose to the contrary that P_1 and P_2 are equivalent. So there exists a symmetry $w \in B_n$ such that $w(P_1) = P_2$. Since $2^{n-2} < k \leq 2^{n-1}$, by Theorem 1.1 we see that P_1 and P_2 are of dimension $n - 1$. For a spanned hyperplane H of Q_n, we use $w(H)$ to denote the hyperplane obtained from H under the action of w. So we have $w(H_i) = H_j$, contradicting the fact that the spanned hyperplanes H_i and H_j are not equivalent. Consequently, the 0/1-polytopes P_1 and P_2 are not equivalent.

It remains to show that Φ is surjective. For any $\mathcal{C} \in \mathcal{H}_n(k)$, we aim to find a partial $0/1$-equivalence class $\mathcal{P}$ such that $\Phi(\mathcal{P}) = \mathcal{C}$. Let P be any $0/1$-polytope in $\mathcal{C}$. Since P is not full-dimensional, there exists a spanned hyperplane H of Q_n such that P is contained in H. It follows that H contains at leat k vertices. Thus there exists a representative H_j ($1 \leq j \leq h(n, k)$) such that H is in the equivalence class of H_j. Assume that $w(H) = H_j$ for some $w \in B_n$. So $w(P)$ is contained in H_j. Let $\mathcal{P}$ be the partial $0/1$-equivalence class of H_j containing $w(P)$. Clearly, we have $\Phi(\mathcal{P}) = \mathcal{C}$. This completes the proof. □

It should also be noted that in the proof of Theorem 3.1, the condition $2^{n-2} < k \leq 2^{n-1}$ is required. When $k \leq 2^{n-2}$, the map Φ is not necessarily an injection while is always a surjection. For a $0/1$-polytope P with $k \leq 2^{n-2}$ vertices contained in a spanned hyperplane of Q_n, it is not always true that $\dim(P) = n - 1$. So there may exist equivalent 0/1-polytopes P and P' with k vertices and nonequivalent spanned hyperplanes H and H' such that P is contained in H and P' is contained in H'. If this is the case, then Φ maps these two partial 0/1-equivalence classes containing P and P' to the same 0/1-equivalence class in $\mathcal{H}_n(k)$.

As a consequence of Theorem 3.1, we obtain the following formula.

Corollary 3.2 *For* $2^{n-2} < k \leq 2^{n-1}$,

$$H_n(k) = \sum_{i=1}^{h(n,k)} N_{H_i}(k). \tag{3.2}$$

By Corollary 3.2, the computation of $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$ is carried out by determining the number of partial 0/1-equivalence classes of every spanned hyperplane of Q_n. We shall explain how to compute the latter in the rest of this section.

For $2^{n-2} < k \leq 2^{n-1}$, let H be a spanned hyperplane of Q_n containing at least k vertices. Let P and P' be two distinct 0/1-polytopes of Q_n with k vertices that are contained in H. Assume that P and P' belong to the same partial 0/1-equivalence class of H. Then there exists a symmetry $w \in B_n$ such that $w(P) = P'$. By Theorem 1.1, both P and P' have dimension $n-1$. Hence we have $w(H) = H$.

Let

$$F(H) = \{w \in B_n \mid w(H) = H\}$$

be the stabilizer subgroup of H, namely, the subgroup of B_n that fixes H. By the above argument, we see that P and P' belong to the same partial 0/1-equivalence class of H if and only if one can be transformed to the other by a symmetry in $F(H)$. So, for $2^{n-2} < k \leq 2^{n-1}$, we can use Pólya's theorem to compute the number $N_H(k)$ of partial 0/1-equivalence classes of H with k vertices.

Denote by $V_n(H)$ the set of vertices of Q_n that are contained in H. Consider the action of $F(H)$ on $V_n(H)$. Assume that each vertex in $V_n(H)$ is assigned one of the two colors, say, black and white. For such a coloring of the vertices in $V_n(H)$, assume that the black vertices are vertices of a 0/1-polytope contained in H. Clearly, for $2^{n-2} < k \leq 2^{n-1}$, this leads to a one-to-one correspondence between partial 0/1-equivalence classes of H with k vertices and equivalence classes of colorings of the vertices in $V_n(H)$ with k black vertices.

Write $Z_H(z)$ for the cycle index of $F(H)$, and let $C_H(u_1, u_2)$ denote the polynomial obtained from $Z_H(z)$ by substituting z_i with $u_1^i + u_2^i$.

Theorem 3.3 *Assume that* $2^{n-2} < k \leq 2^{n-1}$*, and let* H *be a spanned hyperplane of* Q_n *containing at least* k *vertices of* Q_n*. Then we have*

$$N_H(k) = \left[u_1^k u_2^{|V_n(H)|-k}\right] C_H(u_1, u_2).$$

We shall compute the cycle index $Z_H(z)$ in Sects. 4 and 5. Section 4 is devoted to a characterization of the stabilizer group $F(H)$. In Sect. 5, we will give an explicit expression for $Z_H(z)$.

4 The Structure of the Stabilizer $F(H)$

In this section, we aim to characterize the stabilizer $F(H)$ for a given spanned hyperplane H of Q_n.

As mentioned in Sect. 3, for every equivalence class of spanned hyperplanes of Q_n, we can choose a representative of the form

$$H:\ a_1x_1 + a_2x_2 + \cdots + a_tx_t = b, \tag{4.1}$$

where the coefficients a_i are positive integers with $a_1 \le a_2 \le \cdots \le a_t$, and b is a nonnegative integer.

From now on, we shall restrict our attention only to spanned hyperplanes of Q_n in the form of (4.1). We define the type of the spanned hyperplane H in (4.1) to be a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, where α_i is the multiplicity of i occurring in the set $\{a_1, a_2, \ldots, a_t\}$. For example, let

$$H: x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 = 4$$

be a spanned hyperplane of Q_5. Then the type of H is $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$.

For positive integers i and j with $i \le j$, let $[i, j]$ denote the interval $\{i, i+1, \ldots, j\}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be the type of a spanned hyperplane. For $i = 1, 2, \ldots, \ell$, let S_{α_i} be the group of permutations on the interval

$$\big[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i\big], \tag{4.2}$$

where we assume that $\alpha_0 = 0$. We define

$$S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_\ell}, \tag{4.3}$$

where $\times$ denotes the direct product of groups. We also define

$$\overline{S}_\alpha = \overline{S}_{\alpha_1} \times \overline{S}_{\alpha_2} \times \cdots \times \overline{S}_{\alpha_\ell}, \tag{4.4}$$

where $\overline{S}_{\alpha_i}$ is the set of signed permutations on the interval (4.2) for which every element is associated with the minus sign.

Let

$$P(H) = \begin{cases} S_\alpha & \text{if } \sum_{i=1}^t a_i \ne 2b, \\ S_\alpha \cup \overline{S}_\alpha & \text{if } \sum_{i=1}^t a_i = 2b. \end{cases} \tag{4.5}$$

We have the following characterization of the stabilizer of a spanned hyperplane of Q_n.

Theorem 4.1 *Let $H: a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n. Then*

$$F(H) = P(H) \times B_{n,t},$$

where $B_{n,t}$ is the group of signed permutations on the interval $[t + 1, n]$.

To give a proof of Theorem 4.1, we need to describe the action of a symmetry of Q_n on a hyperplane in $\mathbb{R}^n$. Let $H: a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a hyperplane in $\mathbb{R}^n$, and w be a symmetry in B_n. Recall that $w(H)$ is the hyperplane obtained from H under the action of w. Let $s(w)$ be the set of entries of w that are assigned the minus sign. In view of (2.3), we see that $w(H)$ is of the form

$$\sum_{i \notin s(w)} a_{\pi(i)}x_i + \sum_{j \in s(w)} a_{\pi(j)}(1 - x_j) = b, \tag{4.6}$$

where π is the underlying permutation of w. For $1 \le j \le n$, let

$$s(w, j) = \begin{cases} -1 & \text{if } j \in s(w), \\ 1 & \text{otherwise.} \end{cases}$$

Then (4.6) can be rewritten as

$$s(w, 1) \cdot a_{\pi(1)}x_1 + s(w, 2) \cdot a_{\pi(2)}x_2 + \cdots + s(w, n) \cdot a_{\pi(n)}x_n = b - \sum_{j \in s(w)} a_{\pi(j)}. \tag{4.7}$$

For example, let

$$H: x_1 - x_2 - x_3 + 2x_4 = 1$$

be a hyperplane in $\mathbb{R}^4$, and let $w = (1)(\bar{2}\bar{3})(4) \in B_4$. Then $w(H)$ is the following hyperplane:

$$x_1 + x_2 + x_3 + 2x_4 = 3.$$

We are now in a position to prove Theorem 4.1.

Proof Assume that $w \in F(H)$ and π is the underlying permutation of w. We aim to show that $w \in P(H) \times B_{n,t}$. Notice that $w(H)$ can be expressed in the form of (4.7). Since $H = w(H)$, it follows that for $1 \le j \le t$, $s(w, j)$ are either all positive or all negative. So we have the following two cases.
Case 1: $s(w, j)$ is positive for $1 \le j \le t$. In this case, it is clear that $w(H)$ is of the following form:

$$a_{\pi(1)}x_1 + a_{\pi(2)}x_2 + \cdots + a_{\pi(t)}x_t = b,$$

where $a_{\pi(j)} = a_j$ for $1 \le j \le t$. So we deduce that, for any $1 \le j \le t$, $\pi(j)$ is in the interval $\left[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i\right]$ that contains the element j. This implies that $w \in S_\alpha \times B_{n,t}$.
Case 2: $s(w, j)$ is negative for $1 \le j \le t$. Then $w(H)$ is of the following form:

$$-a_{\pi(1)}x_1 - a_{\pi(2)}x_2 - \cdots - a_{\pi(t)}x_t = b - (a_1 + \cdots + a_t).$$

Since $w(H) = H$, we have $a_{\pi(j)} = a_j$ for $1 \le j \le t$ and $b - (a_1 + \cdots + a_t) = -b$. This yields that $w \in \overline{S}_\alpha \times B_{n,t}$.

Combining the above two cases, we deduce that $w \in P(H) \times B_{n,t}$. It remains to show that if w belongs to $P(H) \times B_{n,t}$, then w fixes H. Write $w = \pi\sigma$, where $\pi \in P(H)$ and $\sigma \in B_{n,t}$. We have the following two cases.
Case 1: $\pi \in S_\alpha$. By (4.7), the hyperplane $w(H)$ is of the following form:

$$a_{\pi(1)}x_1 + \cdots + a_{\pi(t)}x_t = b.$$

By the definition of S_α, we see that $a_{\pi(i)} = a_i$ for $1 \le i \le t$. So we have $w(H) = H$.
Case 2: $\pi \in \overline{S}_\alpha$. Let π_0 be the underlying permutation of π. By (4.7), the hyperplane $w(H)$ can be expressed as

$$-a_{\pi_0(1)}x_1 - \cdots - a_{\pi_0(t)}x_t = b - (a_1 + \cdots + a_t).$$

By the definition of $\overline{S}_\alpha$, we see that $a_{\pi_0(i)} = a_i$ for $1 \le i \le t$, which, together with the following relation:

$$2b = a_1 + \cdots + a_t,$$

implies that $w(H) = H$. This completes the proof. □

We conclude this section with a sufficient condition to determine whether two elements in the subgroup $P(H)$ are in the same conjugacy class. Recall that for a group G, two elements g_1 and g_2 are in the same conjugacy class of G if there exists an element $g \in G$ such that $g_1 = gg_2g^{-1}$. This condition will be used in Sect. 5 for the purpose of computing the cycle index of the stabilizer group of a spanned hyperplane H.

