

On Gromov's Method of Selecting Heavily Covered Points

Jiří Matoušek · Uli Wagner

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Abstract A result of Boros and Füredi ($d = 2$) and of Bárány (arbitrary d) asserts that for every d there exists $c_d > 0$ such that for every n -point set $P \subset \mathbb{R}^d$, some point of \mathbb{R}^d is covered by at least $c_d \binom{n}{d+1}$ of the d -simplices spanned by the points of P . The largest possible value of c_d has been the subject of ongoing research. Recently Gromov improved the existing lower bounds considerably by introducing a new, topological proof method. We provide an exposition of the combinatorial component of Gromov's approach, in terms accessible to combinatorialists and discrete geometers, and we investigate the limits of his method. In particular, we give tighter bounds on the *cofilling profiles* for the $(n - 1)$ -simplex. These bounds yield a minor improvement over Gromov's lower bounds on c_d for large d , but they also show that the room for further improvement through the cofilling profiles alone is quite small. We also prove a slightly better lower bound for c_3 by an approach using an additional structure besides the cofilling profiles. We formulate a combinatorial extremal problem whose solution might perhaps lead to a tight lower bound for c_d .

Keywords Higher-dimensional expanders · Selection theorems · Multiplicities of maps · Cofilling profiles

J. Matoušek
Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Praha 1,
Czech Republic

J. Matoušek
Institute of Theoretical Computer Science, ETH Zürich, 8092 Zurich, Switzerland

U. Wagner (✉)
IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria
e-mail: uli@ist.ac.at

1 Introduction

Let $P \subset \mathbb{R}^2$ be a set of n points in general position (i.e., no three points collinear). Boros and Füredi [3] showed that there always exists a point $a \in \mathbb{R}^2$ contained in a positive fraction of all the $\binom{n}{3}$ triangles spanned by P , namely, in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ triangles. (Generally we cannot assume $a \in P$, as the example of points in convex position shows.)

This result was generalized by Bárány [2] to point sets in arbitrary fixed dimension:

Theorem 1 (Bárány [2]) *Let P be a set of n points in general position in \mathbb{R}^d (i.e., no $d + 1$ or fewer of the points are affinely dependent). Then there exists a point in \mathbb{R}^d that is contained in at least*

$$c_d \cdot \binom{n}{d+1} - O(n^d)$$

d -dimensional simplices spanned by the points in P , where the constant $c_d > 0$ (as well as the constant implicit in the O -notation) depend only on d .

The largest possible value of c_d has been the subject of ongoing research. With some abuse of notation, we will henceforth denote by c_d the largest possible constant for which Theorem 1 holds true.

Upper Bounds. Bukh, Matoušek and Nivasch [4] showed that¹

$$c_d \leq \frac{(d+1)!}{(d+1)^{(d+1)}} \sim \frac{\sqrt{2\pi d}}{e^d} = e^{-\Theta(d)} \quad (1)$$

by constructing examples of n -point sets P in \mathbb{R}^d for which no point in \mathbb{R}^d is contained more than $\binom{n}{d+1}^{d+1} - O(n^d)$ many d -simplices spanned by P .

Lower Bounds. The result of Boros and Füredi says that $c_2 \geq 2/9$, which matches the upper bound in (1). For general d , Bárány's proof yields

$$c_d \geq \frac{1}{(d+1)^d},$$

which is smaller than (1) by a factor of $d!$. In [20], the lower bound was improved by a factor of roughly d to

$$c_d \geq \frac{d^2 + 1}{(d+1)^{d+1}},$$

but this narrowed the huge gap between upper and lower bounds only slightly. Moreover, Bukh, Matoušek and Nivasch showed that the method in [20] cannot be pushed farther. An improvement of the lower bound for c_3 by a clever elementary geometric argument was recently achieved by Basit et al. [5].

We refer to [4] for a more detailed discussion of the problem and related results.

¹ In [4], a different normalization is used, in the sense that simplices are counted in the form $c_d \cdot n^{d+1} - O(n^d)$. This means that the bounds on c_d given in [4] have to be multiplied by $(d+1)!$ to match our normalization.

Gromov’s Results. Recently, Gromov [9] introduced a new, topological proof method, which improves on the previous lower bounds considerably and which, moreover, applies to a more general setting, described next.

An n -point set $P \subseteq \mathbb{R}^d$ determines an affine map T from the $(n - 1)$ -dimensional simplex $\Delta^{n-1} \subseteq \mathbb{R}^{n-1}$ to \mathbb{R}^d as follows. Label the vertices of Δ^{n-1} by $V = \{v_1, \dots, v_n\}$, and the points as $P = \{p_1, \dots, p_n\}$. Then T is given by mapping v_i to p_i , $1 \leq i \leq n$, and by interpolating linearly on the faces Δ^{n-1} .

Thus, Bárány’s result can be restated by saying that for any *affine* map $T : \Delta^{n-1} \rightarrow \mathbb{R}^d$, there exists a point in \mathbb{R}^d that is contained in the ψ -images of at least $c_d \binom{n}{d+1} - O(n^d)$ many d -dimensional faces of Δ^{n-1} .

Gromov shows that, more generally², the following is true:

Theorem 2 ([9]) *For an arbitrary continuous map $T : \Delta^{n-1} \rightarrow \mathbb{R}^d$, there exists a point in \mathbb{R}^d that is contained in the T -images of at least $c_d^{\text{top}} \cdot \binom{n}{d+1}$ many d -faces of Δ^{n-1} , where $c_d^{\text{top}} > 0$ is a constant that depends only on d .*

By the same abuse of notation as before, c_d^{top} will also henceforth denote the largest constant for which Theorem 2 holds. In particular, $c_d \geq c_d^{\text{top}}$.

Gromov’s method gives

$$c_d \geq c_d^{\text{top}} \geq \frac{2d}{(d + 1)!(d + 1)}. \tag{2}$$

For $d = 2$, this yields the tight bound $c_2^{\text{top}} = c_2 = 2/9$. For general d , Gromov’s result improves on the earlier bounds by a factor exponential in d , but it is still of order $e^{-\Theta(d \log d)}$ and thus far from the upper bound.

One of the goals of this paper is to provide an exposition of the combinatorial component of Gromov’s approach, in terms accessible to combinatorialists and discrete geometers.

After a preliminary version of this paper was written and circulated, Karasev [12] found a very short and elegant proof of Gromov’s bound (2) for *affine* maps.

Karasev’s proof, which he himself describes as a “decoded and refined” version of Gromov’s proof, combines probabilistic and topological arguments, but he avoids the heavy topological machinery applied in Gromov’s proof and only uses the elementary notion of the degree of a piecewise smooth map between spheres.

