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Some Remarks on the Circumcenter of Mass

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Abstract In this article, we give new proofs for the existence and basic properties of the circumcenter of mass defined by Adler in 1993 and Tabachnikov and Tsukerman in 2013.

We start with definitions.

Definition 1 *The power of a point* $\mathbf{x}$ with respect to a sphere $\omega(\mathbf{o}, R)$ in $\mathbb{R}^d$ is defined as $\mathrm{Pow}(\omega, \mathbf{x}) = \|\mathbf{ox}\|^2 - R^2$. Here $\mathbf{o}$ is the center and R is the radius of the sphere $\omega(\mathbf{o}, R)$.

Definition 2 Given a simplex Δ in $\mathbb{R}^d$, define

$$\mathrm{Pow}(\Delta) = \int_{\Delta} \mathrm{Pow}(\omega_\Delta, \mathbf{x})\mathrm{d}\mathbf{x},$$

where ω_Δ is the circumsphere of Δ.

Remark 1 If the sphere ω is a sphere of higher dimension passing through all vertices of ω_Δ, then the power of any point of Δ with respect to ω is the same as the power

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with respect to ω_Δ. Therefore, in the definition of $\mathrm{Pow}(\Delta)$, the circumscribed sphere could be changed to any sphere passing through the vertices of Δ.

Note also that the value of $\mathrm{Pow}(\Delta)$ is always negative.

Denote the vertices of Δ by $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d$; let $\mathbf{o}_\Delta$ and R_Δ be the center and the radius of the circumsphere, respectively; and let $\mathbf{m}_\Delta$ be the centroid of Δ. Then one has the following formulas for $\mathrm{Pow}(\Delta)$ (see. [5]):

$$-\mathrm{Pow}(\Delta) = \frac{\mathrm{Vol}(\Delta)}{(d+1)(d+2)}\left(\sum_{i=0}^{d}\sum_{j=0}^{i-1}\|\mathbf{v}_i\mathbf{v}_j\|^2\right)$$
$$= \frac{d+1}{d+2}\,\mathrm{Vol}(\Delta)\big(R_\Delta^2 - \|\mathbf{o}_\Delta\mathbf{m}_\Delta\|^2\big),$$

where $\mathrm{Vol}(\Delta)$ is the volume of Δ.

Lemma 1 *Given a simplex Δ in $\mathbb{R}^d$, denote by $\mathbf{n}_i$ the unit normal to the hyperface Δ_i directed in the exterior of Δ. Then*

$$\sum_{i=0}^{d}\mathrm{Pow}(\Delta_i)\mathbf{n}_i = 2\,\mathrm{Vol}(\Delta)\overrightarrow{\mathbf{o}_\Delta\mathbf{m}_\Delta}.$$

Proof Without loss of generality, assume that $\mathbf{o}_\Delta$ is the origin.

Let us use the following variant of the Gauss–Ostrogradsky theorem, also known as the gradient theorem:

$$\int_\Delta \mathrm{grad}\, f(\mathbf{x})\mathrm{d}v = \int_{\partial\Delta} f(\mathbf{x})\mathbf{n}(\mathbf{x})\mathrm{d}s,$$

where $\mathrm{d}v$ and $\mathrm{d}s$ are the volume elements of total space and of the surface of the simplex, respectively, and $\mathbf{n}(\mathbf{x})$ is the unit normal to the surface at a point $\mathbf{x}$.

Apply this equation to the power of a point with respect to the circumsphere, $f(\mathbf{x}) = \|\mathbf{x}\|^2 - R_\Delta^2$. Then $\mathrm{grad}\, f(\mathbf{x}) = 2\mathbf{x}$. Note also that $\int_{\Delta_i} f(\mathbf{x})\mathrm{d}s = \mathrm{Pow}(\Delta_i)$ since the sphere $(\mathbf{o}, R_\Delta)$ passes through the vertices of Δ_i. We obtain

$$2\,\mathrm{Vol}(\Delta)\overrightarrow{\mathbf{o}_\Delta\mathbf{m}_\Delta} = \int_\Delta 2\mathbf{x}\mathrm{d}v = \int_{\partial\Delta} f(\mathbf{x})\mathbf{n}(\mathbf{x})\mathrm{d}s = \sum_{i=0}^{d}\mathrm{Pow}(\Delta_i)\mathbf{n}_i.$$