Let H be a spanned hyperplane of Q_n of type $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Recall that each element π in the subgroup $P(H)$ is either in S_α or in $\overline{S}_\alpha$. Hence π can be expressed as a product $\pi = \pi_1\pi_2\cdots\pi_\ell$, where, for $1 \le i \le \ell$, π_i belongs to S_{α_i} if $\pi \in S_\alpha$, and π_i belongs to $\overline{S}_{\alpha_i}$ if $\pi \in \overline{S}_\alpha$.

Theorem 4.2 *Let $\pi = \pi_1\pi_2\cdots\pi_\ell$ and $\pi' = \pi'_1\pi'_2\cdots\pi'_\ell$ be two elements in $P(H)$ such that π and π' are both in S_α, or π and π' are both in $\overline{S}_\alpha$. If the underlying permutations of π_i and π'_i have the same cycle type for any $1 \le i \le \ell$, then π and π' are in the same conjugacy class of $P(H)$.*

Proof We first consider the case when both π and π' are in S_α. Since π_i and π'_i are permutations of the same cycle type, they are in the same conjugacy class. So there is a permutation $w_i \in S_{\alpha_i}$ such that $\pi_i = w_i\pi'_iw_i^{-1}$. It follows that $\pi = (w_1\pi'_1w_1^{-1})\cdots(w_\ell\pi'_\ell w_\ell^{-1}) = w\pi'w^{-1}$, where $w = w_1\cdots w_\ell \in S_\alpha$. This shows that π and π' are in the same conjugacy class.

It remains to consider the case when both π and π' are in $\overline{S}_\alpha$. Let π_0 (resp., π'_0) be the underlying permutation of π (resp., π'). Then there is a symmetry $w \in S_\alpha$ such that $\pi_0 = w\pi'_0w^{-1}$. We claim that $\pi = w\pi'w^{-1}$. Indeed, it is enough to show that $\pi(x_1, x_2, \dots, x_t) = w\pi' w^{-1}(x_1, x_2, \dots, x_t)$ for any point $(x_1, x_2, \dots, x_t)$ in $\mathbb{R}^t$. Assume that $\pi(x_1, x_2, \dots, x_t) = (y_1, y_2, \dots, y_t)$ and $w\pi' w^{-1}(x_1, x_2, \dots, x_t) = (z_1, z_2, \dots, z_t)$. Since every element of π is associated with the minus sign, by (2.3) we find that $y_i = 1 - x_{\pi_0(i)}$ for $1 \le i \le t$. On the other hand, using (2.3), it is easy to check that $z_i = 1 - x_{w^{-1}\pi_0' w(i)}$ for $1 \le i \le t$. Since $\pi_0 = w\pi_0' w^{-1}$, we deduce that $\pi_0(i) = w^{-1}\pi_0' w(i)$. Therefore, we have $y_i = z_i$ for $1 \le i \le t$. So the claim is justified. This completes the proof. □

5 The Computation of $Z_H(z)$

In this section, we obtain a formula for the cycle index $Z_H(z)$ of the stabilizer group $F(H)$ of a spanned hyperplane H of Q_n.

Let

$$H:\ a_1x_1 + a_2x_2 + \cdots + a_tx_t = b \tag{5.1}$$

be a spanned hyperplane of Q_n. Recall that $V_n(H)$ is the set of vertices of Q_n contained in H. To compute the cycle index $Z_H(z)$, we need to determine the cycle structures of permutations on $V_n(H)$ induced by the symmetries in $F(H)$. By Theorem 4.1, each symmetry in $F(H)$ can be written uniquely as a product πw, where $\pi \in P(H)$ and $w \in B_{n,t}$. We shall define two group actions for the subgroups $P(H)$ and $B_{n,t}$, and we derive an expression for the cycle type of the permutation on $V_n(H)$ induced by πw in terms of the cycle types of the permutations induced by π and w.

Let H be a spanned hyperplane of Q_n as given in (5.1). To define the action of $P(H)$, we should consider H as a hyperplane in $\mathbb{R}^t$. Clearly, if H is regarded as a hyperplane in $\mathbb{R}^t$, it is a spanned hyperplane of Q_t. Denote by $V_t(H)$ the set of vertices of Q_t that are contained in H, namely,

$$V_t(H) = \{(x_1, x_2, \dots, x_t) \in V_t \mid a_1x_1 + a_2x_2 + \cdots + a_tx_t = b\}.$$

Since $P(H)$ stabilizes the set $V_t(H)$, we get an action of the group $P(H)$ on $V_t(H)$.

We also need to describe the action of a symmetry in the group $B_{n,t}$ on the set of vertices of Q_{n-t}. Assume that $w \in B_{n,t}$, namely, w is a signed permutation on the interval $[t+1, n]$. Subtracting each element of w by t, we get a signed permutation on $[1, n-t]$. In this way, each signed permutation in $B_{n,t}$ corresponds to a symmetry of Q_{n-t}. Hence, $B_{n,t}$ is isomorphic to the group B_{n-t} of symmetries of Q_{n-t}. This leads to an action on V_{n-t}.

Let πw be a symmetry in $F(H)$, where $\pi \in P(H)$ and $w \in B_{n,t}$. The following lemma shows that the cycle type of the permutation on $V_n(H)$ induced by πw is determined by the cycle types of the permutations on $V_t(H)$ and V_{n-t} induced by π and w. For an element g in a group G acting on a finite set X, we use $c(g)$ to denote the cycle type of the permutation on X induced by g, which is written as a multiset $\{1^{c_1}, 2^{c_2}, \dots\}$.

Lemma 5.1 *Let $H\colon\ a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n, and πw be a symmetry in $F(H)$, where $\pi \in P(H)$ and $w \in B_{n,t}$. Assume that $c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}$ and $c(w) = \{1^{k_1}, 2^{k_2}, \ldots\}$. Then we have*

$$c(\pi w) = \bigcup_{i\geq 1}\bigcup_{j\geq 1}\left\{\Big(\mathrm{lcm}(i, j)\Big)^{\frac{ijm_ik_j}{\mathrm{lcm}(i,j)}}\right\}, \tag{5.2}$$

where $\bigcup$ denotes the disjoint union of multisets, and lcm(i, j) *denotes the least common multiple of i and j.*

Proof Clearly, each vertex in $V_n(H)$ can be expressed as a vector of the following form

$$(x_1, \ldots, x_t, y_1, \ldots, y_{n-t}),$$

where $(x_1, \ldots, x_t)$ is a vertex in $V_t(H)$ and $(y_1, \ldots, y_{n-t})$ is a vertex of Q_{n-t}. Assume that $|V_t(H)| = m$. Let $V_t(H) = \{u_1, u_2, \ldots, u_m\}$ and $V_{n-t} = \{v_1, v_2, \ldots, v_{2^{n-t}}\}$. Then each vertex in $V_n(H)$ can be expressed as an ordered pair (u_i, v_j), where $1 \leq i \leq m$ and $1 \leq j \leq 2^{n-t}$.

Let $C_i = (s_1, \ldots, s_i)$ be an i-cycle of the permutation on $V_t(H)$ induced by π, that is, C_i maps the vertex u_{s_p} to the vertex $u_{s_{p+1}}$ for $1 \leq p \leq i - 1$, and to the vertex u_{s_1} for $p = i$. Similarly, let $C_j = (t_1, \ldots, t_j)$ be a j-cycle of the permutation on V_{n-t} induced by w, that is, C_j maps the vertex v_{t_q} to the vertex $v_{t_{q+1}}$ for $1 \leq q \leq j - 1$, and to the vertex v_{t_1} for $q = j$. Define $C_{i,j}$ to be the permutation on the subset $\{(u_{s_p}, v_{t_q}) \mid 1 \leq p \leq i,\ 1 \leq q \leq j\}$ of $V_n(H)$ such that

$$C_{i,j}(u_{s_p}, v_{t_q}) = (C_i(u_{s_p}), C_j(v_{t_q})).$$

It is easily seen that the induced permutation of πw on $V_n(H)$ is the direct product of $C_{i,j}$, where C_i (resp., C_j) runs over the cycles of the permutation on $V_t(H)$ (resp., V_{n-t}) induced by π (resp., w).

It can be verified that the cycle type of $C_{i,j}$ is

$$\left\{ (\mathrm{lcm}(i, j))^{\frac{ij}{\mathrm{lcm}(i,j)}} \right\}.$$

Thus the cycle type of the induced permutation of πw on $V_n(H)$ is given by (5.2). This completes the proof. □

For convenience, we introduce the following notation. Let π be a symmetry in $P(H)$. Assume that the cycle type of the permutation on $V_t(H)$ induced by π is

$$c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}.$$

For $j \geq 1$, we define

$$f_{\pi,j}(z) = \prod_{i\geq 1}(z_{\mathrm{lcm}(i,j)})^{\frac{ijm_i}{\mathrm{lcm}(i,j)}}. \tag{5.3}$$

We have the following proposition.

Proposition 5.2 *Let H be a spanned hyperplane of Q_n of type α. Assume that $\pi = \pi_1\pi_2\cdots\pi_\ell$ and $\pi' = \pi'_1\pi'_2\cdots\pi'_\ell$ are two symmetries in $P(H)$ such that π and π' are both in S_α, or π and π' are both in $\overline{S}_\alpha$. If the underlying permutations of π_i and π'_i have the same cycle type for $1 \le i \le \ell$, then, for $j \ge 1$,*

$$f_{\pi,j}(z) = f_{\pi',j}(z). \tag{5.4}$$

Proof It follows from Theorem 4.2 that π and π' are in the same conjugacy class of $P(H)$. Hence the permutations on $V_t(H)$ induced by π and π' are in the same conjugacy class, that is, $c(\pi) = c(\pi')$. Since $f_{\pi,j}(z)$ depends only on the cycle type $c(\pi)$, we deduce that $f_{\pi,j}(z) = f_{\pi',j}(z)$. This completes the proof. □

To compute the cycle index $Z_H(z)$, we recall some notation and terminology on integer partitions. A partition λ of a positive integer n, denoted $\lambda \vdash n$, will be expressed in the multiset form, that is, $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$, where m_i is the number of occurrences of i in λ. Denote by $\ell(\lambda)$ the number of parts of λ, that is, $\ell(\lambda) = m_1 + m_2 + \cdots$. For a partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$, let

$$m_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots .$$

For two partitions λ and μ, define $\lambda \cup \mu$ to be the partition obtained by putting the parts of λ and μ together. For example, for $\lambda = \{1, 2\}$ and $\mu = \{1^2, 3\}$, we have $\lambda \cup \mu = \{1^3, 2, 3\}$.