Karasev’s argument can be modified and extended so that it covers the case of arbitrary continuous maps into \mathbb{R}^d (but not yet into general d -dimensional manifolds). Furthermore, the combinatorial and the topological aspects of the argument, which are intertwined in Karasev’s proof, can be split into two independent parts. In this way, any combinatorial improvement on the cofilling profiles or on the pagoda problem introduced in Sect. 6 immediately implies improved bounds for Theorem 2 also via

² In fact, Gromov’s approach is still much more general than this and applies to continuous maps from arbitrary finite simplicial complexes X to arbitrary d -dimensional manifolds Y . The method yields lower bounds for the maximum number of d -simplices of X whose images share a common point as long as X has certain *expansion properties*. This will be briefly explained in Sect. 3.5 below.

this simpler topological route. We plan to discuss this in more detail in a separate note.

Coboundaries and Cofilling Profiles. We need two basic notions, *coboundaries* and *cofilling profiles*, which have their roots in cohomology but which can be defined in elementary and purely combinatorial terms.

Let V be a fixed set of n elements, w.l.o.g., $V = [n] := \{1, 2, \dots, n\}$. We will always assume that n is sufficiently large.

In topological terms, we think of the $(n - 1)$ -dimensional simplex Δ^{n-1} as a combinatorial object (an abstract simplicial complex), namely, as the system of all subsets of V . Thus, given a subset $f \subseteq V$, we will also sometimes refer to f as a *face* of Δ^{n-1} , and the *dimension* of a face is defined as $\dim f := |f| - 1$.

Let $E \subseteq \binom{V}{d}$ be a system of (unordered) d -tuples, or in other words, of $(d - 1)$ -dimensional faces.

We write $\|E\| := |E|/\binom{n}{d}$ for the *density* (normalized support size) of E ; one can also interpret $\|E\|$ as the probability that a random d -tuple lies in E . The notation $\|E\|$ implicitly refers to d and n , which have to be understood from the context.

The *coboundary* δE is the system of those $(d + 1)$ -tuples $f \in \binom{V}{d+1}$ that contain an odd number of $e \in E$. (For $d = 2$, this notion and some of the following considerations are related to *Seidel switching* and *two-graphs*, which are notions studied in combinatorics—see the end of Sect. 2 for an explanation and references.)

We emphasize that δE also depends on the ground set V , and sometimes we may write $\delta_V E$ instead of δE to avoid ambiguities.

Many different E 's may have the same coboundary. We call E *minimal* if $\|E\| \leq \|E'\|$ for every E' with $\delta E' = \delta E$.

We define the *cofilling profile*³ as follows:

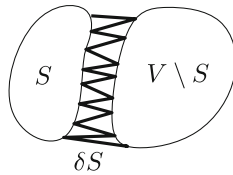
$$\varphi_d(\alpha) := \liminf_{|V|=n \rightarrow \infty} \min \{ \|\delta E\| : E \subseteq \binom{V}{d} \text{ minimal, } \|E\| \geq \alpha \}.$$

Equivalently, one can also view this notion as follows. Suppose we are given a system $F \subseteq \binom{V}{d+1}$, and we are guaranteed that F is a coboundary, i.e., that $F = \delta E$ for some E . Now we want to know the smallest possible density $\|E\|$ of an E with $F = \delta E$, as a function of $\|F\|$. It suffices to consider minimal E 's, and $\varphi(\alpha) \geq \beta$ means that if we are forced to take $\|E\| \geq \alpha$, then we must have $\|F\| \geq \beta$.

We also remark that there is no minimal E with $\|E\| > 1/2$ (see Sect. 2), so formally, $\varphi_d(\alpha) = \infty$ for $\alpha > 1/2$ (since we take the minimum over an empty set).

As a warm-up, let us consider case $d = 1$: In this case, we can view an $S \subseteq \binom{V}{1}$ simply as a subset of $S \subseteq V$, and δS is the set of edges of the complete bipartite graph with color classes S and $V \setminus S$; in graph theory, one also speaks of the *edge cut* determined by S in the complete graph on V .

³ Gromov uses the notation $\|(\partial^{d-1})_{\text{fil}}^{-1}\|(\beta)$ for what we would write as $\varphi_d^{-1}(\beta)/\beta$. Actually, he does not take the \liminf , which we use in order to avoid dealing with small values of n .



The minimality of \$S\$ simply means that \$|S| \le n/2\$. It follows that \$\varphi_1(\alpha) = 2\alpha(1-\alpha)\$, \$0 \le \alpha \le \frac{1}{2}\$.

For general \$d\$, the following basic bound for \$\varphi_d\$ was observed by Gromov, and independently by Linial, Meshulam, and Wallach [15, 17] (and maybe by others). In our terminology, it can be stated as follows:

Lemma 3 (Basic Cofilling Bound) *For every \$d \ge 1\$ and all \$\alpha \in [0, 1]\$,*

$$\varphi_d(\alpha) \ge \alpha.$$

We will recall a simple combinatorial proof of this bound, along with other basic properties of the coboundary operator, in Sect. 2. A simple example shows that the basic bound is attained with equality for \$\alpha = \frac{(d+1)!}{(d+1)^{(d+1)}} \approx e^{-(d+1)}\$, but for smaller \$\alpha\$, improvements are possible, as we will discuss later.

In Sect. 3, we present an outline of topological part of Gromov’s argument, which yields the following general lower bound.

Proposition 4 ([9]) *For every \$d \ge 1\$,*

$$c_d^{\text{top}} \ge \varphi_d\left(\frac{1}{2}\varphi_{d-1}\left(\frac{1}{3}\varphi_{d-2}\left(\dots\frac{1}{d}\varphi_1\left(\frac{1}{d+1}\right)\dots\right)\right)\right).$$

Theorem 2 follows from this proposition by using \$\varphi_1(\alpha) = 2\alpha(1-\alpha)\$ and the basic bound for all \$d \ge 2\$.

Better bounds on the \$c_d^{\text{top}}\$ would immediately follow from Proposition 4 if one could improve on the basic cofilling bound in an appropriate range of \$\alpha\$’s; this is a purely combinatorial question (and a quite nice one, in our opinion).

We establish the following lower bounds for \$\varphi_2\$ and \$\varphi_3\$:

Theorem 5 *For \$d = 2\$ and all \$\alpha \le \frac{1}{4}\$, we have the lower bound*

$$\varphi_2(\alpha) \ge \frac{3}{4}(1 - \sqrt{1 - 4\alpha})(1 - 4\alpha) = \frac{3}{2}\alpha - \frac{9}{2}\alpha^2 - 3\alpha^3 - O(\alpha^4).$$

Figure 1 shows a plot of this lower bound.

Theorem 6 *For \$d = 3\$ and \$\alpha\$ sufficiently small,*

$$\varphi_3(\alpha) \ge \frac{4}{3}\alpha - O(\alpha^2)$$

(with a constant that could be made explicit).

