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A New Topological Helly Theorem and Some Transversal Results

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Abstract We prove that for a topological space X with the property that $H_*(U) = 0$ for $* \geq d$ and every open subset U of X, a finite family of open sets in X has nonempty intersection if for any subfamily of size j, $1 \leq j \leq d+1$, the $(d-j)$-dimensional homology group of its intersection is zero. We use this theorem to prove new results concerning transversal affine planes to families of convex sets.

Keywords Helly theorem · Homology group · Transversal

Mathematics Subject Classification Primary 52A35 · 55N10

1 Introduction and Preliminaries

A prominent role in combinatorial geometry is played by the Helly Theorem [7], which states that a finite family of convex sets in $\mathbb{R}^d$ has nonempty intersection if and only if any subfamily of size at most $d+1$ has nonempty intersection. Helly himself realized in 1930 (see [9]) that a finite family of sets in $\mathbb{R}^d$ has nonempty intersection if for any subfamily of size at most $d+1$, its intersection is homeomorphic to a ball in $\mathbb{R}^d$. In fact, the result is true if we replace the notion of topological ball by the notion of acyclic set, see [3,10]. In 1970, Debrunner [8] proved that a finite family of open sets in $\mathbb{R}^d$ has nonempty intersection if for any subfamily of size j, $1 \leq j \leq d+1$, its intersection is $(d-j)$-acyclic. In fact, these hypothesis imply that each one the open sets are acyclic.

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Unlike all previous topological Helly results in the literature, where the ambient space is the euclidean space and the sets are acyclic, we only require as an ambient space a topological space X in which $H_*(U) = 0$ for $* \geq d$ and for any subfamily of size j of our sets, the $(d - j)$-dimensional homology group of its intersection to be zero.

The fact that this is a non-expensive topological Helly theorem—in the sense that it does not require the open sets to be simple—from the homotopy point of view (we only require its $(d - 1)$-dimensional homology group to be zero), allows us to prove interesting new results concerning transversal planes to families of convex sets.

Throughout this paper we use reduced singular homology with any nonzero coefficient group. Let U be a topological space. We say that $H_{-1}(U) = 0$ if and only if U is nonempty and we say that U is *connected* if and only if $H_0(U) = 0$. For $q = -1$ and $q = 0$, the exactness of the Mayer–Vietoris sequence should be understood as the statement that if $H_q(U)$ occurs there between two vanishing terms, then $H_q(U) = 0$. For an integer $n \geq -1$, we say that U is *n-acyclic* if $H_*(U) = 0$ for $-1 \leq * \leq n$. Furthermore, U is *acyclic* if $H_*(U) = 0$ for $* \geq -1$.

2 A Topological Helly-Type Theorem

We start with the following auxiliary proposition.

Proposition $\mathrm{A}_{(m,\lambda)}$. *Let $F = \{A_1, \dots, A_m\}$ be a family of open subsets of a topological space X and let $\lambda \geq 0$ be an integer. Suppose that for any subfamily $F' \subset F$ of size j, $1 \leq j \leq m$,*

$$H_{m-1-j+\lambda}\Big(\bigcap F'\Big) = 0.$$

Then

$$H_{m-2+\lambda}\Big(\bigcup F\Big) = 0.$$

Proof Proposition $\mathrm{A}_{(2,0)}$ claims that the union of two connected sets with nonempty intersection is a connected set, and Proposition $\mathrm{A}_{(2,\lambda)}$ is just the statement of the exactness of the Mayer–Vietoris sequence: $0 = H_\lambda(A_1) \oplus H_\lambda(A_2) \to H_\lambda(A_1 \cup A_2) \to H_{\lambda-1}(A_1 \cap A_2) = 0$.

The proof is by induction on m. In fact, we shall prove that Proposition $\mathrm{A}_{(m,\lambda)}$ together with Proposition $\mathrm{A}_{(m,\lambda+1)}$ implies Proposition $\mathrm{A}_{(m+1,\lambda)}$.