Let H be a spanned hyperplane of Q_n of type $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. For $1 \le i \le \ell$, let μ^i be a partition of α_i, and let $\mu = \mu^1 \cup \cdots \cup \mu^\ell$. Assume that $\pi = \pi_1\pi_2\cdots\pi_\ell$ (resp., $\pi' = \pi'_1\pi'_2\cdots\pi'_\ell$) is a symmetry in S_α (resp., $\overline{S}_\alpha$) such that the underlying permutation of π_i (resp., π'_i) has cycle type μ^i for $1 \le i \le \ell$. For $j \ge 1$, define

$$g_{\mu,j}(z) = f_{\pi,j}(z)$$

and

$$\overline{g}_{\mu,j}(z) = f_{\pi',j}(z).$$

By Proposition 5.2, the functions $g_{\mu,j}(z)$ and $\overline{g}_{\mu,j}(z)$ are well defined.

Let

$$g_\mu(z) = (g_{\mu,1}(z), g_{\mu,2}(z), \ldots)$$

and

$$\overline{g}_\mu(z) = (\overline{g}_{\mu,1}(z), \overline{g}_{\mu,2}(z), \ldots).$$

In the above notation, we obtain the following formula for the cycle index $Z_H(z)$.

Theorem 5.3 *Let $H: a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n. Assume that H is of type $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Then we have*

$$Z_H(z) = \frac{1}{2^{\delta(H)}} \sum_{(\mu^1,\ldots,\mu^\ell)} \prod_{i=1}^{\ell} m_{\mu^i}^{-1} \left(Z_{n-t}(g_\mu(z)) + \delta(H) Z_{n-t}(\overline{g}_\mu(z))\right), \tag{5.5}$$

where $\mu^i \vdash \alpha_i$, $\mu = \mu^1 \cup \cdots \cup \mu^\ell$, $\delta(H) = 1$ if $\sum_{i=1}^t a_i = 2b$ and $\delta(H) = 0$ otherwise.

Proof Let $\pi \in P(H)$ and $w \in B_{n,t}$, and let

$$c(w) = \{1^{k_1}, 2^{k_2}, \ldots\}$$

be the cycle type of the permutation on V_{n-t} induced by w. In view of Lemma 5.1, we have

$$z^{c(\pi w)} = f_{\pi,1}(z)^{k_1} f_{\pi,2}(z)^{k_2} \cdots \tag{5.6}$$

Summing over signed permutations w in $B_{n,t}$ and using (2.1) and (5.6), we deduce that

$$\begin{aligned}
\sum_{\pi w} z^{c(\pi w)} &= \sum_{w} f_{\pi,1}(z)^{k_1} f_{\pi,2}(z)^{k_2} \cdots \\
&= (n-t)!2^{n-t} Z_{n-t}(f_{\pi,1}(z), f_{\pi,2}(z), \ldots) \\
&= (n-t)!2^{n-t} Z_{n-t}(f_\pi(z)),
\end{aligned}$$

where

$$f_\pi(z) = (f_{\pi,1}(z), f_{\pi,1}(z), \ldots).$$

Thus,

$$\begin{aligned}
Z_H(z) &= \frac{1}{|F(H)|} \sum_{\pi w \in F(H)} z^{c(\pi w)} \\
&= \frac{1}{|F(H)|} \sum_{\pi \in P(H)} (n-t)!2^{n-t} Z_{n-t}(f_\pi(z)) \\
&= \frac{(n-t)!2^{n-t}}{|F(H)|} \left(\sum_{\pi \in S_\alpha} Z_{n-t}(f_\pi(z)) + \delta(H) \sum_{\pi' \in \overline{S}_\alpha} Z_{n-t}(f_{\pi'}(z)) \right),
\end{aligned} \tag{5.7}$$

where $\delta(H) = 1$ if $\sum_{i=1}^t a_i = 2b$ and $\delta(H) = 0$ otherwise.

For a partition $\lambda \vdash n$, there are $\frac{n!}{m_\lambda}$ permutations on $\{1, 2, \ldots, n\}$ that are of type λ, see, for example, Stanley [17, Proposition 1.3.2]. So the number of symmetries

$\pi = \pi_1\pi_2 \ldots \pi_\ell$ in S_α (or, $\overline{S}_\alpha$) such that for $i = 1, 2, \ldots, \ell$, the underlying permutation of π_i is of type μ^i equals

$$\prod_{i=1}^{\ell} \frac{\alpha_i!}{m_{\mu^i}}. \tag{5.8}$$

Combining (5.7), (5.8) and Proposition 5.2, we obtain that

$$Z_H(z) = \frac{(n-t)!2^{n-t}}{|F(H)|} \sum_{(\mu^1,\ldots,\mu^\ell)} \prod_{i=1}^{\ell} \frac{\alpha_i!}{m_{\mu^i}} \big(Z_{n-t}(g_\mu(z)) + \delta(H) Z_{n-t}(\overline{g}_\mu(z))\big), \tag{5.9}$$

where $\mu^i \vdash \alpha_i$ and $\mu = \mu^1 \cup \cdots \cup \mu^\ell$.

It is easily seen that

$$|F(H)| = (n-t)!2^{n-t+\delta(H)} \prod_{i=1}^{\ell} \alpha_i!. \tag{5.10}$$

Substituting (5.10) into (5.9), we are led to (5.5). □

By Theorem 5.3, to compute the cycle index $Z_H(z)$, it suffices to determine the cycle type $c(\pi)$ of the permutation on $V_t(H)$ induced by $\pi \in P(H)$. Let $c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}$. By Theorem 2.1, we have

$$m_i = \frac{1}{i} \sum_{j|i} \mu(i/j) \psi(\pi^j), \tag{5.11}$$

where $\psi(\pi^j)$ is the number of vertices in $V_t(H)$ that are fixed by π^j. The following theorem gives a formula for $\psi(\pi)$, leading to a formula for $\psi(\pi^j)$.

Theorem 5.4 *Let $H: a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n. Assume that $\pi = \pi_1\pi_2 \cdots \pi_\ell$ is a symmetry in $P(H)$ such that the underlying permutation of π_i is of type $\mu^i = \{1^{m_{i1}}, 2^{m_{i2}}, \ldots\}$ for $i = 1, 2, \ldots, \ell$. Then*

$$\psi(\pi) = \begin{cases} [x^b] \prod_{i=1}^{\ell} \prod_{j\geq 1} (1 + x^{ij})^{m_{ij}} & \text{if } \pi \in S_\alpha, \\ \chi(\mu) 2^{\ell(\mu)} & \text{if } \pi \in \overline{S}_\alpha, \end{cases} \tag{5.12}$$

where $\mu = \mu^1 \cup \cdots \cup \mu^\ell$, $\chi(\mu) = 1$ if μ has no odd parts and $\chi(\mu) = 0$ otherwise.

Proof We first consider the case when π is in S_α. Observe that, a vertex $v = (x_1, x_2, \ldots, x_t)$ of Q_t is both fixed by π and contained in $V_t(H)$ if and only if

(1) For $1 \leq i \leq \ell$ and each k-cycle $(j_1, j_2, \ldots, j_k)$ of π_i, we have

$$x_{j_1} = x_{j_2} = \cdots = x_{j_k}.$$

(2) $a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$, or equivalently,

$$b_1 + 2b_2 + \cdots + \ell b_\ell = b,$$

where b_i $(1 \le i \le \ell)$ is the sum of the entries of v equal to 1.

It can be easily deduced that the number of vertices of Q_t satisfying the above conditions is given by

$$[x^b] \prod_{i=1}^{\ell} \prod_{j \ge 1} \left(1 + x^{ij}\right)^{m_{ij}}.$$

This proves (5.12) for the case when $\pi \in S_\alpha$.

We now consider the case when π is in $\overline{S}_\alpha$. Notice that a vertex $v = (x_1, x_2, \ldots, x_t)$ of Q_t is fixed by π if and only if, for any k-cycle $(\overline{j_1}, \overline{j_2}, \ldots, \overline{j_k})$ of π, we have

$$(x_{j_1}, x_{j_2}, \ldots, x_{j_k}) = (1 - x_{j_2}, 1 - x_{j_3}, \ldots, 1 - x_{j_1}). \tag{5.13}$$

Consequently, if a vertex $v = (x_1, x_2, \ldots, x_t)$ of Q_t is fixed by π, then, for any k-cycle $(\overline{j_1}, \overline{j_2}, \ldots, \overline{j_k})$ of π, the vector $(x_{j_1}, x_{j_2}, \ldots, x_{j_k})$ is either $(0, 1, \ldots, 0, 1)$ or $(1, 0, \ldots, 1, 0)$. This implies that k is even. Thus π does not have any fixed points if π contains an odd cycle.

We now assume that π has only even cycles. In this case, the number of vertices of Q_t fixed by π equals $2^{\ell(\mu)}$. To prove $\psi(\pi) = 2^{\ell(\mu)}$, we need to demonstrate that any vertex of Q_t fixed by π is in $V_t(H)$. Let $v = (x_1, x_2, \ldots, x_t)$ be a vertex of Q_t fixed by π. Since, for each cycle $(\overline{j_1}, \overline{j_2}, \ldots, \overline{j_k})$ of π, the vector $(x_{j_1}, x_{j_2}, \ldots, x_{j_k})$ is either $(0, 1, \ldots, 0, 1)$ or $(1, 0, \ldots, 1, 0)$, we deduce that $a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ by applying the relation $a_1 + \cdots + a_t = 2b$. Hence the vertex v is in $V_t(H)$. This completes the proof. □

Based on Theorem 5.4, we can compute $\psi(\pi^j)$ since the cycle structure of π^j is easily determined by the cycle structure of π. Let $\pi = \pi_1\pi_2 \ldots \pi_\ell$ be a symmetry in $P(H)$ such that for $1 \le i \le \ell$, the underlying permutation of π_i is of type $\mu^i = \{1^{m_{i1}}, 2^{m_{i2}}, \ldots\}$. Clearly, we have $\pi^j = \pi_1^j \pi_2^j \ldots \pi_\ell^j$. Moreover, we see that π^j belongs to S_α if π is in S_α or π is in $\overline{S}_\alpha$ and j is even, and π^j belongs to $\overline{S}_\alpha$ otherwise. Let $\gcd(i, j)$ denote the greatest common divisor of i and j. Then the cycle type of the underlying permutation of π_i^j is given by

$$\left\{1^{m_{i1}}, \gcd(2, j)^{\frac{2m_{i2}}{\gcd(2,j)}}, \gcd(3, j)^{\frac{3m_{i3}}{\gcd(3,j)}}, \ldots\right\}.$$

6 $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \le 2^{n-1}$

This section is devoted to the computation of $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \le 2^{n-1}$. This requires the cycle index $Z_H(z)$ for every spanned hyperplane H of Q_n for $n = 4, 5, 6$ that contains more than 2^{n-2} vertices of Q_n.