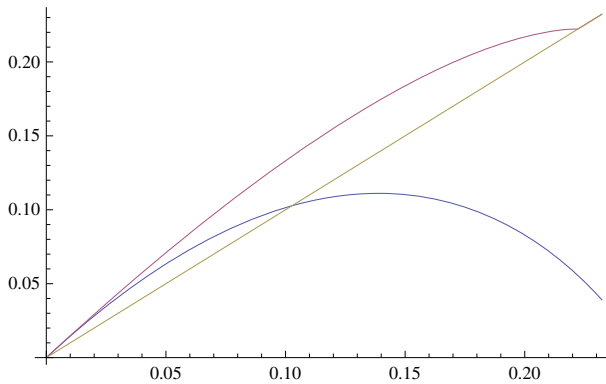


Fig. 1 The lower and upper bounds on $\varphi_2(\alpha)$: the straight line is the basic bound, the top one the upper bound (Proposition 7), and the bottom (most curved) is the lower bound (Theorem 5)

These theorems will be proved in Sects. 4 and 5, respectively.

They do not improve on Gromov's lower bounds for c_3 , for example, since they do not beat the basic bound for the values of α needed in Proposition 4 for d small. However, they do apply if we take d sufficiently large in Proposition 4, and so they at least show that Gromov's lower bound on c_d^{top} is not tight for large d .

After the research reported in this paper was completed, Král' et al. [13] proved the lower bound $\varphi_2(\alpha) \geq \frac{9}{7}\alpha(1 - \alpha)$. This is better than the bound of Theorem 5 for α larger than approximately 0.0626, and it does improve on Gromov's lower bound for c_3 .

The bounds in Theorems 5 and 6 may look like only minor improvements over the basic bound, but it turns out that they have the right order of magnitude for α tending to 0. Indeed, we have the following upper bound on the φ_d 's.

Proposition 7 For all $d \geq 1$ and $\alpha \leq \frac{1}{d+1}$, let $\sigma \in [0, 1)$ be the smallest positive number with $\alpha = d!\sigma(\frac{1-\sigma}{d})^{d-1}$. Then

$$\varphi_d(\alpha) \leq \frac{d+1}{d}\alpha(1 - \sigma),$$

and consequently, $\varphi_d(\alpha) \leq \frac{d+1}{d}\alpha$.

These bounds are plotted in Fig. 2.

Proposition 7 follows from a simple example, whose special case with $\alpha = \frac{1}{d+1}$ has already been noted in [9] and in [15, 17]. We present the example and its analysis in Sect. 2.

We conjecture that the bound in Proposition 7 is the truth, and moreover, that the example mentioned above is essentially the only possible extremal example. However, a proof may be challenging even for the $d = 2$ case. On the other hand, we believe that a suitable extension of the proof of Theorem 6 may provide a bound of the form $\varphi_d(\alpha) \geq \frac{d+1}{d}\alpha - o(\alpha)$ as $\alpha \rightarrow 0$. At present it seems that such an extension would be highly technical and complicated.

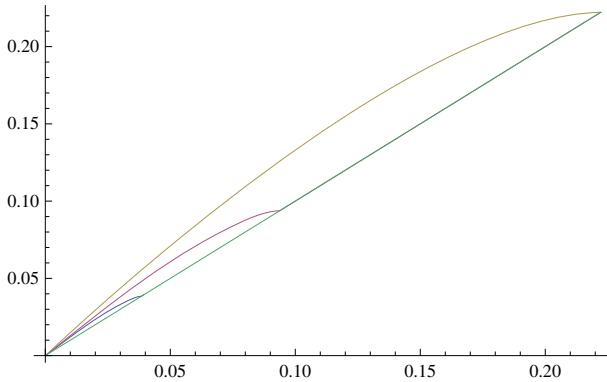


Fig. 2 Plots of the upper bounds from Proposition 7 for $d = 2, 3, 4$, together with the basic bound.

In view of the upper bound $\varphi_d(\alpha) \leq \frac{d+1}{d}\alpha$, Gromov’s lower bound on c_d cannot be improved by more than a factor of roughly d using Proposition 4 alone.

In Sect. 6, we introduce a somewhat different approach, which goes beyond Proposition 4 and uses additional combinatorial structure, and we show that it can provide a slightly better lower bound for c_3 . We formulate a combinatorial extremal problem whose solution might perhaps lead to a tight lower bound for the c_d^{top} .

2 Basics

Linearity of the Coboundary. For systems $E, E' \subseteq \binom{V}{d}$ of d -tuples, $E + E'$ means the symmetric difference.

The coboundary operator is (well known and easily checked to be) *linear* with respect to this operation, i.e.,

$$\delta(E + E') = \delta E + \delta E',$$

and we have

$$\delta\delta E = 0,$$

where 0 means the empty system of $(d + 2)$ -tuples.

Cochains, Coboundaries, and Cocycles. A system $E \subseteq \binom{V}{d}$ of d -tuples is also sometimes called a $(d - 1)$ -dimensional *cochain*,⁴ or simply $(d - 1)$ -cochain. Later on, when working with systems of tuples of various arities, it will be convenient to use this terminology. A cochain E is called a *cocycle* if $\delta E = 0$, and it is a *coboundary* if it can be written as $E = \delta D$ for some $(d - 2)$ -chain D .

In algebraic terms, a $(d - 1)$ -cochain $E \subseteq \binom{V}{d}$ can be identified with a 0/1-vector indexed by the elements of $\binom{V}{d}$, which we can interpret as an element of the vector

⁴ Strictly speaking, a cochain with \mathbb{Z}_2 -coefficients in the simplex Δ^{n-1} .

space $\mathbb{Z}_2 \binom{V}{d}$ over the 2-element field \mathbb{Z}_2 , and the symmetric difference $+$ corresponds to the usual addition in this vector space. The coboundary operators are linear maps between these spaces (mapping $(d-1)$ -cochains to d -cochains). The $(d-1)$ -cocycles are precisely the elements of the kernel of this linear map, and the d -coboundaries are the elements of the image. The property $\delta\delta = 0$ is usually called the *chain complex property*.

Minimality. As we have seen, every coboundary is also a cocycle, i.e., if $F = \delta E \subseteq \binom{V}{d+1}$, then $\delta F = 0$.

In our setting, this is a complete characterization, i.e., $F \subseteq \binom{V}{d+1}$ is a coboundary if and only if $\delta F = 0$. Topologically speaking, this is because the $(n-1)$ -simplex has zero cohomology.

We stress that there is one exceptional case, namely, $d = 0$. There are two 0-cocycles $F \subseteq \binom{V}{1}$, namely, $F = V$, the set of all vertices, and $F = \emptyset$, the empty set of vertices. However, by definition, \emptyset is considered to be a coboundary, but V is not. In topological terms, this is because we are working with ordinary, *non-reduced* cohomology.⁵ This will be formally important in Sect. 3.1.