Suppose $F = \{A_0, \dots, A_m\}$ is a finite collection of $m + 1$ open subsets of X such that for $1 \leq j \leq m+1$ and any subfamily $F' \subset F$ of size j, $H_{m-j+\lambda}\big(\bigcap F'\big) = 0$. We will prove that $H_{m-1+\lambda}\big(\bigcup F\big) = 0$. Let us first prove, using Proposition $\mathrm{A}_{(m,\lambda+1)}$, that $H_{m-2+\lambda}(A_1 \cup \cdots \cup A_m) = 0$. This is so because for any subfamily $F' \subset \{A_1, \dots, A_m\}$ of size j, $1 \leq j \leq m$, we have $H_{m-1-j+(\lambda+1)}\big(\bigcap F'\big) = 0$.

Let us consider the Mayer–Vietoris exact sequence of the pair $((A_1 \cup \cdots \cup A_m), A_0)$:

$$0 = H_{m-1+\lambda}(A_0) \oplus H_{m-1+\lambda}(A_1 \cup \cdots \cup A_m) \to H_{m-1+\lambda}\big(\bigcup F\big) \\ \to H_{m-2+\lambda}(A_0 \cap (A_1 \cup \cdots \cup A_m)) = 0.$$

Since by hypothesis, $H_{m-1+\lambda}(A_0) = 0$, in order to conclude the proof of Proposition $\mathrm{A}_{(m+1,\lambda)}$ it is sufficient to prove that $H_{m-2+\lambda}(A_0 \cap (A_1 \cup \cdots \cup A_m)) = 0$. For that purpose, let $G = \{B_1, \ldots, B_m\}$ be the family of open subsets of X given by $B_i = A_0 \cap A_i$, $1 \le i \le m$. Note that for any subfamily $G' \subset G$ of size j, $1 \le j \le m$, $H_{m-1-j+\lambda}\big(\bigcap G'\big) = 0$. This is so because the homology group $H_{m-1-j+\lambda}\big(\bigcap G'\big) = H_{m-(j+1)+\lambda}\big(\bigcap F'\big) = 0$, where F' is the corresponding subfamily of F of size $j+1$. Then by Proposition $\mathrm{A}_{(m,\lambda)}$, $0 = H_{m-2+\lambda}\big(\bigcup G\big) = H_{m-2+\lambda}(A_0 \cap (A_1 \cup \cdots \cup A_m))$. This completes the proof. □

We now give the Topological Berge's Theorem. See [1].

Theorem $\mathbf{B}_{(m,\lambda)}$ *Let $F = \{A_1, \ldots, A_m\}$ be a family of open subsets of a topological space X and let $\lambda \ge 0$ be an integer. Suppose that*

(a) $H_{m-2+\lambda}\big(\bigcup F\big) = 0$;

(b) *for $1 \le j \le m-1$ and any subfamily $F' \subset F$ of size j,*

$$H_{m-2-j+\lambda}\big(\bigcap F'\big) = 0.$$

Then

$$H_{\lambda-1}\big(\bigcap F\big) = 0.$$

Proof The proof is by induction. Theorem $\mathrm{B}_{(2,0)}$ claims that two nonempty open sets whose union is connected must have a point in common and Theorem $\mathrm{B}_{(2,\lambda)}$ is just the statement of the exactness of the Mayer–Vietoris sequence: $0 = H_\lambda(A_1 \cup A_2) \to H_{\lambda-1}(A_1 \cap A_2) \to H_{\lambda-1}(A_1) \oplus H_{\lambda-1}(A_2) = 0$.