Recall that $h(n,k)$ denotes the number of equivalence classes of spanned hyperplanes of Q_n containing at least k vertices. Let $H_1, H_2, \ldots, H_{h(n,k)}$ be the representatives of these equivalence classes. When $2^{n-2} < k \le 2^{n-1}$, combining relation (1.1), Corollary 3.2 and Theorem 3.3, we deduce that

$$\begin{aligned} F_n(k) &= A_n(k) - H_n(k) \\ &= A_n(k) - \sum_{i=1}^{h(n,k)} N_{H_i}(k) \\ &= A_n(k) - \sum_{i=1}^{h(n,k)} \left[u_1^k u_2^{|V_n(H_i)|-k}\right] C_{H_i}(z_1, z_2). \end{aligned} \tag{6.1}$$

Using formula (6.1), we proceed to compute $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \le 2^{n-1}$. We start with the computation of $F_4(k)$ for $4 < k \le 8$. For $t \le n$, we use H_n^t to denote the following hyperplane in $\mathbb{R}^n$

$$x_1 + x_2 + \cdots + x_t = \lfloor t/2 \rfloor .$$

In this notation, representatives of equivalence classes of spanned hyperplanes of Q_4 containing more than 4 vertices are as follows:

$$\begin{aligned} &H_4^1 : x_1 = 0, \\ &H_4^2 : x_1 + x_2 = 1, \\ &H_4^3 : x_1 + x_2 + x_3 = 1, \\ &H_4^4 : x_1 + x_2 + x_3 + x_4 = 2. \end{aligned}$$

Employing the techniques in Sect. 5, we obtain the cycle indices $Z_{H_4^1}(z)$ and $Z_{H_4^2}(z)$ as given below.

$$\begin{aligned} Z_{H_4^1}(z) &= Z_3(z), \\ Z_{H_4^2}(z) &= \frac{1}{16}\left(9z_2^4 + 4z_4^2 + 2z_1^4 z_2^2 + z_1^8\right). \end{aligned}$$

For the remaining two hyperplanes $H = H_4^3$ and H_4^4, it is easily checked that $N_H(k) = 1$ for $k = 5, 6$, and $N_H(k) = 0$ for $k = 7, 8$. Thus, applying (6.1) we can determine $F_4(k)$ for $k = 5, 6, 7, 8$. These values are given in Table 4, which agree with the computation of Aichholzer [1].

Observing that $F_4(k) = 0$ for $k \le 4$, thus we have completed the enumeration of full-dimensional 0/1-equivalence classes of Q_4.

We now compute $F_5(k)$ for $8 < k \le 16$. Representatives of equivalence classes of spanned hyperplanes of Q_5 containing more than 8 vertices are $H_5^1, H_5^2, H_5^3, H_5^4, H_5^5$. By utilizing the techniques in Sect. 5, we obtain that

Table 4 $F_4(k)$ for $k = 5, 6, 7, 8$

	H_4^1	H_4^2	H_4^3	H_4^4	$F_4(k)$
5	3	5	1	1	17
6	3	5	1	1	40
7	1	1			54
8	1	1			72

Table 5 $F_5(k)$ for $8 < k \leq 16$

	H_5^1	H_5^2	H_5^3	H_5^4	H_5^5	$F_5(k)$
9	56	159	9	7	1	8781
10	50	135	5	5	1	19767
11	27	68	1	1		37976
12	19	43	1	1		65600
13	6	12				98786
14	4	7				133565
15	1	1				158656
16	1	1				159110

$$\begin{aligned}
Z_{H_5^1}(z) &= Z_4(z),\\
Z_{H_5^2}(z) &= \frac{1}{96}\big(z_1^{16} + 6z_1^8 z_2^4 + 33z_2^8 + 8z_1^4 z_3^4 + 24z_4^4 + 24z_2^2 z_6^2\big),\\
Z_{H_5^3}(z) &= \frac{1}{48}\big(12z_2^6 + 8z_4^3 + 2z_1^6 z_2^3 + z_1^{12} + 6z_1^2 z_2^5 + 3z_1^4 z_2^4 + 6z_6^2 + 4z_{12} + 4z_3^2 z_6\\
&\qquad + 2z_3^4\big),\\
Z_{H_5^4}(z) &= \frac{1}{96}\big(z_1^{12} + 27z_2^6 + 9z_1^4 z_2^4 + 8z_3^4 + 24z_6^2 + 18z_2^2 z_4^2 + 6z_1^4 z_4^2 + 3z_1^8 z_2^2\big),\\
Z_{H_5^5}(z) &= \frac{1}{120}\big(24z_5^2 + 30z_2 z_4^2 + 20z_1 z_3 z_6 + 20z_1 z_3^3 + 15z_1^2 z_2^4 + 10z_1^4 z_2^3 + z_1^{10}\big).
\end{aligned}$$

Consequently, the values $F_5(k)$ for $8 < k \leq 16$ can be derived from (6.1), and they agree with the computation of Aichholzer [1], see Table 5.

The main objective of this section is to compute $F_6(k)$ for $16 < k \leq 32$. As mentioned in Sect. 4, there are 6 representatives of equivalence classes of spanned hyperplanes of Q_6 containing more than 16 vertices, namely, $H_6^1, H_6^2, H_6^3, H_6^4, H_6^5, H_6^6$. Again, by applying the techniques in Sect. 5, we obtain that

$$\begin{aligned}
Z_{H_6^1}(z) &= Z_5(z),\\
Z_{H_6^2}(z) &= \frac{1}{768}\begin{pmatrix} z_1^{32} + 12z_1^{16} z_2^8 + 12z_1^8 z_2^{12} + 127z_2^{16} + 32z_1^8 z_3^8 \\ + 48z_1^4 z_2^2 z_4^6 + 168z_4^8 + 224z_2^4 z_6^4 + 96z_8^4 + 48z_2^4 z_4^6 \end{pmatrix},\\
Z_{H_6^3}(z) &= \frac{1}{288}\begin{pmatrix} z_1^{24} + 6z_1^{12} z_2^6 + 52z_2^{12} + 18z_3^8 + 48z_4^6 + 32z_2^3 z_6^3 + 3z_1^8 z_2^8 \\ + 18z_1^4 z_2^{10} + 24z_1^2 z_3^2 z_2^2 z_6^2 + 8z_1^6 z_3^6 + 12z_3^4 z_6^2 + 42z_6^4 + 24z_{12}^2 \end{pmatrix},
\end{aligned}$$

Table 6 $F_6(k)$ for $16 < k \leq 32$

	H_6^1	H_6^2	H_6^3	H_6^4	H_6^5	H_6^6	$F_6(k)$
17	158658	767103	1464	1334	12	5	30063520396
18	133576	642880	657	630	5	3	78408664654
19	98804	474635	220	216	1	1	189678190615
20	65664	312295	81	86	1	1	426539396250
21	38073	179829	19	20			893345853436
22	19963	92309	7	8			1745593621167
23	9013	40948	1	1			3186944223591
24	3779	16335	1	1			5443544457875
25	1326	5500					8708686176141
26	472	1753					13061946974320
27	131	441					18382330104124
28	47	129					24289841497705
29	10	23					30151914536900
30	5	9					35176482187384
31	1	1					38580161986424
32	1	1					39785643746724

$$Z_{H_6^4}(z) = \frac{1}{384}\begin{pmatrix} z_1^{24} + 81z_2^{12} + +2z_1^{12}z_2^6 + 18z_1^4z_2^{10} + 15z_1^8z_2^8 + 72z_6^4 + 32z_{12}^2 \\ + 64z_4^6 + 16z_3^4z_6^2 + 8z_3^8 + 54z_2^4z_4^4 + 12z_1^4z_2^2z_4^4 + 6z_1^8z_4^4 + 3z_1^{16}z_2^4 \end{pmatrix},$$

$$Z_{H_6^5}(z) = \frac{1}{240}\begin{pmatrix} z_1^{20} + 24z_{10}^2 + 60z_2^2z_4^4 + 26z_2^{10} + 20z_1^2z_3^2z_6^2 \\ + 20z_1^2z_3^6 + 15z_1^4z_2^8 + 10z_1^8z_2^6 + 40z_2z_6^3 + 24z_5^4 \end{pmatrix},$$

$$Z_{H_6^6}(z) = \frac{1}{1440}\begin{pmatrix} z_1^{20} + 144z_5^4 + 144z_{10}^2 + 320z_2z_6^3 + 270z_2^2z_4^4 + 76z_2^{10} \\ + 90z_1^4z_4^4 + 30z_1^8z_2^6 + 45z_1^4z_2^8 + 240z_1^2z_3^2z_6^2 + 80z_1^2z_3^6 \end{pmatrix}.$$

Using (6.1), we can compute $F_6(k)$ for $16 < k \leq 32$. These values are listed in Table 6.

7 $H_6(k)$ for $k = 13, 14, 15, 16$

In this section, we compute $H_6(k)$ for $k = 13, 14, 15, 16$. Together with the computation of Aichholzer for $n = 6$ and $k \leq 12$, we complete the enumeration of full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube. In fact, we can compute $H_n(k)$ when $n > 4$ and k is close to 2^{n-2}.

Let us recall the map Φ defined in Sect. 3. Let $H_1, H_2, \ldots, H_{h(n,k)}$ be the representatives of equivalence classes of spanned hyperplanes of Q_n containing at least k vertices. As before, we use $\mathcal{P}(H_i, k)$ to denote the set of partial 0/1-equivalence classes of H_i with k vertices, and use $N_{H_i}(k)$ to denote the cardinality of $\mathcal{P}(H_i, k)$. Let $\mathcal{P}$ be a partial 0/1-equivalence class in the (disjoint) union of $\mathcal{P}(H_i, k)$ where

$1 \le i \le h(n,k)$. Then Φ maps $\mathcal{P}$ to the unique 0/1-equivalence class in $\mathcal{H}_n(k)$ that contains $\mathcal{P}$.

When $k \le 2^{n-2}$, it is possible that there exist equivalent 0/1-polytopes P and P' that are contained respectively in H_i and H_j, where $1 \le i \ne j \le h_{n,k}$. Let $\mathcal{P}$ and $\mathcal{P}'$ be the partial 0/1-equivalence classes of H_i and H_j that contain P and P' respectively. Then we have $\Phi(\mathcal{P}) = \Phi(\mathcal{P}')$. So Φ is not necessarily an injection when $k \le 2^{n-2}$. Note that when restricted to $\mathcal{P}(H_i, k)$, Φ is always an injection. Thus, in order to compute $H_n(k)$ for $k \le 2^{n-2}$, we need to compute the number $N_{H_i}(k)$ of partial 0/1-equivalence classes of each spanned hyperplane H_i as well as the number of partial 0/1-equivalence classes with k vertices that are contained in the intersection of distinct spanned hyperplanes.