Because of the equivalence of cocycles and coboundaries, for $d-1 \geq 1$, minimality of E can be equivalently characterized as follows: E is minimal if $\|E\| \leq \|E + \delta D\|$ for every $D \subseteq \binom{V}{d-1}$. Thus, minimality means that E contains at most half of the d -tuples from each system of the form δD . Consequently, if E is minimal and $E' \subseteq E$, then E' is minimal as well.

It also follows from the alternative characterization of minimality that every minimal E satisfies $\|E\| \leq 1/2$. To see this, let $D = \{x\}$ be a singleton set consisting of a single $(d-1)$ -tuple x . Then $|E \cap \delta D| \leq 1/2|\delta D|$ means that x is incident to at most $|\delta D|/2 = \frac{n-d+1}{2}$ many d -tuples $e \in E$. Summing up over all $x \in \binom{V}{d-1}$, we get $d|E| \leq \binom{n}{d-1}(n-d+1)/2$, and hence $|E| \leq \binom{n}{d}/2$.

As was remarked above, minimality refers to a fixed vertex set V . If a minimal $E \subseteq \binom{V}{d}$ happens to be contained in $\binom{W}{d}$ for some subset $W \subset V$ then, considered as a set of subsets of W , it needs no longer be minimal. For example, suppose we partition V into three vertex sets V_1, V_2, V_3 of size $n/3$ each and let $E = \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$, the (edge set of the) complete bipartite graph between V_1 and V_2 . Then it is not hard to check that E is minimal as a subset of $\binom{V}{2}$, but not as a subset of $\binom{V_1 \cup V_2}{2}$.

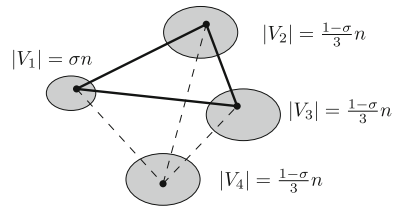
Links of Vertices. For system $E \subseteq \binom{V}{d}$ of d -tuples and a vertex $v \in V$, let us write $E_v := \{e \in E : v \in e\}$. The *link* of v in E is the system of $(d-1)$ -tuples $\text{lk}(v, E) := \{e \setminus \{v\} : e \in E_v\}$.

It is easy to check the following formula for the coboundary of the link:

$$\delta \text{lk}(v, E) = E_v + \text{lk}(v, \delta E_v) \quad (3)$$

⁵ We remark that from a combinatorial point of view, it may be more natural also to consider the subsets of $\binom{V}{0}$. There are two such subsets, namely $\{\varepsilon\}$ (the singleton set containing the unique 0-tuple ε of vertices of V) and \emptyset (the empty set of 0-tuples). For reduced cohomology, one defines $\delta\{\varepsilon\} = V$, so that the exceptional case in the characterization of cocycles disappears, but for the topological theorem that Gromov applies, it is important to work with non-reduced cohomology.

Fig. 3 The upper bound example in the case $d = 3$. The solid triangle depicts a typical element of E , and the dashed segments complete it to a tetrahedron that is a typical element of δE



(the $+$ on the right-hand side is actually a disjoint union in this case).

The Basic Bound for the Filling Profile. We now recall a combinatorial proof of the basic lower bound for φ_d .

Proof of Lemma 3 Let $F \subseteq \binom{V}{d+1}$ be a coboundary, and let $\beta := \|F\|$. We will show that there exists an $E \in \binom{V}{d}$ with $\delta E = F$ and $\|E\| \leq \beta$. It follows, in particular, that any minimal E' with $\delta E' = F$ satisfies $\|E'\| \leq \beta$, which proves the lemma.

We define the *normalized degree* of $v \in V$ as $\|\text{lk}(v, F)\| = |\text{lk}(v, F)| / \binom{n}{d}$.

A simple counting argument shows that the average normalized degree of a vertex equals $\beta = \|F\|$. In particular, there exists a vertex v of normalized degree at most β ; so we fix such a vertex v and set $E := \text{lk}(v, F)$. We will check that $\delta E = F$.

Let $F_{\setminus v} := F \setminus F_v$. Since F is a coboundary, we have $\delta F = 0$, and thus $\delta F_v = \delta F_{\setminus v}$.

Let us consider an arbitrary $(d + 1)$ -tuple f and distinguish two cases.

Case 1. If $v \in f$, then it is easily seen that $f \in \delta E$ is equivalent to $f \in F$.

Case 2. Next, assume $v \notin f$. By definition of E , d -tuples $e \in E$ are in one-to-one correspondence with the $(d + 1)$ -tuples $e \cup \{v\} \in F_v$. It follows that, $f \in \delta E$ iff $f^+ := f \cup \{v\} \in \delta F_v$. This in turn is the case iff $f^+ \in \delta F_{\setminus v}$ because $\delta F_v = \delta F_{\setminus v}$. Since the sets of $F_{\setminus v}$ avoid v , only one $(d + 1)$ -tuple of $F_{\setminus v}$ may be contained in f^+ , namely f . Thus, $f^+ \in \delta F_{\setminus v}$ iff $f \in F_{\setminus v}$ iff $f \in F$ (since we assume $v \notin f$). Hence, $f \in \delta E$ iff $f \in F$ also in this case. \square

The Upper Bound Example. Here we prove Proposition 7, the upper bound on the cofilling profile φ_d .

Let $\sigma \in [0, 1]$ be a parameter. We partition the vertex set V into V_1, \dots, V_{d+1} , where $|V_1| = \sigma n$ and the remaining vertices are divided evenly, i.e., $|V_i| = \frac{1-\sigma}{d}n$, $i > 1$ (we ignore divisibility issues). Let E consist of all d -tuples that use exactly one point from each of V_1, \dots, V_d ; see Fig. 3. We have $\|E\| =: \alpha = d! \sigma (\frac{1-\sigma}{d})^{d-1}$.

Then $F := \delta E$ is the complete $(d + 1)$ -partite system on V_1, \dots, V_{d+1} , and $\|\delta E\| = \frac{d+1}{d} \alpha (1 - \sigma)$. For a suitable choice of σ , this matches the quantitative bounds in Proposition 7, and it remains to check the minimality of E , which is easy (and stated by Gromov and by Meshulam et al. without proof): Let us consider some E' with $F = \delta E'$. Every $f \in F$ contains at least one $e \in E'$, while every $e \in E'$ is contained in at most $M := \max_i |V_i|$ many $f \in F$. So $|E'| \geq |F|/M = |E|$. \square

On Seidel’s Switching. In combinatorics, a *two-graph* (see the references below) is a set $F \subseteq \binom{V}{3}$ of triples with $\delta F = 0$, i.e., a cocycle in our terminology). As we have mentioned, this is equivalent to F being a coboundary, i.e., to the existence of some

$E \subseteq \binom{V}{2}$ with $F = \delta E$. The system of all possible E' with $\delta E' = F$ is called the *Seidel switching class* of E .