Let us prove that Theorem $\mathrm{B}_{(m,\lambda)}$ implies Theorem $\mathrm{B}_{(m+1,\lambda)}$. Let $F = \{A_0, A_1, \ldots, A_m\}$ as in Theorem $\mathrm{B}_{(m+1,\lambda)}$. That is, $H_{m-1+\lambda}\big(\bigcup F\big) = 0$ and for $1 \le j \le m$ and any subfamily $F' \subset F$ of size j we have $H_{m-1-j+\lambda}\big(\bigcap F'\big) = 0$. Let $G = \{B_1, \ldots, B_m\}$, where $B_i = A_0 \cap A_i$, $1 \le i \le m$. In order to prove that $H_{\lambda-1}\big(\bigcap F\big) = H_{\lambda-1}\big(\bigcap G\big) = 0$, it is enough to show that the family $G = \{B_1, \ldots, B_m\}$ satisfies properties (a) and (b) of Theorem $\mathrm{B}_{(m,\lambda)}$.

Proof of (a) We need to prove that $H_{m-2+\lambda}(A_0 \cap (A_1 \cup \cdots \cup A_{m-1})) = H_{m-2+\lambda}\big(\bigcup G\big) = 0$. Note that $H_{m-1+\lambda}\big(\bigcup F\big) = 0$ and $H_{m-2+\lambda}(A_0) = 0$. Furthermore, by Proposition $\mathrm{A}_{(m,\lambda)}$, for the family $\{A_1, \ldots, A_m\}$, we have that $H_{m-2+\lambda}(A_1 \cup \cdots \cup A_m) = 0$. Thus the conclusion follows from the Mayer–Vietoris exact sequence of the pair $(A_0, (A_1 \cup \cdots \cup A_m))$;

$$0 = H_{m-1+\lambda}(A_0 \cup \cdots \cup A_m) \to H_{m-2+\lambda}(A_0 \cap (A_1 \cup \cdots \cup A_m)) \\ \to H_{m-2+\lambda}(A_0) \oplus H_{m-2+\lambda}(A_1 \cup \cdots \cup A_m) = 0.$$

Proof of (b) For $1 \leq j \leq m-1$ and any subfamily $G^{'} \subset G$ of size j, $\bigcap G^{'} = \bigcap F^{'}$, where $F^{'} \subset F$ has size $j+1$. Thus $H_{m-1-(j+1)+\lambda}\big(\bigcap F^{'}\big) = H_{m-2-j+\lambda}\big(\bigcap G^{'}\big) = 0$. This completes the proof of Theorem $\mathrm{B}_{(m+1,\lambda)}$. □

We now state our main theorem.

Topological Helly Theorem *Let F be a finite family of open subsets of a topological space X. Let $d > 0$ be and integer such that $H_i(U) = 0$ for $i \geq d$ and every open subset U of X.*

Suppose that

$$H_{d-j}\big(\bigcap F^{'}\big) = 0$$

for any subfamily $F^{'} \subset F$ of size j, $1 \leq j \leq d+1$. Then

$$\bigcap F \neq \emptyset.$$

Furthermore, $\bigcap F$ is acyclic.

Proof Suppose the size of F is m. Take an integer $3 \leq n \leq m+1$. Using Theorem $\mathrm{B}_{(n-1-\lambda,\lambda)}$, from $\lambda = 0$ up to $\lambda = n-3$, we can prove the following:

Claim C_n. *Suppose that for every $1 \leq j \leq n-1$ and any subfamily $F^{'} \subset F$ of size j,*

$$H_{n-3}\big(\bigcup F^{'}\big) = 0, \text{ and}$$

for every $1 \leq j \leq n-2$ and any subfamily $F^{'} \subset F$ of size j,

$$H_{n-j-3}\big(\bigcap F^{'}\big) = 0.$$

Then for every $1 \leq j \leq n-1$ and any subfamily $F^{'} \subset F$ of size j,

$$H_{n-j-2}\big(\bigcap F^{'}\big) = 0.$$

Assume now $H_*(U) = 0$ for every $* \geq d$ and every open $U \subset X$ and suppose that $d \leq n-3$. By repeating the use of Claim C_n, from $n = d+3$ up to $n = m+1$, we obtain that $\bigcap F \neq \emptyset$.