The objective of this section is to find a way to compute $N_{H_i}(k)$ when k is close to 2^{n-2}. As will be seen, when $2^{n-3} < k \le 2^{n-2}$, to compute $N_{H_i}(k)$ we need to consider all possible symmetries $w \in B_n$ such that the intersections of H_i and $w(H_i)$ contain at least k vertices. To be more specific, we need to determine the number of partial 0/1-equivalence classes with k vertices that are contained in the intersection $H_i \cap w(H_i)$. Moreover, when k is close to 2^{n-2}, there are only a few symmetries w such that the intersection $H_i \cap w(H_i)$ contains at least k vertices. This makes it possible to compute $N_{H_i}(k)$ when k is close to 2^{n-2}.

When k is close to 2^{n-2}, the same technique can be applied to determine the number of partial 0/1-equivalence classes with k vertices that are contained in the intersection of distinct spanned hyperplanes.

Notice that

$$\mathcal{H}_n(k) = A_1 \cup A_2 \cup \cdots \cup A_{h(n,k)},$$

where

$$A_i = \Phi(\mathcal{P}(H_i, k)).$$

By the principle of inclusion–exclusion, we have the following expression for $H_n(k)$.

Lemma 7.1 *Let H be a spanned hyperplane of Q_n. Then we have*

$$\begin{aligned} H_n(k) = &\sum_{1\le i\le h(n,k)} |A_i| - \sum_{1\le i_1<i_2\le h(n,k)} |A_{i_1} \cap A_{i_2}| \\ &+ \sum_{1\le i_1<i_2<i_3\le h(n,k)} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots \end{aligned} \tag{7.1}$$

By Lemma 7.1, the computation of $H_n(k)$ reduces to the evaluation of the cardinalities of $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$, where $1 \le i_1 < \cdots < i_m \le h(n,k)$. Since Φ is an injection when restricted to $\mathcal{P}(H_i, k)$, we have $|A_i| = N_{H_i}(k)$. Moreover, as will be shown, when $2^{n-3} < k \le 2^{n-2}$ and $m \ge 2$, the computation of $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ can be transformed to the determination of partial 0/1-equivalence classes contained in the intersection of distinct spanned hyperplanes.

We now focus on the computation of $N_H(k)$, where H is a spanned hyperplane of Q_n and k is close to 2^{n-2}. Let $S \subseteq H$ be a subset of H. In Sect. 3, we have defined the partial 0/1-equivalence relation on the set of 0/1-polytopes of Q_n contained in S. Here we need another equivalence relation on this set, that is, two 0/1-polytopes are said to be equivalent if one can be transformed to the other by a symmetry in the stabilizer $F(H)$ of H. The associated equivalence classes are called local 0/1-equivalence classes of S. Since $F(H)$ is a subgroup of B_n, each local 0/1-equivalence class of S is contained in a unique partial 0/1-equivalence class of S.

Denote by $\mathcal{L}(S,k)$ the set of local 0/1-equivalence classes of S with k vertices. To compute $N_H(k)$ when k is close to 2^{n-2}, we need to compute the cardinality of $\mathcal{L}(H,k)$ and the cardinality of $\mathcal{L}(S,k)$, where S can be expressed as $S = H \cap w(H)$ for a symmetry w in B_n satisfying certain conditions. The cardinality of $\mathcal{L}(H,k)$ can be obtained from the cycle index $Z_H(z)$ of the stabilizer $F(H)$. In the following formula, $C_H(u_1,u_2)$ denotes the polynomial obtained from $Z_H(z)$ by substituting z_i with $u_1^i + u_2^i$, as defined in Sect. 3.

Lemma 7.2 *For any* $1 \le k \le 2^{n-1}$, *we have*

$$|\mathcal{L}(H,k)| = \left[u_1^k u_2^{|V_n(H)-k|}\right] C_H(u_1,u_2). \tag{7.2}$$

In the remaining of this section, we assume that $2^{n-3} < k \le 2^{n-2}$. Keep in mind that $N_H(k)$ is the cardinality of the set $\mathcal{P}(H,k)$ of partial 0/1-equivalence classes of H. To compute $|\mathcal{P}(H,k)|$, we shall define a subset $\mathcal{L}_1(H,k)$ of $\mathcal{L}(H,k)$ and a subset $\mathcal{P}_1(H,k)$ of $\mathcal{P}(H,k)$, which satisfy the following relation:

$$|\mathcal{P}(H,k)| = |\mathcal{L}(H,k)| - |\mathcal{L}_1(H,k)| + |\mathcal{P}_1(H,k)|.$$

We first define the subset $\mathcal{L}_1(H,k)$, which depends on a map Ψ from the set of local 0/1-equivalence classes of certain intersections $H \cap w(H)$ to the set $\mathcal{L}(H,k)$. To define Ψ, let $E(H,k)$ denote the set of affine subspaces $H \cap w(H)$, where w ranges over symmetries in B_n such that

(1) $H \ne w(H)$, that is, the symmetry w of Q_n does not fix H;
(2) $H \cap w(H)$ contains at least k vertices of Q_n.

Consider the equivalence classes of $E(H,k)$ under the symmetries in $F(H)$. This means that two elements $H \cap w(H)$ and $H \cap w'(H)$ in $E(H,k)$ are equivalent if there exists a symmetry $\sigma \in F(H)$ such that

$$H \cap w(H) = \sigma(H \cap w'(H)).$$

Denote by $h_1(H,k)$ the number of equivalence classes of $E(H,k)$ under the symmetries in $F(H)$. Let

$$E_1(H,k) = \{H \cap w_i(H) \mid 1 \le i \le h_1(H,k)\}$$

be the set of representatives of these equivalence classes of $E(H,k)$.

The map Ψ is defined from the (disjoint) union of $\mathcal{L}(H \cap w_i(H), k)$, where $1 \leq i \leq h_1(H, k)$, to $\mathcal{L}(H, k)$. Let $\mathcal{L}$ be a local 0/1-equivalence class in $\mathcal{L}(H \cap w_i(H), k)$. Define $\Psi(\mathcal{L})$ to be the unique local 0/1-equivalence class in $\mathcal{L}(H, k)$ containing $\mathcal{L}$. We have the following property.

Theorem 7.3 *For $n > 4$ and $2^{n-3} < k \leq 2^{n-2}$, the map Ψ is an injection.*

Proof Let $\mathcal{L}$ and $\mathcal{L}'$ be two distinct local 0/1-equivalence classes with k vertices. Assume that $\mathcal{L}$ is contained in $\mathcal{L}(H \cap w_i(H), k)$ and $\mathcal{L}'$ is contained in $\mathcal{L}(H \cap w_j(H), k))$, where $1 \leq i, j \leq h_1(H, k)$. To prove that Ψ is an injection, we need to show that $\Psi(\mathcal{L}) \neq \Psi(\mathcal{L}')$. If $i = j$, from the definition of the local 0/1-equivalence relation, it is clear that $\Psi(\mathcal{L}) \neq \Psi(\mathcal{L}')$.

We now consider the case $i \neq j$. Let P and P' be two 0/1-polytopes contained in $\mathcal{L}$ and $\mathcal{L}'$, respectively. We claim that $\dim(P) = \dim(P') = n - 2$. We only give a proof of the assertion that $\dim(P) = n - 2$. The relation $\dim(P') = n - 2$ can be justified by the same argument.

Since P has more than 2^{n-3} vertices, it follows from Theorem 1.1 that $\dim(P) \geq n - 2$. On the other hand, since P is contained in the intersection $H \cap w_i(H)$, we see that $\dim(P) \leq n - 2$. Hence we have $\dim(P) = n - 2$.

Based on the above claim, it can be shown that $\Psi(\mathcal{L}) \neq \Psi(\mathcal{L}')$. Suppose to the contrary that $\Psi(\mathcal{L}) = \Psi(\mathcal{L}')$. Then there is a symmetry $w \in F(H)$ such that $P = w(P')$. Since $\dim(P) = \dim(P') = n - 2$, we deduce that $H \cap w_i(H) = w(H \cap w_j(H))$, which contradicts the fact that $H \cap w_i(H)$ and $H \cap w_j(H)$ are not equivalent under the symmetries in $F(H)$. This completes the proof. □

We can now give the definition of the subset $\mathcal{L}_1(H, k)$ of $\mathcal{L}(H, k)$. Notice that for each $1 \leq i \leq h_1(H, k)$, $\Psi(\mathcal{L}(H \cap w_i(H), k))$ is a subset of $\mathcal{L}(H, k)$. By Theorem 7.3, these subsets are disjoint. We define $\mathcal{L}_1(H, k)$ to be the union of $\Psi(\mathcal{L}(H \cap w_i(H), k))$, where $1 \leq i \leq h_1(H, k)$.

We proceed to define the subset $\mathcal{P}_1(H, k)$ of $\mathcal{P}(H, k)$. Let $\overline{\mathcal{L}}_1(H, k)$ be the complement of $\mathcal{L}_1(H, k)$, that is,

$$\overline{\mathcal{L}}_1(H, k) = \mathcal{L}(H, k) \backslash \mathcal{L}_1(H, k). \tag{7.3}$$

In the above notation, for any local 0/1-equivalence class $\mathcal{L} \in \overline{\mathcal{L}}_1(H, k)$ and any 0/1-polytope $P \in \mathcal{L}$, if $w \in B_n$ is a symmetry such that $w(P)$ is contained in H, then $w(H) = H$. This yields that $\mathcal{L}$ is also a partial 0/1-equivalence class of H. Consequently, when $2^{n-3} < k \leq 2^{n-2}$, $\overline{\mathcal{L}}_1(H, k)$ is a subset of $\mathcal{P}(H, k)$. Define

$$\mathcal{P}_1(H, k) = \mathcal{P}(H, k) \backslash \overline{\mathcal{L}}_1(H, k). \tag{7.4}$$

From (7.3) and (7.4), we see that $N_H(k)$ can be expressed in terms of the cardinalities of $\mathcal{L}(H, k)$, $\mathcal{L}_1(H, k)$ and $\mathcal{P}_1(H, k)$. More precisely,

$$\begin{aligned} N_H(k) &= |\mathcal{P}(H, k)| \\ &= |\overline{\mathcal{L}}_1(H, k)| + |\mathcal{P}_1(H, k)| \\ &= |\mathcal{L}(H, k)| - |\mathcal{L}_1(H, k)| + |\mathcal{P}_1(H, k)|. \end{aligned} \tag{7.5}$$

By Lemma 7.2, $|\mathcal{L}(H,k)|$ can be computed from the cycle index $Z_H(z)$. From Theorem 7.3, $|\mathcal{L}_1(H,k)|$ can be derived from the cardinalities of $\mathcal{L}(H\cap w(H),k)$, where $H\cap w(H)\in E_1(H,k)$. To compute $|\mathcal{P}_1(H,k)|$, we need a map Γ defined as follows.

Let $h_2(H,k)$ denote the number of equivalence classes of $E(H,k)$ under the symmetries in B_n, and let

$$E_2(H,k)=\{H\cap w_i(H)\,|\,1\le i\le h_2(H,k)\}$$

be the set of representatives of these equivalence classes of $E(H,k)$. We define a map Γ from the (disjoint) union of $\mathcal{P}(H\cap w_i(H),k)$, where $1\le i\le h_2(H,k)$, to $\mathcal{P}_1(H,k)$. Let $\mathcal{P}$ be a partial 0/1-equivalence class in $\mathcal{P}(H\cap w_i(H),k)$. Then Γ maps $\mathcal{P}$ to the unique partial 0/1-equivalence class in $\mathcal{P}_1(H,k)$ that contains $\mathcal{P}$.

