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On Packing $\mathbb{R}^3$ with Thin Tori

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Abstract We show that $\mathbb{R}^3$ can be packed at a density of $0.222\ldots$ with tori whose minor radius goes to zero. Furthermore, we show that the same torus arrangement yields an asymptotically optimal number of pairwise-linked tori.

Keywords Packing · Non-convex bodies · Linkage · Torus · Three space

Mathematics Subject Classification (2000) MSC 52C17

1 Introduction and Main Result

Geometric packing problems in $\mathbb{R}^3$ received huge attention over many decades (see [1,3,10–13,15,16,20–22,27] for books on packings). Still, the sphere is the only body which does not tile $\mathbb{R}^3$ and for which we know the exact packing density [17]. For other bodies such as platonic solids [5,23] and ellipsoids [2,9], dense packing constructions are known, but no proof of optimality exists and a vast amount of related questions remain open (see the books [4] and [8]).

On the other hand, there is only a very limited amount of the literature studying packings involving non-convex objects, such as the work of Jiao et al. [19,24,25]. We would thus like to extend this line of research by considering packings with the possibly simplest non-convex shape, the torus. Note that although Conway and Hopcroft showed in [6] that using the axiom of choice it is possible to fill $\mathbb{R}^3$ with unit circles,

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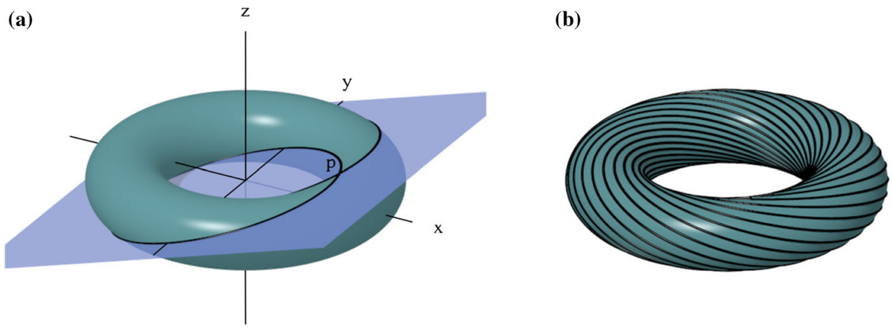

Fig. 1 **a** An illustration [14] of a bitangential plane and its two Villarceau circles through p. **b** A family of Villarceau circles obtained by rotating the bitangential cutting plane in discrete steps around the z-axis

the problem addressed here is of different types, since we are dealing with solid bodies and not with one-dimensional curve.

Throughout this note we refer to a torus with major radius 1 and minor radius going to zero as a *thin torus*.

Theorem 1 $\mathbb{R}^3$ *can be packed with thin tori at a density of* $0.222\ldots$.

Theorem 2 *It is possible to pairwise link* $\Theta(1/s^2)$ *many tori of major radius* 1 *and minor radius* $s < 1/3$ *and this is asymptotically optimal.*

2 Proof of Theorem 1

Given a torus lying in a plane H and a point p on its surface, there are three different types of non-trivial circles through p. One can draw a circle through p which lies in a plane parallel to the plane H and one can draw a circle through p in the plane perpendicular to H. A third type of circles is obtained by cutting the torus open at p along a plane which is bitangential to the torus and then draw two circles on the surface of the torus along the cut as shown in Fig. 1a. These circles have a radius which corresponds to the major radius of the torus and were first observed by Yvon Villarceau thus they are called *Villarceau circles* [26]. Most prominently, Villarceau circles appear in topology, when the Hopf fibration [18] of a 3-sphere is stereographically projected into $\mathbb{R}^3$.

In order for the torus arrangement to pack $\mathbb{R}^3$ with positive density, it is necessary that some tori are linked. This holds, since the volume of the bounding box of a torus with minor radius s is $\Theta(s)$, while the volume of the torus is $\Theta(s^2)$. Thus, for $s \to 0$, the packing density of an arrangement of unlinked thin tori goes to 0. Therefore, it is not a priori clear that a packing of thin tori with positive density exists.

Throughout this note we denote by $T(R, r)$ a torus of major radius R and minor radius r. The basic idea of the torus arrangement is to first pack $\mathbb{R}^3$ with an auxiliary lattice packing of fat tori $T_F = T(1, r)$ for some constant r. All such tori are placed parallel to the xy-plane and are centered at points of the lattice, generated by the vectors