Arguing as above and using Theorem $\mathrm{B}_{(m-1-\lambda,\lambda)}$, from $\lambda = 0$ up to $\lambda = m-3$, we obtain that $H_0\big(\bigcap F\big) = 0$. The conclusion of acyclicity can be achieved by repeating the use of Theorem $\mathrm{B}_{(n,\lambda)}$, $2 \leq n \leq m-1$, $1 \leq \lambda \leq m-3$. Note that in the case $m = d+2$, our argument does not produce $H_{d-1}\big(\bigcap F\big) = 0$, so we need to continue to $\lambda = m-2$. This concludes the proof of our main theorem. □

For completeness, we include here a Topological Breen's Theorem.

Theorem Σ_m. *Let $F = \{A_1, \ldots, A_m\}$ be a family of open subsets of a topological space X. Suppose that for $1 \leq j \leq m$ and any subfamily $F^{'} \subset F$ of size j,*

$$H_{j-2}\big(\bigcup F^{'}\big) = 0.$$

Then

$$\bigcap F \neq \emptyset.$$

Proof The proof is by induction. Theorem Σ_2 claims that two nonempty open sets whose union is connected must have a point in common. Suppose Theorem Σ_m is true and let $F = \{A_0, A_1, \ldots, A_m\}$ be a family of open subsets of X such that for $1 \leq j \leq m$ and any subfamily $F^{'} \subset F$ of size j, $H_{j-2}\big(\bigcup F^{'}\big) = 0$.

Let us prove first that for any subfamily $F^{'} \subset \{A_2, \ldots, A_m\}$ of size j, $0 \leq j \leq m-1$,

$$H_{j-1}\big((A_0 \cap A_1) \cup \bigcup F^{'}\big) = 0.$$

To do so, simply consider the Mayer–Vietoris exact sequence of the pair $\big(A_0 \cup \bigcup F^{'}, A_1 \cup \bigcup F^{'}\big)$:

$$\begin{aligned} 0 = H_j\big(A_0 \cup A_1 \cup \bigcup F^{'}\big) &\to H_{j-1}\big((A_0 \cap A_1) \cup \bigcup F^{'}\big) \\ &\to H_{j-1}\big(A_0 \cup \bigcup F^{'}\big) \oplus H_{j-1}\big(A_1 \cup \bigcup F^{'}\big) = 0. \end{aligned}$$

This implies that the family $\{A_0 \cap A_1, A_2, \ldots, A_m\}$ satisfies the hypothesis of Theorem Σ_m, and therefore by induction that $A_0 \cap A_1 \cap \cdots \cap A_m \neq \emptyset$. This completes the proof of this theorem. □

As an immediate consequence, we have the following theorem:

Topological Breen Theorem *Let F be a finite family of open subsets of a topological space X. Let $d > 0$ be and integer such that $H_*(U) = 0$ for $* \geq d$ and every open subset U of X.*

Suppose that

$$H_{j-2}\big(\bigcup F^{'}\big) = 0$$

for $1 \leq j \leq d+1$ and any subfamily $F^{'} \subset F$ of size j. Then

$$\bigcap F \neq \emptyset.$$

Remark The corresponding theorems for Čech cohomology groups are also true. Furthermore, the theorems in this section are true for a class of sets A_i in which the Mayer–Vietoris sequence of the pair $(A_0, (A_1 \cup \cdots \cup A_m))$ is exact. For example, when X is a polyhedron and every $A_i \subset X$ is a subpolyhedron.

3 Transversal Theorems

3.1 Preliminary Lemmas

We start with some notation.
Let $G(n, d)$, be the Grassmannian space of all n -planes in $\mathbb{R}^d$ through the origin and let $M(n, d)$ be the space of all affine n-planes in $\mathbb{R}^d$ as an open subset of $G(n+1, d+1)$.