When $2^{n-3}<k\le 2^{n-2}$, it has been shown that each 0/1-polytope with k vertices contained in the intersection $H\cap w_i(H)$ has dimension $n-2$. This enables us to use the same argument as in the proof Theorem 7.3 to reach the following assertion.

Theorem 7.4 *For $n>4$ and $2^{n-3}<k\le 2^{n-2}$, the map Γ is a bijection.*

Combining Lemma 7.2, Theorem 7.3 and Theorem 7.4, formula (7.5) can be rewritten as

$$N_H(k)=\left[u_1^k u_2^{|V_n(H)|-k}\right]C_H(u_1,u_2)-\sum_{H\cap w(H)\in E_1(H,k)}|\mathcal{L}(H\cap w(H),k)| + \sum_{H\cap w(H)\in E_2(H,k)}|\mathcal{P}(H\cap w(H),k)|. \tag{7.6}$$

So, to compute $N_H(k)$, it is enough to determine $|\mathcal{L}(H\cap w(H),k)|$ and $|\mathcal{P}(H\cap w(H),k)|$. We can compute $|\mathcal{L}(H\cap w(H),k)|$ and $|\mathcal{P}(H\cap w(H),k)|$ by applying Pólya's theorem.

We first consider $|\mathcal{L}(H\cap w(H),k)|$. Let P and P' be any two 0/1-polytopes belonging to the same local 0/1-equivalence class in $\mathcal{L}(H\cap w(H),k)$. Then there exists a symmetry σ in $F(H)$ such that $\sigma(P)=P'$. It is clear from Theorem 1.1 that both P and P' have dimension $n-2$. So we deduce that $w'(H\cap w(H))=H\cap w(H)$.

Let $F_1(H,w)$ be the subgroup of $F(H)$ that stabilizes $H\cap w(H)$, that is,

$$F_1(H,w)=\{\sigma\in F(H)\,|\,\sigma\,(H\cap w(H))=H\cap w(H)\}\,.$$

Denote by $V_n(H\cap w(H))$ the set of vertices of Q_n contained in $H\cap w(H)$. Consider the action of $F_1(H,w)$ on $V_n(H\cap w(H))$. Assume that each vertex in $V_n(H\cap w(H))$ is assigned one of the two colors, say, black and white. Clearly, when $2^{n-3}<k\le 2^{n-2}$, this leads to a one-to-one correspondence between local 0/1-equivalence classes in $\mathcal{L}(H\cap w(H),k)$ and equivalence classes of colorings of the vertices in $V_n(H\cap w(H))$ with k black vertices.

Denote by $Z_{(H,w)}(z)$ the cycle index of $F_1\,(H,w)$ acting on $V_n(H\cap w(H))$. Write $C_{(H,w)}(u_1,u_2)$ for the polynomial obtained from $Z_{(H,w)}(z)$ by substituting z_i with $u_1^i+u_2^i$. For $2^{n-3}<k\le 2^{n-2}$, we obtain that

$$|\mathcal{L}(H \cap w(H), k)| = \left[u_1^k u_2^{|V_n(H \cap w(H))|-k}\right] C_{(H,w)}(u_1, u_2). \tag{7.7}$$

Similarly, we can use Pólya's theorem to compute $|\mathcal{P}(H \cap w(H), k)|$. Let $F_2\,(H, w)$ be the subgroup of B_n that stabilizes $H \cap w(H)$, that is,

$$F_2\,(H, w) = \left\{\sigma \in B_n \mid \sigma\,(H \cap w(H)) = H \cap w(H)\right\}.$$

Denote by $Z_{H \cap w(H)}(z)$ the cycle index of $F_2\,(H, w)$ acting on $V_n(H \cap w(H))$. Write $C_{H \cap w(H)}(u_1, u_2)$ for the polynomial obtained from $Z_{H \cap w(H)}(z)$ by substituting z_i with $u_1^i + u_2^i$. For $2^{n-3} < k \leq 2^{n-2}$, we have

$$|\mathcal{P}(H \cap w(H), k)| = \left[u_1^k u_2^{|V_n(H \cap w(H))|-k}\right] C_{H \cap w(H)}(u_1, u_2). \tag{7.8}$$

Now, plugging (7.7) and (7.8) into (7.6), we arrive at the following formula for $N_H(k)$.

Theorem 7.5 *Assume that $n > 4$ and $2^{n-3} < k \leq 2^{n-2}$. Let H be a spanned hyperplane of Q_n containing at least k vertices of Q_n. Let $q(w) = |V_n(H \cap w(H))|$. Then we have*

$$N_H(k) = \left[u_1^k u_2^{|V_n(H)|-k}\right] C_H(u_1, u_2) - \sum_{H \cap w(H) \in E_1(H,k)} \left[u_1^k u_2^{q(w)-k}\right] C_{(H,w)}(u_1, u_2) + \sum_{H \cap w(H) \in E_2(H,k)} \left[u_1^k u_2^{q(w)-k}\right] C_{H \cap w(H)}(u_1, u_2). \tag{7.9}$$

For $n = 6$ and $k = 13, 14, 15, 16$, we can use Theorem 7.5 to compute $N_H(k)$, where H is a spanned hyperplane of Q_6 containing more than 12 vertices. By the computation of Aichholzer [2], in addition to the spanned hyperplanes $H_6^1, H_6^2, H_6^3, H_6^4, H_6^5, H_6^6$, there are 8 representatives of equivalence classes of spanned hyperplanes of Q_6 containing more than 12 vertices, namely,

$$\begin{aligned}
&H_1: x_1 + x_2 + x_3 + 2x_4 = 2,\\
&H_2: x_1 + x_2 + x_3 + x_4 = 1,\\
&H_3: x_1 + x_2 + x_3 + x_4 + 2x_5 = 3,\\
&H_4: x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 = 3,\\
&H_5: x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2,\\
&H_6: x_1 + x_2 + x_3 + x_4 + 2x_5 = 2,\\
&H_7: x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3,\\
&H_8: x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 = 4.
\end{aligned}$$

Using a Maple program, when $k = 13, 14, 15, 16$, it is routine to check that $E(H, k) = \emptyset$ for $H = H_6^3, H_6^4, H_6^5, H_6^6$ and $H = H_1, H_2, \ldots, H_8$. Therefore, for these spanned hyperplanes, by Theorem 7.5 we obtain that

$$N_H(k) = \left[u_1^k u_2^{|V_n(H)|-k}\right] C_H(u_1, u_2). \tag{7.10}$$

The cycle indices $Z_H(z)$ for $H = H_6^3, H_6^4, H_6^5, H_6^6$ are given in Sect. 6. Using the techniques in Sect. 5, we can derive the cycle indices for $H_1, H_2, \ldots, H_5$, which are given below.

$$Z_{H_1}(z) = \frac{1}{48}\begin{pmatrix} z_1^{16} + 4z_{12}z_4 + 4z_3^2 z_6 z_1^2 z_2 + 2z_3^4 z_1^4 \\ +12z_2^8 + 8z_4^4 + 6z_1^4 z_2^6 + 5z_1^8 z_2^4 + 6z_6^2 z_2^2 \end{pmatrix},$$

$$Z_{H_2}(z) = \frac{1}{192}\begin{pmatrix} z_1^{16} + 68z_4^4 + 24z_6^2 z_2^2 + 16z_{12}z_4 + 8z_3^4 z_1^4 \\ + 39z_2^8 + 12z_1^4 z_2^6 + 8z_1^8 z_2^4 + 16z_3^2 z_6 z_1^2 z_2 \end{pmatrix},$$

$$Z_{H_3}(z) = \frac{1}{96}\left(z_1^{16} + 24z_6^2 z_2^2 + 8z_3^4 z_1^4 + 33z_2^8 + 6z_1^8 z_2^4 + 24z_4^4\right),$$

$$Z_{H_4}(z) = \frac{1}{120}\left(z_1^{15}+24z_5^3+30z_2 z_4^3 z_1+20z_1 z_3^2 z_6 z_2+20z_1^3 z_3^4+15z_1^3 z_2^6+10z_1^7 z_2^4\right),$$

$$Z_{H_5}(z) = \frac{1}{720}\begin{pmatrix} z_1^{15} + 120z_3 z_6^2 + 144z_5^3 + 40z_3^5 + 180z_1 z_2 z_4^3 \\ + 40z_1^3 z_3^4 + 60z_1^3 z_2^6 + 15z_1^7 z_2^4 + 120z_1 z_2 z_3^2 z_6 \end{pmatrix}.$$

For $H = H_6, H_7, H_8$, we obtain that $N_H(13) = 2$, $N_H(14) = 1$, and $N_H(15) = N_H(16) = 0$ without computing the cycle index $Z_H(z)$. For example, for $H = H_6$, since H_6 contains 14 vertices of Q_6, we have $N_H(14) = 1$ and $N_H(15) = N_H(16) = 0$. On the other hand, there are 14 0/1-polytopes with 13 vertices contained in H_6. It is easy to check that these 14 0/1-polytopes form two partial 0/1-equivalence classes. So we have $N_H(13) = 2$. Similarly, we get $N_H(13) = 2$, $N_H(14) = 1$, and $N_H(15) = N_H(16) = 0$ for $H = H_7, H_8$.