Two-graphs and Seidel's switching were introduced by Van Lint and Seidel [19] and further studied by many authors, because of their connections with equiangular lines, strongly regular graphs, and interesting finite groups, for example (for surveys see, e.g., Seidel and Taylor [18] or Hage [10]).

Numerous authors investigated the computational complexity of various problems related to Seidel's switching (we refer to [11, 14] for citations). For us, the following result is of particular interest: It is NP-complete to decide if a given $E \subseteq \binom{V}{2}$ is minimal (in its Seidel switching class), as was proved by Jelínková [11]. Her reduction produces only E 's with $\|E\| \geq \frac{1}{2}$ and $\|E'\| \geq \frac{1}{4}$ for all E' in the same switching class; however, recently Jelínek (private communication, September 2010) was able to modify the reduction, showing that the problem remains NP-complete even if we restrict only to E 's with $\|E\| \leq c$, for every fixed $c > 0$. This shows that minimal sets have a complicated structure, and one cannot expect to find a reasonable characterization.

3 An Outline of Gromov's Topological Approach

In this section, we give a rather informal and elementary outline of the topological part of Gromov's approach (Sects. 2.2, 2.4, 2.5, and 2.6 of [9]). This outline is not directly related to the new (combinatorial) results of the present paper.

We strive to keep the discussion as elementary as possible for as long as possible. For this reason, we restrict ourselves to the most basic (affine) setting of finite set P of n points in general position in \mathbb{R}^d , which allows us to describe most steps in the argument in simple geometric terms.

As remarked above, Gromov's method applies in much more general situations. In Sect. 3.5, we briefly discuss the more general setting and also include some remarks as to how our elementary discussion would be formulated in more standard topological terms.

We begin with an outline of our outline, by means of an example.

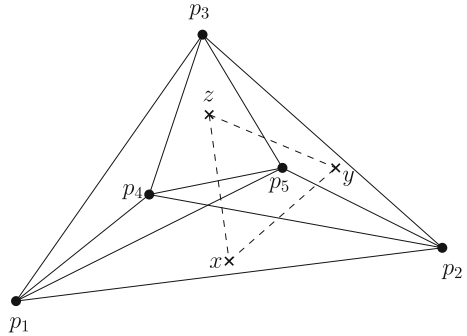
Example 8 Let $P = \{p_1, \dots, p_5\}$ be the set of five points in \mathbb{R}^2 depicted by bold dots in Fig. 4, and let $V = [5] := \{1, 2, 3, 4, 5\}$.

Consider the three points x, y, z marked by crosses in the picture. These three points are in general position w.r.t. P , in the sense that they do not lie on any line segment spanned by P , and no point of P lies on any of the line segments spanned by x, y, z .

Let $F_x = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$ be the set of all triples $\{i, j, k\} \in \binom{V}{3}$ such that x lies in the triangle $p_i p_j p_k$, and let $F_y = \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}\}$ and $F_z = \{\{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}\}$ be defined analogously as the (index sets of) triangles containing y and z , respectively.

Let $F_{xy} = \{\{2, 4\}, \{2, 5\}\}$ be the set of pairs $\{i, j\} \in \binom{V}{2}$ such that the line segment $p_i p_j$ intersects the line segment xy , and let $F_{yz} = \{\{3, 5\}\}$ and $F_{xz} = \{\{1, 5\}, \{2, 4\}, \{4, 5\}\}$ be defined analogously.

Fig. 4 A set of five labeled points in general position in \mathbb{R}^2 (the image of Δ^4 under an affine map)



Finally, let $F_{xyz} = \{5\}$ be the set of indices $i \in V$ such that p_i lies in the triangle xyz (here, we identify elements $i \in V$ with singleton sets $\{i\} \subseteq V$ to simplify notation).

The basic observation is that these sets satisfy the relations

$$\delta F_x = \delta F_y = \delta F_z = 0,$$

$$F_x + F_y = \delta F_{xy}, \quad F_y + F_z = \delta F_{yz}, \quad F_x + F_z = \delta F_{xz},$$

and

$$F_{xy} + F_{yz} + F_{xz} = \delta F_{xyz}.$$

It is straightforward to verify this in the specific example at hand, but it may in fact be easier—and an instructive exercise—to check that these relations are not a coincidence, but hold in general for any finite set $P \subseteq \mathbb{R}^2$ and any triple of points x, y, z , assuming only general position. Moreover, similar facts hold in \mathbb{R}^d . We discuss a proof of the general case below.

Informally speaking and in very general terms, the structure of Gromov’s topological approach can be summarized as follows. Each step of the argument will be discussed in more detail in a separate subsection below. We fix $d \geq 1$ (the target dimension) and $V = [n]$ (the vertex set of the $(n - 1)$ -simplex Δ^{n-1}).

1. We define a topological space $\mathcal{Z}^d = \mathcal{Z}^d(\Delta^{n-1})$, the space of d -dimensional cocycles (of the $(n - 1)$ -simplex).

This space is a *simplicial set*, i.e., a space built of vertices, edges, triangles, and higher-dimensional simplices like a simplicial complex, but simplices are allowed to be glued to each other and to themselves in more general ways (in a first approximation, simplicial sets can be thought of as higher-dimensional analogues of multi-graphs with loops, while simplicial complexes are higher-dimensional analogues of simple graphs).

The vertices of \mathcal{Z}^d are the d -dimensional cocycles $F \subseteq \binom{V}{d+1}$. The edges of \mathcal{Z}^d correspond to relations of the form $F_1 + F_2 = \delta F_{12}$, where $F_{12} \subseteq \binom{V}{d}$. The triangles of \mathcal{Z}^d correspond to a triple of d -dimensional cocycles F_1, F_2, F_3 , edges between them, and a relation of the form $F_{12} + F_{23} + F_{31} = \delta F_{123}$, where $F_{123} \subseteq \binom{V}{d-1}$, etc.

We stress that \mathcal{Z}^d depends only on n and d and is defined purely combinatorially. Moreover, as a combinatorial object, it is huge. For instance, the number of vertices of \mathcal{Z}^d (d -dimensional cocycles of the simplex) is $2^{\binom{n-d}{d}}$.

2. With every labeled n -point set $P \subseteq \mathbb{R}^d$, we associate a particular subspace⁶ $\mathcal{W} = \mathcal{W}(P) \subseteq \mathcal{Z}^d$.