Let F be a collection of nonempty convex sets in euclidean d-space $\mathbb{R}^d$ and let $0 \leq n < d$ be an integer. We denote by $T_n(F) \subset G(n, d)$ the topological space of all n-planes in $\mathbb{R}^d$ transversals to F; that is, the space of n-planes that intersect all members of F. We say that F is *separated* if for every $2 \leq n \leq d$ and every subfamily $F^{'} \subset F$ of size n, there is no $(n-2)$-plane transversal to $F^{'}$.

Let $F = \{A^1, \ldots, A^n\}$ be a collection of closed subsets of a metric space X and let $\varepsilon > 0$ be a real number. We denote by $F_\varepsilon = \{A^1_\varepsilon, \ldots, A^n_\varepsilon\}$ the collection of open subsets of X, where A_ε denotes the open ε-neighborhood of $A \subset X$.

Lemma 3.1 *Let A be a nonempty convex set in $\mathbb{R}^d$ and let $1 \leq n < d$. Then $T_n(\{A\})$ is homotopically equivalent to $G(n, d)$, the Grassmannian space of all n-planes in $\mathbb{R}^d$ through the origin.*

Proof Let $\Upsilon : T_n(\{A\}) \to G(n, d)$ be given as follows: for every $H \in T_n(\{A\})$, let $\Upsilon(H)$ be the unique n-plane through the origin parallel to H. Then if $\Gamma \in G(n, d)$, $\Upsilon^{-1}(\Gamma)$ is homeomorphic to $\pi(A)$, where $\pi : \mathbb{R}^d \to \Gamma^\perp$ is the orthogonal projection and $\Gamma^\perp \in G(d-n, d)$ is orthogonal to Γ. Since Υ has contractible fibers, it is a homotopy equivalence. □

Lemma 3.2 *Let $F = \{A_1, A_2, \ldots, A_n\}$ be a separated family of nonempty convex sets in $\mathbb{R}^d$, $2 \leq n \leq d$, and let $n \leq m \leq d$ be an integer. Then $T_{m-1}(F)$ is homotopically equivalent to $G(m-n, d-n+1)$.*

Proof We start by proving that $T_{n-1}(F)$ is contractible. For this purpose let $\Psi : A_1 \times \cdots \times A_n \to T_{n-1}(F)$ given by $\Psi((a_1, \ldots, a_n))$ be equal to the unique $(n-1)$-plane in $\mathbb{R}^d$ through $\{a_1, \ldots, a_n\}$, for every $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. Note that Ψ is well defined because F is a separated family of sets. Furthermore, if $H \in T_{n-1}(F)$, then $\Psi^{-1}(H) = (H \cap A_1) \times \cdots \times (H \cap A_n)$ is contractible. This implies that Ψ is a homotopy equivalence and hence that $T_{n-1}(F)$ is contractible.

Let $E = \{(H, \Gamma) \mid H$ is a $(n-1)$-plane of $\mathbb{R}^d$, Γ is a $(m-1)$-plane of $\mathbb{R}^d$ and $H \subset \Gamma\}$. Then $\gamma : E \to M(n-1, d)$, given by the projection in the first coordinate, is a classical fiber bundle with fiber $G(m-n, d-n+1)$. Now let $Y = \{(H, \Gamma) \in T_{n-1}(F) \times T_{m-1}(F) \mid H \subset \Gamma\}$. Clearly, the restriction $\gamma \mid: Y \to T_{n-1}(F)$ is a fiber bundle with fiber $G(m-n, d-n+1)$ and contractible base space $T_{n-1}(F)$. Therefore $\gamma \mid: Y \to T_{n-1}(F)$ is a trivial fiber bundle and hence Y is homotopically equivalent to $G(m-n, d-n+1)$.