□

Corollary 1 *Let $\mathcal{C}$ be a d-dimensional piece-wise linear simplicial cycle in $\mathbb{R}^d$. Let $\mathbf{o}_i$ and $\mathbf{m}_i$ be circumcenters and centroids of d-dimensional simplices $\Delta_i \in \mathcal{C}$, respectively. Then*

$$\sum_{\Delta_i\in\mathcal{C}}\overrightarrow{\mathbf{o}_i\mathbf{m}_i}\,\mathrm{Vol}(\Delta_i) = \mathbf{0}.$$

For the centroid, one has $\sum_{\Delta_i \in \mathcal{C}} \mathbf{m_i} \operatorname{Vol}(\Delta_i) = \mathbf{0}$, because each point is counted the same number of times with positive and negative sign. So, we obtain the following corollary.

Corollary 2 (*V. E. Adler, S. Tabachnikov, E. Tsukerman*) *Let $\mathcal{C}$ be a d-dimensional piece-wise linear simplicial cycle in $\mathbb{R}^d$. Suppose that $\mathbf{o}_i$ are the circumcenters of d-dimensional simplices $\Delta_i \in \mathcal{C}$. Then*

$$\sum_{\Delta_i \in \mathcal{C}} \mathbf{o}_i \operatorname{Vol}(\Delta_i) = \mathbf{0}.$$

Following [6], we give the following definition.

Definition 3 Let $\mathcal{K}$ be a d-dimensional piece-wise linear simplicial chain. Let $(\mathbf{o}_i, \operatorname{Vol}(\Delta_i))$ be the weighted point located at the circumcenter of $\Delta_i \in \mathcal{K}$ with the weight $\operatorname{Vol}(\Delta_i)$. The center of mass of points $(\mathbf{o}_i, \operatorname{Vol}(\Delta_i))$ of all simplices of $\mathcal{K}$ is called the *circumcenter of mass* of $\mathcal{K}$.

Remark 2 We can define the circumcenter of mass of any $(d-1)$-dimensional piece-wise linear simplicial cycle $\mathcal{C}$ in $\mathbb{R}^d$ as the circumcenter of mass of any its filling, that is $\mathcal{K}$, such that $\partial\mathcal{K} = \mathcal{C}$. Due to Corollary 2, the choice of filling for $\mathcal{C}$ does not matter.

It seems that Giusto Bellavitis was the first who noted the existence of the circumcenter of mass of a planar polygon in 1834 (see the book [3], pages 150–151).

In 1993, it was independently noticed by Adler in [1] for the case of triangulation of planar polygon by diagonals and in the private correspondence of G.C. Shephard and B. Grünbaum. They also noted that the circumcenter could be replaced by any point on the Euler line, that is, by a fixed affine combination of the centroid and the circumcenter (for example, the orthocenter or the center of the Euler circle).

Myakishev in [4] proved the existence of Euler (and also Nagel) line for a quadrilateral.

Tabachnikov and Tsukerman in [6] proved the correctness of definition of circumcenter of mass for any simplicial polytope and the existence of the Euler line in a high-dimensional polytope.

The case of central triangulation of a tetrahedron was posed on the student contest IMC 2009 (Problem 5).

In the planar case, we can take a polygon as a cycle. Using Lemma 1, we can give a short proof of the following theorem proved by S. Tabachnikov and E. Tsukerman.

Theorem 1 ([6]) *Let $P = \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n$ be an equilateral polygon. Then its circumcenter of mass coincides with centroid of the polygonal lamina.*

Proof Denote by $\mathbf{o}$ and $\mathbf{m}$ the circumcenter of mass and the centroid, respectively. Note that, for all i, the values $\operatorname{Pow}(\mathbf{a}_i\mathbf{a}_{i+1})$ are equal to each other. Denote this quantity by p, and let l be the length of the sides. We have $\operatorname{Vol}(P)\overrightarrow{\mathbf{mo}} = \frac{1}{2}\sum_{i=1}^{n} p\mathbf{n}_i$. Note that this sum is equal to zero because each vector $\mathbf{n}_i$ is the vector $\frac{1}{l}\overrightarrow{a_i a_{i+1}}$ rotated by 90°, but $\sum_{i=1}^{n} a_i a_{i+1} = 0$. □

Note that using the equation on $\mathrm{Pow}(\Delta)$, we can generalize this theorem to a higher dimension.