A concrete way of doing this is to choose a triangulation \mathcal{T} of \mathbb{R}^d that is in general position w.r.t. P (i.e., no k -dimensional simplex of \mathcal{T} intersects any ℓ -dimensional simplex spanned by P if $k + \ell < d$). With every vertex x of \mathcal{T} we associate the set

$$F_x := \{f = \{i_0, i_1, \dots, i_d\} \in \binom{V}{d+1} : x \in p_{i_0}p_{i_1} \cdots p_{i_d}\}$$

of (indices of) d -simplices spanned by P that contain x . As indicated by the example, each such F_x is a cocycle, i.e., a vertex of \mathcal{Z}^d (but not all vertices of \mathcal{Z}^d may be of this special form).

With every edge xy of \mathcal{T} , we associate the set F_{xy} of d -tuples from V such that the corresponding points of P span a $(d - 1)$ -simplex that intersects xy . As in the example, we get the relation $F_x + F_y = \delta F_{xy}$, and hence an edge of \mathcal{Z}^d .

Similarly, each k -dimensional simplex of the triangulation gives rise to a k -simplex of \mathcal{Z}^d (but not all k -simplices in \mathcal{Z}^d may be of this form). We define the subspace \mathcal{W} to consist of all simplices of \mathcal{Z}^d that are obtained from an odd number of simplices of \mathcal{T} using this construction. (In principle, different simplices of the triangulation \mathcal{T} may yield the same k -simplex in \mathcal{Z}^d .)

3. It follows from a theorem in algebraic topology, the Almgren–Dold–Thom Theorem, that the subspace \mathcal{W} is not contractible inside \mathcal{Z}^d .
4. If we choose the triangulation \mathcal{T} sufficiently finely, then for every point $a \in \mathbb{R}^d$, there is a vertex x of \mathcal{T} with $F_a = F_x$. Thus, if no point of \mathbb{R}^d is covered by “many” d -simplices of P , then all sets F_x are “small.” If this is so, then by purely combinatorial means, we can define a concrete way of contracting the subspace \mathcal{W} to a single point inside \mathcal{Z}^d ; see Fig. 5. This is a contradiction. Thus, some point must be covered by many d -simplices.

We now proceed to discuss the above steps in more detail.

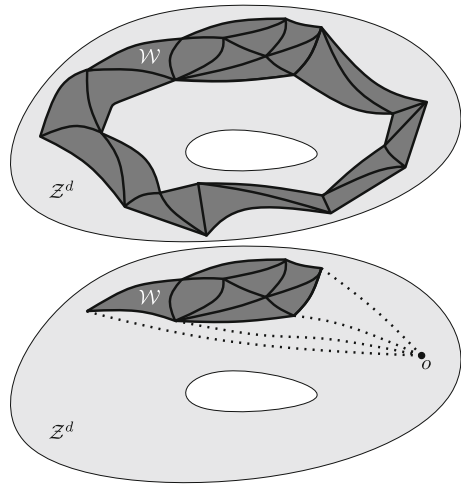
3.1 Simplicial Sets and the Space of Cocycles

*Simplicial sets*⁷ are a generalization of simplicial complexes. As in the case of a simplicial complex, a simplicial set is built from 0-simplices (vertices), 1-simplices (edges), 2-simplices (triangles), and higher-dimensional simplices.

⁶ Formally, it would be more precise to regard \mathcal{W} as a d -dimensional \mathbb{Z}_2 -homology cycle, i.e., as a formal \mathbb{Z}_2 -linear combination of d -simplices in \mathcal{Z}^d such that $\partial\mathcal{W} = 0$. However, since we are working with \mathbb{Z}_2 -coefficients, we can simply think of \mathcal{W} as a subspace, given as the union of those d -simplices that appear an odd number of times in the formal sum.

⁷ These objects also have many other names commonly found in the literature, including *complete semi-simplicial complexes* [7]. Gromov uses the terminology *semisimplicial spaces*.

Fig. 5 A schematic illustration of the last two steps of the argument: \mathcal{W} is not contractible inside \mathbb{Z}^d , but if no point in \mathbb{R}^d were covered by sufficiently many d -simplices of P , then we could contract \mathcal{W} inside \mathbb{Z}^d to a single point



One starts with the vertices, then glues each edge to one or two vertices by its endpoints, then one attaches triangles to vertices or edges along their boundaries, etc. In contrast to simplicial complexes, the attaching may involve various identifications. For instance, both endpoints of an edge may be attached to the same vertex, and two or more i -simplices in a simplicial set may have the same boundary. In this respect, simplicial sets can be thought of, in a first approximation, as higher-dimensional analogues of multigraphs with loops. On the other hand, in contrast to general cell complexes, there are restrictions as to what kind of attaching maps are allowed,⁸ which makes simplicial sets more combinatorial than general cell complexes. We refer the reader to the article by Friedman [8] for a very clear and accessible introduction to simplicial sets and to the book by May [16] for a detailed treatment (further references can be found in Friedman’s article).

The key object in Gromov’s method is the *space of d -dimensional cocycles*, which we denote⁹ by $\mathbb{Z}^d = \mathbb{Z}^d(\Delta^{n-1})$ and which is a simplicial set defined as follows.

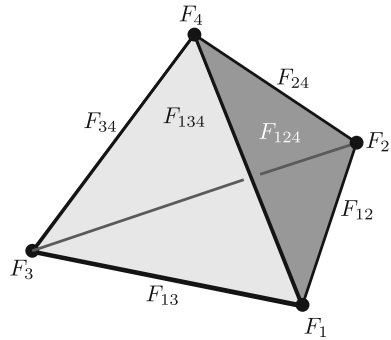
The vertices of \mathbb{Z}^d are the d -dimensional cocycles (of the simplex Δ^{n-1}), i.e., subsets $F \subseteq \binom{V}{d+1}$ such that $\delta F = 0$.

The edges of \mathbb{Z}^d are given by two (not necessarily distinct) d -cocycles F_1 and F_2 and a set $F_{12} \subseteq \binom{V}{d}$ such that $\delta F_{12} = F_1 + F_2$. We stress that F_{12} and at least one of the F_i are necessary in order to uniquely define an edge in \mathbb{Z}^d . If there is a F'_{12} with $\delta F'_{12} = F_1 + F_2$ then it defines a different edge of \mathbb{Z}^d connecting the same pair of vertices. On the other hand, if F'_1 and F'_2 are another pair of d -cocycles with $\delta F_{12} = F'_1 + F'_2$, then the same F_{12} yields a different edge of \mathbb{Z}^d connecting a different pair of vertices.

⁸ Roughly speaking, one can think of each of the original simplices as having an ordered set of vertices. The attaching maps are linear maps induced by weakly monotone (order-preserving) maps between the vertex sets of the simplices.

⁹ For those readers who wish to read [9] in conjunction with the present one, we remark that Gromov uses the notation cl^d or cl^d_{sims} for the space we denote by \mathbb{Z}^d .