Consider now the projection $\pi : Y \to T_{m-1}(F)$. Note that for every $\Gamma \in T_{m-1}(F)$, the fiber $\pi^{-1}(\Gamma)$ is equal to $T_{n-1}(\{A_1 \cap \Gamma, \ldots, A_n \cap \Gamma\})$. By the first part of this proof, the fibers of π are contractible, hence π is a homotopy equivalence and $T_{m-1}(F)$ is homotopically equivalent to $G(m-n, d-n+1)$. □

Lemma 3.3 *Let A, B, and C be three nonempty convex sets in $\mathbb{R}^d$ such that $A \cap B = \emptyset$. Then*

$$H_1(T_1(\{A, B, C\})) = 0.$$

Proof Since $A \cap B \cap C = \emptyset$, by Theorem 3 of [4], $T_1(\{A, B, C\})$ has the homotopy type of the space $\mathcal{C}_1(\{A, B, C\}) \subset \mathcal{C}_2^1$ of all affine configurations of three points in the line, achieved by transversal lines to $\{A, B, C\}$. Note now that the space of affine configuration of three points in a line, $\mathcal{C}_2^1$, is $\mathbb{S}^1$ and note further that since $A \cap B = \emptyset$, the space of all affine configurations of three points in the line achieved by transversal lines to $\{A, B, C\}$ is a subset $\mathbb{S}^1 - \{\infty\}$, where $\infty \in \mathbb{S}^1$ is the affine configuration in which the first and the second points coincide. This implies that $H_1(T_1(\{A, B, C\})) = 0$. □

Lemma 3.4 *Let $F = \{A^1, \dots, A^{d+1}\}$ be a separated family of closed convex sets in $\mathbb{R}^d$. Suppose that $H_0(T_{d-1}(F)) = 0$. Then there is $\varepsilon_0 > 0$ with the property that if $0 < \varepsilon < \varepsilon_0$, then $H_0(T_{d-1}(F_\varepsilon)) = 0$.*

Proof By Theorem 1 of [4], the space of transversals $T_{d-1}(F)$ of a separated family of convex sets in $\mathbb{R}^d$ has finitely many components and each one of them is contractible. In fact, each component corresponds precisely to a possible order type, of $d-1$ points in affine $(d-1)$-space, achieved by the transversal hyperplanes when they intersect the family F. In our case, since $H_0(T_{d-1}(F)) = 0$, we have that $T_{d-1}(F)$ is contractible and that the transversal hyperplanes intersect the family F consistently with a precise order type Ω.

Suppose now the lemma is not true, then there exist an order type Ω_0, different from Ω, and a collection of hyperplanes H_i that intersect F_{ε_i} consistently with the order type Ω_0. Since we may assume that $\{\varepsilon_i\} \to 0$ and $\{H_i\} \to H$, where H is a transversal hyperplane to F consistently with the order type Ω_0, we have a contradiction. □

3.2 Transversal Lines in the Plane

A family of sets is called *semipairwise disjoint* if, given any three elements of F, two of them are disjoint.

Theorem 3.1 *Let F be a semipairwise disjoint family of at least 6 open convex sets in $\mathbb{R}^2$. Suppose that for every subfamily $F' \subset F$ of size 5, $T_1(F') \neq \emptyset$ and for every subfamily $F' \subset F$ of size 4, $T_1(F')$ is connected. Then $T_1(F) \neq \emptyset$.*

Proof Let X be the space of all lines in $\mathbb{R}^2$. Hence $H_*(U) = 0$ for $* \geq 2$ and every open subset $U \subset X$. We are interested in applied the Topological Helly Theorem when $d = 4$. Note first that $H_3(T_1(\{A\}) = 0$ for every $A \in F$, and $H_2(T_1(\{A, B\}) = 0$ for $A \neq B \in F$. By Lemma 3.3 and the fact that F is semipairwise disjoint, we have that $H_1(T_1(F')) = 0$, for any subfamily $F' \subset F$ of size 3. By hypothesis, $H_0(T_1(\{F'\}) = 0$, for any subfamily $F' \subset F$ of size 4, and $H_{-1}(T_1(\{F'\}) = 0$, for any subfamily $F' \subset F$ of size 5. This implies, by the Topological Helly Theorem, that $T_1(F)$ is nonempty. □

3.3 Transversal Lines in 3-Space

In this section we study transversal lines to families of convex sets in $\mathbb{R}^3$.