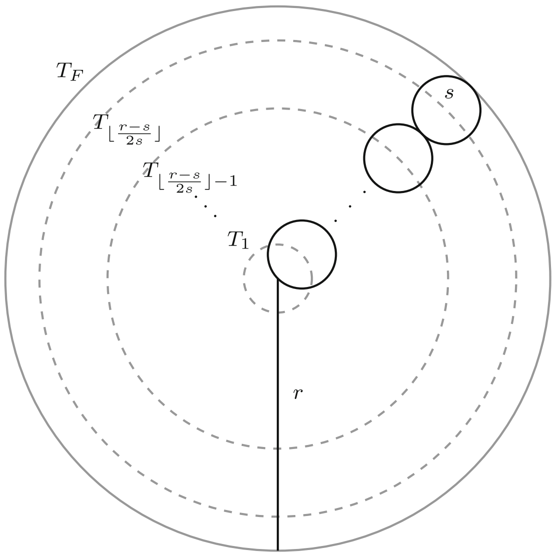

Fig. 2 A *vertical* cut through T_{F}, the nested sequence of T_k tori and a schematic representation of the $T(1, s)$ tori through one *point* on each T_k

$$\begin{pmatrix} 2+\sqrt{3}r \\ 0 \\ r \end{pmatrix}, \quad \begin{pmatrix} 1+r \\ \sqrt{3}(1+r) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 2r \end{pmatrix}.$$

Since the parallelepiped of this lattice has volume $2\sqrt{3}\left(2+\sqrt{3}r\right)(1+r)\,r$, the lattice packing of $\mathbb{R}^3$ with T_{F} tori has a density of

$$\delta_L(r) = \frac{\pi^2 r}{\sqrt{3}\left(2+\sqrt{3}r\right)\left(1+r\right)}. \tag{1}$$

Inside every T_{F} torus we build a second auxiliary structure by forming a nested sequence of concentric tori $T_{\lfloor \frac{r-s}{2s} \rfloor}, \ldots, T_1$ centered at lattice points, with $T_k = T(1, 2ks)$. Here, s denotes the minor radius of the thin tori which we will ultimately use to pack $\mathbb{R}^3$. Next, a $2s$ neighborhood of the surface of each T_k torus gets packed with thin $T(1, s)$ tori as illustrated in Fig. 2. For this we construct a family of Villarceau circles by rotating the bitangential cutting plane of T_k in discrete steps around the z-axis in such a way that the smallest distance between any two Villarceau circles is at least $2s$ (see Fig. 2). Note that we only chose one of the two resulting Villarceau circles associated with each cutting plane. Replacing every Villarceau circle by a $T(1, s)$ torus yields a packing of the $2s$ neighborhood of the surface of T_k. Denoting by V_k the volume of the union of all $T(1, s)$ tori on T_k, we obtain a total volume of

$$V = \sum_{k=1}^{\lfloor \frac{r-s}{2s} \rfloor} V_k$$

for the nested arrangement of thin tori inside T_{F} and thus a packing density of $\delta_T(r) = V/(2\pi^2 r^2)$ with respect to T_{F}. Therefore, the packing density of the $T(1, s)$ torus arrangement with respect to $\mathbb{R}^3$ is $\delta(r) = \delta_T(r)\delta_L(r)$.

The remaining part of this section is used to calculate the volume V.

Lemma 1 *Given a torus $T(R, r)$ lying in the xy-plane and two Villarceau circles c_0, c_1 lying in the bitangential planes H_0, H_1, respectively. In order for the minimum distance between c_0 and c_1 to be at least d, an angular distance between H_0 and H_1 of $\psi = \arcsin \frac{dR}{r(R-r)}$ around the z-axis suffices, given $r \leq R - d$.*

Proof A Villarceau circle of a torus $T(R, r)$ centered at the origin and lying in the xy-plane can be parameterized (see [7]) as

$$\mathbf{c}(\psi, t) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{R^2 - r^2}\cos t \\ r + R\sin t \\ r\cos t \end{pmatrix}.$$

Here ψ denotes the rotation angle of the bitangential cutting plane around the z-axis and t defines the location on the Villarceau circle. At $\psi = 0$, the derivatives of $\mathbf{c}$ are

$$\frac{\partial \mathbf{c}}{\partial t} = \begin{pmatrix} -\sqrt{R^2 - r^2}\sin t \\ R\cos t \\ -r\sin t \end{pmatrix}$$

and

$$\frac{\partial \mathbf{c}}{\partial \psi} = \begin{pmatrix} r + R\sin t \\ -\sqrt{R^2 - r^2}\cos t \\ 0 \end{pmatrix}.$$

The closest distance between the circle $c_0 = \mathbf{c}(0, t)$ and a circle c_1 rotated around the z-axis appears on c_0, when $\frac{\partial \mathbf{c}}{\partial \psi}$ assumes the smallest value in the direction perpendicular to $\frac{\partial \mathbf{c}}{\partial t}$, i.e., at the point where

$$\left|\frac{\partial \mathbf{c}}{\partial \psi}\right| \sin\gamma$$

attains its minimum, with γ denoting the angle between $\frac{\partial \mathbf{c}}{\partial t}$ and $\frac{\partial \mathbf{c}}{\partial \psi}$.