Fig. 6 An illustration of a 3-dimensional simplex in \mathcal{Z}^d . It is given by four 2-dimensional cocycles $F_i \in \binom{V}{3}$, six sets $F_{ij} \in \binom{V}{2}$, and four sets $F_{ijk} \in \binom{V}{1}$ (with pairwise distinct indices i, j, k running between 1 and 4) that satisfy the relations $\delta F_{ij} = F_i + F_j$, $\delta F_{ijk} = F_{ij} + F_{jk} + F_{ik}$, and $\sum_{ijk} F_{ijk} = 0$



In the next step, a triangle of \mathcal{Z}^d is given by a triple of d -cocycles F_1, F_2, F_3 , a triple of sets $F_{ij} \subseteq \binom{V}{d}$ and a set $F_{123} \subseteq \binom{V}{d-1}$ such that

- (i) $\delta F_{ij} = F_i + F_j, 1 \leq i < j \leq 3$, and
- (ii) $\delta F_{123} = F_{12} + F_{13} + F_{23}$.

The F_i and the F_{ij} define three (not necessarily distinct) vertices and three (not necessarily distinct) edges of \mathcal{Z}^d that form the boundary of a triangle, and together with this other data, F_{123} defines a triangle glued in along that boundary. Again, there may be other F'_{123} with the same coboundary, which define different triangles glued to the same boundary (a higher-dimensional analogue of a multiedge), and there may be a different set of F'_i and/or F'_{ij} which also satisfy conditions (i) and (ii); if so, they yield, together with F_{123} , a different triangle of \mathcal{Z}^d , glued to a different boundary.

One can continue this definition inductively¹⁰ for simplices of arbitrary dimension r . The case of $(d + 1)$ -simplices in \mathcal{Z}^d , deserves special attention, however (due to the exceptional behavior of the coboundary operator in dimension zero, i.e., the fact that V is not considered a coboundary, which was mentioned in Sect. 2). A $(d + 1)$ -simplex in \mathcal{Z}^d is given by the following data (see Fig. 6): for each $0 \leq i \leq d$ and each $A \in \binom{[d+2]}{i+1}$, there is a set $F_A \in \binom{V}{d+1-i}$ (i.e., a set of $(d - i)$ -dimensional faces) such that

$$\delta F_A = F_{\partial A} := \sum_{B \in \partial A} F_B$$

and

$$\sum_{A \in \binom{[d+2]}{d+1}} F_A = 0.$$

Here, by definition, $\partial A := \{A \setminus \{a\} : a \in A\}$, called the *boundary* of A .

¹⁰ In the beginning of [9, Sect. 2.2 Gromov also gives an equivalent definition along the lines of the usual formal viewpoint of simplicial sets as functors from the category of finite totally ordered sets and monotone maps to the category of sets, as in [16].

3.2 Intersections and Cocycles

Let $P = \{p_1, p_2, \dots, p_n\} \subseteq \mathbb{R}^d$ be a labeled set of n points in general position. We think of $V = [n]$ as the set of “labels” of P . The goal of this section is to define the subspace $\mathcal{W} = \mathcal{W}(P) \subseteq \mathcal{Z}^d$ for this set P .

Let $A = a_0a_1 \dots a_k$ be a k -dimensional simplex in \mathbb{R}^d that is in general position w.r.t. P , i.e., no i -face of A intersects any $(d - i - 1)$ -simplex spanned by P , $0 \leq i \leq k$. We define

$$F_A := \{\{i_0, i_1, \dots, i_{d-k}\} \in \binom{V}{d+1-k} : A \cap p_{i_0}p_{i_1} \dots p_{i_{d-k}} \neq \emptyset\}.$$

That is, we consider the $(d - k)$ -simplices spanned by P that are intersected by A . Each such simplex is of the form $p_{i_0}p_{i_1} \dots p_{i_{d-k}}$ for some $(d + 1 - k)$ -tuple $\{i_0, i_1, \dots, i_{d-k}\} \in \binom{V}{d+1-k}$ of labels, and F_A consists precisely of these tuples. For simplicity, we will also say that F_A corresponds to the set of $(d - k)$ -simplices of P intersected by A .

Thus, for a point $x \in \mathbb{R}^d$ in general position w.r.t. P , $F_x = \{\{i_0, i_1, \dots, i_d\} \in \binom{V}{d+1} : x \in p_{i_0}p_{i_1} \dots p_{i_d}\}$ corresponds to the set of d -simplices of P that contain x . If xy is a segment in general position, then F_{xy} corresponds to the set of $(d - 1)$ -simplices of P that intersect xy , etc.

As remarked above, the sets F_x are always cocycles, i.e., $\delta F_x = 0$, and the sets F_{xy} satisfy $F_x + F_y = \delta F_{xy}$. More generally, we have:

Lemma 9 *Let $A = a_0a_1 \dots a_k$ be a k -simplex in \mathbb{R}^d that is in general position w.r.t. P . Then*

$$\delta F_A = F_{\partial A} := F_{A_0} + F_{A_1} + \dots + F_{A_k},$$

where $A_i = a_0 \dots a_{i-1}a_{i+1} \dots a_k$ is the $(k - 1)$ -face of A obtained by dropping vertex a_i .

Proof Consider a $(d - k + 2)$ -tuple in $f \subseteq V$ corresponding to a $(d - k + 1)$ -dimensional simplex σ spanned by P . By general position, this $(d - k + 1)$ -dimensional simplex σ is either disjoint from A , or it intersects A in a line segment. Each endpoint of this line segment $\sigma \cap A$ is of one of two types: either such an endpoint arises as the intersection of $\sigma \cap A_i$ of σ with some facet of A , or as the intersection $\sigma_j \cap A$ of A with some facet of σ . If the intersection $\sigma \cap A$ is empty or if both endpoints are of the same kind then f does not contribute to either side of the claimed identity. If there is one endpoint of each type, then f contributes to both sides of the identity. \square

If we apply the preceding lemma to a $(k - 1)$ -face A_i of A , we see that

$$\delta F_{A_i} = \sum_{j \neq i} F_{A_{ij}},$$

where A_{ij} is the $(k - 2)$ -face of A obtained by dropping vertices y_i and y_j . Iterating this, we see that A , together with all its faces, defines a k -dimensional simplex in \mathcal{Z}^d ,

which we denote by $\Delta_P(A)$. Note that, in particular, the vertices of $\Delta_P(A)$ correspond to the d -cocycles F_{a_i} associated with the vertices a_i of A .

Now we proceed to define the space \mathcal{W} . We fix a d -dimensional bounding simplex B that contains P in its interior. We also choose and fix a triangulation of B that is in general position¹¹ with respect to P and that is *sufficiently fine*, in the sense that

- (a) for every point $q \in \mathbb{R}^d$ there is a vertex x of the triangulation with $F_q = F_x$, and
- (b) any simplex A in the triangulation of dimension $\dim A = k > 0$ intersects $O(n^{d-k})$ of the $(d-k)$ -simplices of P , i.e., $\|F_A\| = o(1)$ as $n \rightarrow \infty$.