It remains to compute $N_H(k)$ for $H = H_6^1, H_6^2$ and $k = 13, 14, 15, 16$. We first consider H_6^1. Keep in mind that H_6^1 is the spanned hyperplane $x_1 = 0$. Thus, for H_6^1 and $k = 13, 14, 15, 16$, it is easily seen that the intersections $H_6^1 \cap w(H_6^1)$ in $E(H_6^1, k)$ form only one equivalence class under the symmetries in $F(H_6^1)$ or B_n. A representative of this equivalence class can be chosen as $H_6^1 \cap w(H_6^1)$, where $w = (1, 2)(3)(4)(5)(6)$. So we have

$$E_1(H_6^1, k) = E_2(H_6^1, k) = \left\{(x_1, x_2, \ldots, x_6) \in \mathbb{R}^6 \mid x_1 = x_2 = 0\right\}.$$

Moreover, for $k = 13, 14, 15, 16$, it is easy to check that if two 0/1-polytopes in $H_6^1 \cap w(H_6^1)$ with k vertices are equivalent under the symmetries in B_n, then they are equivalent under the symmetries in $F(H_6^1)$. This implies that each local 0/1-equivalence class of $H_6^1 \cap w(H_6^1)$ is also a partial 0/1-equivalence class of $H_6^1 \cap w(H_6^1)$ and vice versa. Hence we obtain

$$\mathcal{L}(H_6^1 \cap w(H_6^1), k) = \mathcal{P}(H_6^1 \cap w(H_6^1), k).$$

Therefore, for $k = 13, 14, 15, 16$, by formula (7.6) we have

$$N_{H_6^1}(k) = \left[u_1^k u_2^{32-k}\right] C_{H_6^1}(u_1, u_2). \tag{7.11}$$

We now compute $N_{H_6^2}(k)$ for $k = 13, 14, 15, 16$. Recall that H_6^2 is the spanned hyperplane $x_1 + x_2 = 1$. It is not hard to check that the intersections $H_6^2 \cap w(H_6^2)$ in $E(H_6^2, k)$ form two equivalence classes under the symmetries in $F(H_6^2)$ or B_n. Moreover, each equivalence class in $E(H_6^2, k)$ under the symmetries in $F(H_6^2)$ is an equivalence class in $E(H_6^2, k)$ under the symmetries in B_n and vice versa. The representatives of these two equivalence classes can be chosen as $H_6^2 \cap w_1(H_6^2)$ and $H_6^2 \cap w_2(H_6^2)$, where $w_1 = (1, 3, 2)(4)(5)(6)$ and $w_2 = (1, 3)(2, 4)(5)(6)$. Notice that the intersections $H_6^2 \cap w_1(H_6^2)$ and $H_6^2 \cap w_2(H_6^2)$ are of the following form:

$$H_6^2 \cap w_1(H_6^2) = \left\{(x_1, x_2, \dots, x_6) \in \mathbb{R}^6 \,|\, x_1 + x_2 = 1 \text{ and } x_2 + x_3 = 1\right\},$$
$$H_6^2 \cap w_2(H_6^2) = \left\{(x_1, x_2, \dots, x_6) \in \mathbb{R}^6 \,|\, x_1 + x_2 = 1 \text{ and } x_3 + x_4 = 1\right\}.$$

Since the set of vertices contained in $H_6^2 \cap w_1(H_6^2)$ is

$$\{(1, 0, 1, x_4, x_5, x_6),\ (0, 1, 0, x_4, x_5, x_6) \,|\, x_i = 0 \text{ or } 1 \text{ for } i = 4, 5, 6\},$$

it is easy to check that for $k = 13, 14, 15, 16$, if two 0/1-polytopes contained in $H_6^2 \cap w_1(H_6^2)$ with k vertices are equivalent under the symmetries in B_n, then they are equivalent under the symmetries in $F(H_6^2)$. This means that each local 0/1-equivalence class of $H_6^2 \cap w_1(H_6^2)$ is also a partial 0/1-equivalence class of $H_6^2 \cap w_1(H_6^2)$ and vice versa. So, we have

$$\mathcal{L}(H_6^2 \cap w_1(H_6^2), k) = \mathcal{P}(H_6^2 \cap w_1(H_6^2), k).$$

Therefore, by formula (7.6) we obtain that for $k = 13, 14, 15, 16$,

$$N_{H_6^2}(k) = \left[u_1^k u_2^{32-k}\right] C_{H_6^2}(u_1, u_2) + |\mathcal{P}(H_6^2 \cap w(H_6^2), k) - |\mathcal{L}(H_6^2 \cap w(H_6^2), k), \tag{7.12}$$

where $w = (1, 3)(2, 4)(5)(6)$.

Combining (7.10), (7.11) and (7.12) , for $n = 6$ and $k = 13, 14, 15, 16$, we obtain that

$$\sum_{i=1}^{h(6,k)} |A_i| = \sum_{i=1}^{6} \left[u_1^k u_2^{|V_6(H_6^i)|-k}\right] C_{H_6^i}(u_1, u_2) + \sum_{i=1}^{8} \left[u_1^k u_2^{|V_6(H_i)|-k}\right] C_{H_i}(u_1, u_2) + |\mathcal{P}(H_6^2 \cap w(H_6^2), k) - |\mathcal{L}(H_6^2 \cap w(H_6^2), k), \tag{7.13}$$

where $w = (1, 3)(2, 4)(5)(6)$.

By Lemma 7.1, to determine $H_6(k)$ for $k = 13, 14, 15, 16$, we still need to compute $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ for $m \geq 2$. We first consider the case $m = 2$. The computation of the general case can be carried out in the same way.

We now demonstrate how to compute $|A_i \cap A_j|$ for $1 \leq i < j \leq h(n, k)$. Let $E(H_i, H_j, k)$ be the set of affine subspaces $H_i \cap w(H_j)$ that contain at least k vertices of Q_n. Denote by $h(H_i, H_j, k)$ the number of equivalence classes in $E(H_i, H_j, k)$ under the symmetries in B_n, and let

$$E_1(H_i, H_j, k) = \{H_i \cap w_t(H_j) \mid 1 \le t \le h(H_i, H_j, k)\}$$

be the set of representatives of equivalence classes in $E(H_i, H_j, k)$.

We consider the union of the sets $\mathcal{P}(H_i \cap w_t(H_j), k)$ of partial 0/1-equivalence classes of $H_i \cap w_t(H_j)$ with k vertices, where $1 \le t \le h(H_i, H_j, k)$, and we define a map Υ from this set of partial 0/1-equivalence classes to $A_i \cap A_j$. Let $\mathcal{P}$ be a partial 0/1-equivalence class in $\mathcal{P}(H_i \cap w_t(H_j), k)$. Then there is a unique 0/1-equivalence class $\mathcal{P}'$ in $A_i \cap A_j$ that contains $\mathcal{P}$. Define $\Upsilon(\mathcal{P}) = \mathcal{P}'$. We have the following property. The proof is omitted since it is similar to that of Theorem 7.3.

Theorem 7.6 *For $n > 4$ and $2^{n-3} < k \le 2^{n-2}$, the map Υ is a bijection.*

As a consequence of Theorem 7.6, for $n > 4$ and $2^{n-3} < k \le 2^{n-2}$, we have

$$|A_i \cap A_j| = \sum_{t=1}^{h(H_i,H_j,k)} |\mathcal{P}(H_i \cap w_t(H_j), k)|. \tag{7.14}$$

The above approach can be used to determine $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ for $m \ge 3$. Let

$$E(H_{i_1}, \ldots, H_{i_m}, k)$$

be the set of affine subspaces $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$, where $w_2, \ldots, w_m$ are symmetries in B_n such that $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ contains at least k vertices of Q_n. Denote by $E_1(H_{i_1}, \ldots, H_{i_m}, k)$ the set of representatives of equivalence classes of $E(H_{i_1}, \ldots, H_{i_m}, k)$ under the symmetries in B_n.

Consider the union of the sets $\mathcal{P}(H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}), k)$ of partial 0/1-equivalence classes, where

$$H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}) \in E_1(H_{i_1}, \ldots, H_{i_m}, k).$$

We define a map Ω from this set of partial 0/1-equivalence classes to $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$. Let $\mathcal{P}$ be a partial 0/1-equivalence of $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$. Then Ω maps $\mathcal{P}$ to the unique 0/1-equivalence class in $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$ that contains $\mathcal{P}$. Using the same argument as in the proof of Theorem 7.3, we obtain the following property.

Theorem 7.7 *For $n > 4$ and $2^{n-3} < k \le 2^{n-2}$, the map Ω is a bijection.*

As a consequence of Theorem 7.7, we see that for $n > 4$ and $2^{n-3} < k \le 2^{n-2}$,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = \sum |\mathcal{P}(H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}), k)|, \tag{7.15}$$

where the sum ranges over the representatives $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ of equivalence classes in $E(H_{i_1}, \ldots, H_{i_m}, k)$.

The following theorem shows that for $m \geq 3$, the set $E(H_{i_1}, \ldots, H_{i_m}, k)$ is empty under certain conditions. When $n = 6$ and $k = 13, 14, 15, 16$, this property allows us to deduce that for any $m \geq 4$ and any spanned hyperplanes $H_{i_1}, \ldots, H_{i_m}$, the set $E(H_{i_1}, \ldots, H_{i_m}, k)$ is empty.

Theorem 7.8 *Let $n > 4$ and $2^{n-3} < k \leq 2^{n-2}$. If there exist $1 \leq p < q \leq m$ such that $E(H_{i_p}, H_{i_q}, k)$ is empty, then $E(H_{i_1}, \ldots, H_{i_m}, k)$ is empty.*

Proof Assume that there exist $1 \leq p < q \leq m$ such that $E(H_{i_p}, H_{i_q}, k)$ is empty. Suppose to the contrary that $E(H_{i_1}, \ldots, H_{i_m}, k)$ is nonempty. Let

$$S = H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$$

be an affine space belonging to $E(H_{i_1}, \ldots, H_{i_m}, k)$. Let w_1 be the identity element e in B_n. We claim that

$$S = w_p(H_{i_p}) \cap w_q(H_{i_q}). \tag{7.16}$$

Clearly, $S \subseteq w_p(H_{i_p}) \cap w_q(H_{i_q})$. Since $\dim(w_p(H_{i_p}) \cap w_q(H_{i_q})) = n-2$, to prove (7.16), it suffices to show that $\dim(S) = n-2$. Since S contains more than 2^{n-3} vertices of Q_n, by Theorem 1.1, we deduce that $\dim(S) \geq n-2$. But $S \subseteq w_p(H_{i_p}) \cap w_q(H_{i_q})$, so we have $\dim(S) = n-2$. This proves the claim.

Let $w = (w_p)^{-1}$. By (7.16), we see that $w(S)$ is an affine space in $E(H_{i_p}, H_{i_q}, k)$, contradicting the assumption that $E(H_{i_p}, H_{i_q}, k)$ is empty. This completes the proof. □

Using formulas (7.14) and (7.15), we can compute $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ for $n = 6$, $k = 13, 14, 15, 16$ and $m \geq 2$. We first consider the case when $m = 2$. Using a Maple program, it can be checked that there are only four pairs for which $E(H_i, H_j, k)$ is nonempty. Recall that for $t \leq n$, H_n^t denotes the hyperplane $x_1 + \cdots + x_t = \lfloor t/2 \rfloor$ in $\mathbb{R}^n$.