Since

$$\sin\gamma = \sqrt{1 - \left(\frac{\frac{\partial \mathbf{c}}{\partial \psi} \cdot \frac{\partial \mathbf{c}}{\partial t}}{\left|\frac{\partial \mathbf{c}}{\partial \psi}\right|\left|\frac{\partial \mathbf{c}}{\partial t}\right|}\right)^2}$$

we want to minimize

$$\sqrt{\left|\frac{\partial \mathbf{c}}{\partial \psi}\right|^2 - \left(\frac{\frac{\partial \mathbf{c}}{\partial \psi}\cdot\frac{\partial \mathbf{c}}{\partial t}}{\left|\frac{\partial \mathbf{c}}{\partial t}\right|}\right)^2},$$

which simplifies to

$$\frac{r}{R}\sqrt{\left(2\,r\,R\sin\left(t\right)+r^2+R^2+\left(\left(\sin\left(t\right)\right)^2-1\right)r^2\right)}$$

and attains its minimum at $t = -\arcsin R/r$. Since $r < R$, the minimum occurs when $\sin t = -1$. If we denote this point $(0, r - R, 0)$ by q, its minimum distance to the cutting plane H_1 provides a lower bound for the minimum distance d between c_0 and c_1. Since the normal unit vector of H_1 is $\mathbf{n_1} = (r/R\cos\psi, -r/R\sin\psi, \sqrt{1 - r^2/R^2})$,

$$d \geq \mathbf{n_1}\cdot q = \frac{r(R-r)}{R}\sin\psi.$$

Thus, an angular distance between H_0 and H_1 around the z-axis of at least

$$\psi = \arcsin\frac{dR}{r(R-r)}$$

results in a minimum distance of at least d between c_0 and c_1. □

It follows from Lemma 1 that the $2s$ neighborhood of the surface of a T_k torus can be packed with $\left\lfloor \frac{2\pi}{\arcsin\frac{1}{k(1-2ks)}}\right\rfloor$ many $T(1, s)$ tori. Using the fact that $11x/7 \geq \arcsin x$ for $0 \leq x \leq 1$, the volume of the union of these tori is

$$V_k \geq 2\pi^2 s^2 \left\lfloor \frac{14}{11}\pi k(1-2ks)\right\rfloor.$$

For $s \to 0$, the volume of the union of all $T(1, s)$ tori inside one T_F torus becomes

$$\overline{V} = \lim_{s\to 0}\sum_{k=1}^{\lfloor\frac{r-s}{2s}\rfloor} V_k \geq \tfrac{7}{66}\pi^3 r^2\left(3-2\,r\right),$$

and thus the packing density $\delta_T(r)$ with respect to T_F is

$$\delta_T(r) = \frac{7}{132}\pi(3-2r). \tag{2}$$

Combining Eqs. (1) and (2), the packing density of the thin tori with respect to $\mathbb{R}^3$ evaluates to

$$\delta(r) = \delta_L(r)\delta_T(r) = \frac{7\pi^3 r\left(3-2\,r\right)}{132\sqrt{3}\left(2+\sqrt{3}r\right)\left(1+r\right)}$$

which, at $r = 0.441\ldots$, obtains the maximum value of $0.222\ldots$.

3 Proof of Theorem 2

In order to prove Theorem 2, we show that in the nested torus construction of the previous section all thin tori are pairwise linked. Let c_0 and c_1 be two Villarceau circles on two nested (auxiliary) unit tori T_0 and T_1 with minor radius r_0 and r_1, respectively, and assume $r_0 < r_1$. We prove the claim by showing that c_1 and T_0 are linked, which, since Villarceau circles are fully contained in the surface of a torus, is equivalent to showing that c_1 is linked with the unit circle c defining the axis of revolution of T_0. Furthermore, wlog we may assume that the cutting plane defining c_1 is unrotated (around the z-axis), since otherwise we may just rotate the whole coordinate system around the z-axis. It thus follows that c_1 intersects the y-axis at distances $-1 + r_1$ and $1 + r_1$. In order to show that c_1 and c are linked, it suffices to note that y-axis intersects c inside the interval $(-1+r_1, 1+r_1)$ exactly once, namely at 1. Since all Villarceau circles on the same torus are linked (see [7]), the claim follows.

We set the minor radius r of T_F, defined in the previous section, equal to 1, i.e., T_F becomes a Horn torus and we note that the outermost torus of the nested structure inside T_F has minor radius $1 - s$. As argued in the previous section, the number of pairwise-linked tori of minor radius s in the nested construction is

$$\sum_{k=1}^{\lfloor \frac{1-s}{2s} \rfloor} \left\lfloor \tfrac{14}{11}\pi k(1 - 2ks) \right\rfloor,$$

which is lower bounded by

$$\frac{1}{132}\,\frac{(3\,s-1)\big(-7\,\pi-21\,\pi\,s+66\,s+28\,\pi\,s^2\big)}{s^2}. \tag{3}$$

It is easy to see that expression (3) is contained in $\Omega(1/s^2)$ for $s \in (0, 1/3)$. On the other hand, a simple area argument shows that at most $O(1/s^2)$ tori of major radius 1 and minor radius s can be linked with a single such torus, thus implying that the construction is asymptotically optimal. Obviously, the construction is also asymptotically optimal for linking many tori with one single torus.

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