Theorem 3.2 *Let F be a pairwise disjoint family of at least 6 open, convex sets in $\mathbb{R}^3$. Suppose that for any subfamily $F^{'} \subset F$ of size 5, $T_1(F^{'}) \neq \emptyset$, and for any subfamily $F^{'} \subset F$ of size 4, $T_1(F^{'})$ is connected. Then, $T_1(F) \neq \emptyset$.*

Proof Let X be the space of all lines in $\mathbb{R}^3$, hence X is an open 4-dimensional manifold and therefore $H_*(U) = 0$ for $* \geq 4$ and every open subset $U \subset X$. We are interested in applied the Topological Helly Theorem for $d = 4$. By Lemma 3.1, $H_3(T_1(\{A\}) = 0$, for every $A \in F$, since $T_1(\{A\})$ has the homotopy type of $G(1, 3) = \mathbb{RP}^2$. By Lemma 3.2, $H_2(T_1(\{A, B\})) = 0$, for every $A \neq B \in F$. By Lemma 3.3 and the fact that F is pairwise disjoint, we have that $H_1(T_1(F^{'})) = 0$, for any subfamily $F^{'} \subset F$ of size 3. By hypothesis, $H_0(T_1(\{F^{'}\}) = 0$, for any subfamily $F^{'} \subset F$ of size 4, and $H_{-1}(T_1(\{F^{'}\}) = 0$, for any subfamily $F^{'} \subset F$ of size 5. This implies, by the Topological Helly Theorem, that $T_1(F)$ is nonempty. □

3.4 Transversal Hyperplanes

This section is devoted to stating and proving a theorem concerning transversal hyperplanes to families of separated convex sets in d-space.

Theorem 3.3 *Let F be a separated family of at least $d + 3$ closed, convex sets in $\mathbb{R}^d$. Suppose that for any subfamily $F^{'} \subset F$ of size $d + 2$, $T_{d-1}(F^{'}) \neq \emptyset$ and for any subfamily $F^{'} \subset F$ of size $d + 1$, $T_{d-1}(F^{'})$ is connected. Then $T_{d-1}(F) \neq \emptyset$.*

Proof Let us first prove the theorem for a separated family of open convex sets. We are going to use the Topological Helly Theorem. Let X be the space of all hyperplanes of $\mathbb{R}^d$. Note that $H_*(U) = 0$ for $* \geq d$ and every open subset $U \subset X$. In particular, $H_*(U) = 0$ for every $* \geq d + 1$.

By Lemma 3.2, for every subfamily $F^{'} \subset F$ of size j, $1 \leq j \leq d$, $T_{d-1}(F^{'})$ is homotopically equivalent to $G(d - j, d - j + 1)$ and hence $H_{d-j+1}(T_{d-1}(F^{'})) = H_{d-j+1}\big(\bigcap\{T_{d-1}(\{A\}) \mid A \in F^{'}\}\big) = 0$. Furthermore, by hypothesis, the same is true for $j = d + 1$ and $j = d + 2$. Consequently, by our Topological Helly Theorem, $T_{d-1}(F) \neq \emptyset$.

By Lemma 3.4, there is $\varepsilon > 0$, such that F_ε is a separated family of open, convex sets in $\mathbb{R}^d$ and for any subfamily $F_\varepsilon^{'} \subset F_\varepsilon$ of size $d+1$, $T_{d-1}(F_\varepsilon^{'})$ is connected. By the above, this implies that $T_{d-1}(F_\varepsilon)) \neq \emptyset$. Hence, by completeness of the Grassmannian spaces, $T_{d-1}(F) \neq \emptyset$. □

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