Theorem 2 *Let P be a simplicial polytope in $\mathbb{R}^d$ such that for each face of P the sum of squares of its edges is a constant. Then the circumcenter of mass and centroid of solid polytope P coincide.*

Proof Denote this constant by c. Note that for each facet Δ_i of P we have

$$\mathrm{Pow}(\Delta_i) = \mathrm{Vol}(\Delta_i)\frac{-c}{(d+1)(d+2)}.$$

From Minkowski's theorem, it follows that

$$\sum_{\Delta_i \in (\text{facets of } P)} \mathrm{Vol}(\Delta_i)\mathbf{n}_i = 0,$$

where $\mathbf{n}_i$ is a unit normal to the facet Δ_i.

Therefore,

$$\begin{aligned}\mathrm{Vol}(P)\overrightarrow{\mathbf{mo}} &= \frac{1}{2}\sum_{\Delta_i \in (\text{facets of } P)} \mathrm{Pow}(\Delta_i)\mathbf{n}_i \\ &= \frac{-c}{2(d+1)(d+2)}\sum_{\Delta_i \in (\text{facets of } P)} \mathrm{Vol}(\Delta_i)\mathbf{n}_i = 0.\end{aligned}$$

Remark 3 Using another formula for $\mathrm{Pow}(\Delta_i)$, we can reformulate the requirements on the facets of the polytope P in the following way: for each facet Δ_i the value $R^2_{\Delta_i} - \|\mathbf{o}_{\Delta_i}\mathbf{m}_{\Delta_i}\|^2$ is a constant.

As the authors have mentioned in [6], if the vertices of $\mathcal{C}$ lie on a sphere ω, then the circumcenter of mass coincides with center of the sphere. Indeed, there is a filling of $\mathcal{C}$ with the same set of vertices as $\mathcal{C}$. The circumcenters of the simplices of this filling coincide with the center of the sphere ω.

In the same article, Tabachnikov and Tsukerman have given a definition of circumcenter of mass in the spherical geometry. Using the previous observation, we can give another explanation of the existence of this point.

Consider the unit sphere $\mathcal{S}^d$ with the center at the origin $\mathbf{o}$ of $\mathbb{R}^{d+1}$. By a weighted point $(\mathbf{x}, m)$ we mean a pair consisting of a point $\mathbf{x}$ and a number m, which is natural to interpret as vector $m\mathbf{x}$ in $\mathbb{R}^{d+1}$. A set of weighted points $(\mathbf{x}_i, m_i)$ has the centroid at point $\frac{\sum m_i\mathbf{x}_i}{\|\sum m_i\mathbf{x}_i\|}$ and the total mass $\|\sum m_i\mathbf{x}_i\|$ (See [2]).

For each spherical d-simplex $\Delta_i = \mathbf{v}_0\mathbf{v}_1\ldots\mathbf{v}_d$ of a spherical simplicial chain $\mathcal{C}$, consider a point $\mathbf{o}'_i$ which is the circumcenter of the simplex $\Delta'_i = \mathbf{o}\mathbf{v}_0\mathbf{v}_1\ldots\mathbf{v}_d$ in $\mathbb{R}^{d+1}$. Now we can define the weighted circumcenter as the point $\mathbf{o}_i = \left(\frac{\mathbf{o}'_i}{\|\mathbf{o}'_i\|}, \mathrm{Vol}(\Delta'_i)\|\mathbf{o}'_i\|\right)$[1].

[1] Using simple calculation, it is easy to show that $\mathrm{Vol}(\Delta'_i)\|\mathbf{o}'_i\| = \frac{\mathrm{Vol}(\Delta_i)}{2(d+1)}$. So the circumcenter of mass from [6] is the same as here.

The $(d+1)$-dimensional complex in $\mathbb{R}^{d+1}$ formed by the simplices Δ'_i is denoted by $\mathcal{C}'$.

Corollary 3 *Suppose* $\mathcal{C}$ *is a* d*-dimensional simplicial cycle* $\mathcal{C}$ *in* $\mathcal{S}^d \subset \mathbb{R}^{d+1}$. *Then its spherical circumcenter of mass coincides with* **o** (*has zero weight*).

Proof By definition, the spherical circumcenter of mass $\mathcal{C}$ coincides with the Euclidean circumcenter of mass of $\mathcal{C}'$. But its circumcenter of mass coincides with the circumcenter of mass of $\partial\mathcal{C}'$ which is the origin, because $\partial\mathcal{C}'$ is inscribed in $\mathcal{S}^d$.

As in Remark 2, we can define the circumcenter of mass of a $(d-1)$-dimensional spherical simplicial cycle in $\mathcal{S}^d$ as the circumcenter of mass of its filling.

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