Case 1: (H_6^1, H_6^2). In this case, it can be easily checked that the affine subspaces in $E(H_6^1, H_6^2, k)$ form two equivalence classes under the symmetries in B_n. The representatives can be chosen as $H_6^1 \cap H_6^2$ and $H_6^1 \cap w(H_6^2)$, where $w = (1, 3, 2)(4)(5)(6)$. Notice that $w(H_6^2)$ is the hyperplane $x_2 + x_3 = 1$. So we have

$$E_1(H_6^1, H_6^2, k) = \{H_6^1 \cap H_6^2,\ H_6^1 \cap H_6^3\}. \tag{7.17}$$

Case 2: (H_6^1, H_6^3). In this case, the affine subspaces in $E(H_6^1, H_6^3, k)$ form only one equivalence class under the symmetries in B_n. The representative can be chosen as $H_6^1 \cap H_6^3$, and hence

$$E_1(H_6^1, H_6^3, k) = \{H_6^1 \cap H_6^3\}. \tag{7.18}$$

Case 3: (H_6^2, H_6^3). This case is similar to Case 2. We have

$$E_1(H_6^2, H_6^3, k) = \{H_6^1 \cap H_6^3\}. \tag{7.19}$$

Case 4: (H_6^2, H_6^4). In this case, it can be verified that

$$E_1(H_6^2, H_6^4, k) = \{H_6^2 \cap H_6^4\}. \tag{7.20}$$

By (7.17)–(7.20), we obtain that for $n = 6$ and $k = 13, 14, 15, 16$,

$$\sum_{1\le i<j\le h(6,k)} |A_i \cap A_j| = |\mathcal{P}(H_6^1 \cap H_6^2, k)| + 3|\mathcal{P}(H_6^1 \cap H_6^3, k)| + |\mathcal{P}(H_6^2 \cap H_6^4, k)|. \tag{7.21}$$

Finally, we compute $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ for $n = 6$, $k = 13, 14, 15, 16$ and $m \ge 3$. We claim that $E(H_{i_1}, \ldots, H_{i_m}, k)$ is empty for any $m \ge 4$. If this is not the case, then, by Theorem 7.8, for any $1 \le p < q \le m$, $E(H_{i_p}, H_{i_q}, k)$ is nonempty. Since $m \ge 4$, there are at least six pairs (H_i, H_j) with $1 \le i < j \le h(6, k)$ for which $E(H_i, H_j, k)$ is nonempty. However, as shown before, there are only four pairs (H_i, H_j) with $1 \le i < j \le h(6, k)$ for which $E(H_i, H_j, k)$ is nonempty, leading to a contradiction. So the claim is proved.

When $m = 3$, it is easy to check that $E(H_{i_1}, H_{i_2}, H_{i_3}, k)$ is nonempty if and only if

$$(H_{i_1}, H_{i_2}, H_{i_3}) = (H_6^1, H_6^2, H_6^3).$$

Moreover, we have

$$E_1(H_6^1, H_6^2, H_6^3, k) = \{H_6^1 \cap H_6^3\}.$$

Thus, for $n = 6$, $k = 13, 14, 15, 16$ and $m \ge 3$, we have

$$\sum_{1\le i_1<\cdots<i_m\le h(6,k)} |A_{i_1} \cap \cdots \cap A_{i_m}| = \begin{cases} |\mathcal{P}(H_6^1 \cap H_6^3, k)| & \text{if } m = 3, \\ 0 & \text{if } m > 3. \end{cases} \tag{7.22}$$

By Lemma 7.1 and formulas (7.13), (7.21) and (7.22), we deduce that for $n = 6$ and $k = 13, 14, 15, 16$,

$$\begin{aligned} H_6(k) = &\sum_{i=1}^{6} \left[u_1^k u_2^{|V_6(H_6^i)|-k}\right] C_{H_6^i}(u_1, u_2) + \sum_{i=1}^{8} \left[u_1^k u_2^{|V_6(H_i)|-k}\right] C_{H_i}(u_1, u_2) \\ &+ |\mathcal{P}(H_6^2 \cap w(H_6^2), k)| - |\mathcal{P}(H_6^1 \cap H_6^2, k)| - 2|\mathcal{P}(H_6^1 \cap H_6^3, k)| \\ &- |\mathcal{P}(H_6^2 \cap H_6^4, k)| - |\mathcal{L}(H_6^2 \cap w(H_6^2), k)|, \end{aligned} \tag{7.23}$$

where $w = (1, 3)(2, 4)(5)(6)$. Notice that for $w = (1, 3)(2, 4)(5)(6)$,

$$H_6^2 \cap w(H_6^2) = H_6^2 \cap H_6^4 = \{(x_1, x_2, \ldots, x_6) \in \mathbb{R}^6 \mid x_1 + x_2 = 1 \text{ and } x_3 + x_4 = 1\}.$$

Thus, (7.23) can be rewritten as

$$H_6(k) = \sum_{i=1}^{6} \left[u_1^k u_2^{|V_6(H_6^i)|-k}\right] C_{H_6^i}(u_1, u_2) + \sum_{i=1}^{8} \left[u_1^k u_2^{|V_6(H_i)|-k}\right] C_{H_i}(u_1, u_2)$$
$$- |\mathcal{P}(H_6^1 \cap H_6^2, k)| - 2|\mathcal{P}(H_6^1 \cap H_6^3, k)| - |\mathcal{L}(H_6^2 \cap w(H_6^2), k)|, \quad (7.24)$$

where $w = (1, 3)(2, 4)(5)(6)$.

As for $|\mathcal{P}(H_6^1 \cap H_6^2, k)|$, we notice that

$$H_6^1 \cap H_6^2 = \{(0, 1, x_3, x_4, x_5, x_6) \mid x_i = 0 \text{ or } 1 \text{ for } i = 3, 4, 5, 6\}.$$

Thus the vertices of Q_6 contained in $H_6^1 \cap H_6^2$ are in one-to-one correspondence with the vertices of Q_4. To be more specific, given a vertex $(0, 1, x_3, x_4, x_5, x_6)$ of Q_6 contained in $H_6^1 \cap H_6^2$, we get a vertex (x_3, x_4, x_5, x_6) of Q_4 and vice versa. Moreover, the partial 0/1-equivalence classes of $H_6^1 \cap H_6^2$ are in one-to-one correspondence with the 0/1-equivalence classes of Q_4. Hence, for $n = 6$ and $k = 13, 14, 15, 16$, we have

$$|\mathcal{P}(H_6^1 \cap H_6^2, k)| = \left[u_1^k u_2^{16-k}\right] C_4(u_1, u_2). \quad (7.25)$$

We now compute $|\mathcal{P}(H_6^1 \cap H_6^3, k)|$. Since

$$H_6^1 \cap H_6^3 = \{(0, x_2, x_3, x_4, x_5, x_6) \mid x_2 + x_3 = 1\},$$

we see that each vertex $(0, x_2, x_3, x_4, x_5, x_6)$ of Q_6 contained in $H_6^1 \cap H_6^3$ corresponds to a vertex $(x_2, x_3, x_4, x_5, x_6)$ of Q_5 contained in the spanned hyperplane H_5^2 of Q_5 and vice versa. Hence the partial 0/1-equivalence classes of $H_6^1 \cap H_6^3$ are in one-to-one correspondence with the partial 0/1-equivalence classes of the spanned hyperplane H_5^2 of Q_5. Therefore, for $n = 6$ and $k = 13, 14, 15, 16$, we have

$$|\mathcal{P}(H_6^1 \cap H_6^3, k)| = \left[u_1^k u_2^{16-k}\right] C_{H_5^2}(u_1, u_2). \quad (7.26)$$

Finally, we determine $|\mathcal{L}(H_6^2 \cap w(H_6^2), k)|$ for $w = (1, 3)(2, 4)(5)(6)$. By (7.7), we see that $|\mathcal{L}(H_6^2 \cap w(H_6^2), k)|$ can be obtained from the cycle index $Z_{(H_6^2, w)}(z)$. Using the technique in Sect. 5, we obtain that

$$Z_{(H_6^2, w)}(z) = \frac{1}{32} \left(z_1^{16} + 21 z_2^8 + 8 z_4^4 + 2 z_1^8 z_2^4\right).$$

Hence

$$|\mathcal{L}(H_6^2 \cap H_6^4, k)| = \left[u_1^k u_2^{16-k}\right] C_{(H_6^2, w)}(u_1, u_2), \quad (7.27)$$

where $C_{(H_6^2, w)}(u_1, u_2)$ is the polynomial obtained from $Z_{(H_6^2, w)}(z)$ by substituting z_i with $u_1^i + u_2^i$.

Table 7 $F_6(k)$ for $k = 13, 14, 15, 16$

k	13	14	15	16
$F_6(k)$	290159817	1051410747	3491461629	10665920350

Using (7.24)–(7.27), we can compute $H_6(k)$ for $k = 13, 14, 15, 16$. Since $F_6(k) = A_6(k) - H_6(k)$, we obtain $F_6(k)$ for $k = 13, 14, 15, 16$ as given in Table 7.

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References

1. Aichholzer, O.: Extreme properties of 0/1-polytopes of dimension 5. In: Kalai, G., Ziegler, G.M. (eds.) Polytopes: Combinatorics and Computation, vol. 29, DMV Seminar, pp. 111–130. Birkhäuser, Basel (2000)
2. Aichholzer, O.: Hyperebenen in Hyperkuben: Eine Klassifizierung und Quantifizierung. Diplomarbeit am Institut für Grundlagen der Informationsverarbeitung, TU Graz (1992). http://www.ist.tugraz.at/staff/aichholzer/research/publications/
3. Aichholzer, O.: http://www.ist.tugraz.at/staff/aichholzer//research/rp/rcs/info01poly/
4. Aichholzer, O., Aurenhammer, F.: Classifying hyperplanes in hypercubes. SIAM J. Discrete Math. **9**, 225–232 (1996). Accessed 2014
5. Aguila, R.P.: Enumerating the configuration in the n-dimensional orthogonal polytopes through Pólya's countings and a concise representation. In: 3rd International Conference on Electrical and Electronics Engineering, pp. 1–4. IEEE Computer Society, México (2006)
6. Bárány, I., Pór, A.: On 0–1 polytopes with many facets. Adv. Math. **161**, 209–228 (2001)
7. Billera, L.J., Sarangarajan, A.: All 0–1 polytopes are traveling salesman polytopes. Combinatorica **16**, 175–188 (1996)
8. Brualdi, R.A.: Introductory Combinatorics, 5th edn. Pearson Prentice Hall, Upper Saddle River (2010)
9. Chen, W.Y.C.: Induced cycle structures of the hyperoctahedral group. SIAM J. Discrete Math. **6**, 353–362 (1993)
10. Chen, W.Y.C., Stanley, R.P.: Derangements on the n-cube. Discrete Math. **115**, 65–75 (1993)
11. Fleiner, T., Kaibel, V., Rote, G.: Upper bounds on the maximal number of facets of 0/1-polytopes. Eur. J. Combin. **21**, 121–130 (2000)
12. Gillmann, R., Kaibel, V.: Revlex-initial 0/1-polytopes. J. Combin. Theory Ser. A **113**, 799–821 (2006)
13. Haiman, M.: A simple and relatively efficient triangulation of the n-cube. Discrete Comput. Geom. **6**, 287–289 (1991)
14. Harrison, M.A., High, R.G.: On the cycle index of a product of permutation group. J. Combin. Theory **4**, 277–299 (1968)
15. Kortenkamp, U., Richter-Gebert, J., Sarangarajan, A., Ziegler, G.M.: Extremal properties of 0/1-polytopes. Discrete Comput. Geom. **17**, 439–448 (1997)
16. Pólya, G.: Sur les types des propositions composées. J. Symbolic Logic **5**, 98–103 (1940)
17. Stanley, R.P.: Enumerative Combinatorics, vol. 1. Cambridge University Press, Cambridge (1999)
18. Ziegler, G.M.: Lectures on 0/1-polytopes. In: Kalai, G., Ziegler, G.M. (eds.) Polytopes: Combinatorics and Computation, vol. 29, DMV Seminar, pp. 1–41. Birkhäuser, Basel (2000)
19. Zong, C.M.: What is known about unit cubes. Bull. Amer. Math. Soc. **42**, 181–211 (2005)