To see that we can choose such a sufficiently fine triangulation, note first that if $\sigma_1, \dots, \sigma_r$ is a finite collection of pairwise distinct simplices, each spanned by P , with $\sigma_1 \cap \dots \cap \sigma_r = \emptyset$ then, by compactness, there exists some $\varepsilon = \varepsilon(\sigma_1, \dots, \sigma_r) > 0$ such that no matter how we select points $x_i \in \sigma_i$, $1 \leq i \leq r$, some pair x_i and x_j of points have distance at least ε . Since P is finite there are only finitely many such collections of simplices, so there is some $\varepsilon > 0$ that works for all of them.

Consequently, if we choose the triangulation of B so that all of its simplices have sidelength at most $\varepsilon/2$, say, then we have the following property: If A is a k -simplex of the triangulation and if $F_A = \{\sigma_1, \dots, \sigma_r\}$ is the collection of $(d-k)$ -simplices spanned by P that intersect A , then $\sigma_1 \cap \dots \cap \sigma_r \neq \emptyset$. We claim that such a triangulation is sufficiently fine.

Let us say that $\sigma_i, \sigma_j \in F_A$ are *independent* if they are vertex-disjoint. By relabeling, if necessary, we may assume that $\{\sigma_1, \dots, \sigma_\ell\}$ is an inclusion-maximal independent subset of F_A , for some suitable index $1 \leq \ell \leq r$. That is, $\sigma_1, \dots, \sigma_\ell$ are pairwise independent and every σ_j , $j > \ell$, shares a vertex with some σ_i , $1 \leq i \leq \ell$. It follows that $|F_A| \leq \ell(d-k+1) \binom{n-1}{d-k}$.

Moreover, if P is in sufficiently general position and if $k > 0$ then $\ell \leq d/k$ since $\sigma_1 \cap \dots \cap \sigma_\ell$ is of codimension $\ell \cdot k$ but also assumed to be nonempty. Thus, $|F_A| \leq \frac{d}{k}(d-k+1) \binom{n-1}{d-k} = O(n^{d-k})$, as desired.

Now we complete this triangulation of B to a triangulation \mathcal{T} of the d -dimensional sphere¹² \mathbb{S}^d by adding a point at infinity and coning from this point over the boundary of B ; see Fig. 7.

We define a subspace \mathcal{W} of \mathcal{Z}^d by taking the formal sum of the d -simplices $\Delta_P(A)$ over all d -simplices A of \mathcal{T} (note that for all simplices A involving the vertex at infinity, we have $F_A = 0$). In other words, a d -simplex of \mathcal{Z}^d is included in \mathcal{W} if it is equal to $\Delta_P(A)$ for an odd number of d -simplices A of \mathcal{T} . (Formally, in homological terminology, \mathcal{W} is a d -dimensional simplicial cycle in \mathcal{Z}^d .) We stress that \mathcal{W} is determined by the d -simplices of \mathcal{T} , not by the vertices of \mathcal{T} .

As we have described it, the subspace \mathcal{W} of \mathcal{Z}^d depends not only on P , but also on the triangulation \mathcal{T} that we have chosen. It turns out that for any two choices of triangulations, the corresponding subspaces \mathcal{W} are equivalent in a suitable sense (any two such cycles are homologous); see Sect. 3.5.

¹¹ That is, if A is a k -simplex of \mathcal{T} and if $\sigma = p_{i_0} \dots p_{i_\ell}$ is an ℓ -simplex spanned by P , then $A \cap \sigma = \emptyset$ for $k + \ell < d$; this can always be ensured by a small perturbation of the vertices of \mathcal{T} .

¹² The somewhat ad-hoc device of introducing a bounding simplex and passing to a triangulation of the sphere can be avoided by using so-called homology with infinite supports (as Gromov does), but we opted for the ad-hoc method to keep the treatment more elementary.

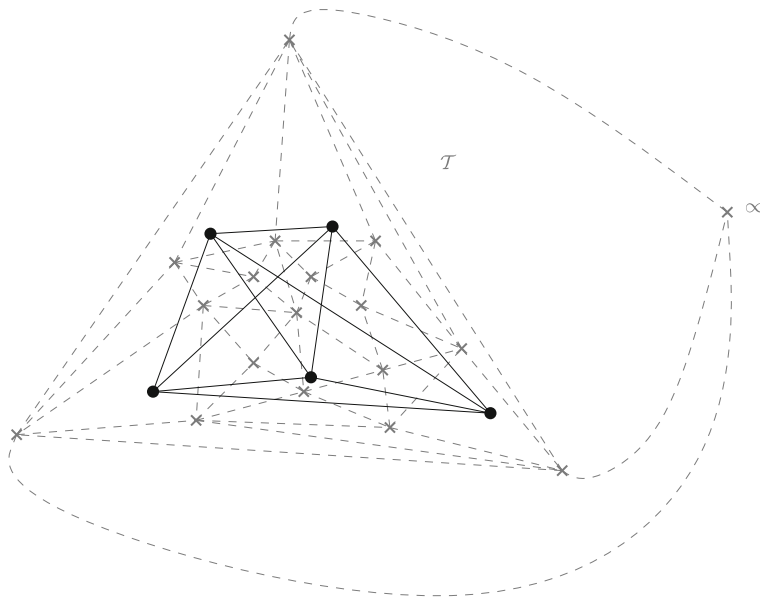


Fig. 7 A set of five points in the plane and the line segments spanned by them (depicted by bold dots and solid segments), and a triangulation \mathcal{T} of a bounding simplex B plus a vertex at infinity (depicted by crosses and dashed lines)

3.3 Nontriviality

The key fact upon which Gromov’s method hinges is that the subspace \mathcal{W} defined in the previous subsection is always nontrivial, in the following sense:

Key Fact. The subspace \mathcal{W} defined above cannot be contracted inside \mathcal{Z}^d .

(More formally, one has the stronger statement that \mathcal{W} is homologically nontrivial, i.e., that it is not a homological boundary inside \mathcal{Z}^d .)

Gromov deduces this fact from what he calls *the algebraic version of the Almgren–Dold–Thom theorem* (see [9, Sect. 2.2]), but we have not been able to locate the ADT theorem in a suitable form and with proof in the literature.¹³ So we will treat the Key Fact as a black box in our presentation.

3.4 Coning in the Space of Cocycles and the Proof of Proposition 4

To prove Proposition 4, one argues that if $\|F_y\|$ were “too small” for all vertices y of the triangulation \mathcal{T} , then the space \mathcal{W} could be contracted to a point inside \mathcal{W} —a contradiction.

¹³ The paper by Almgren [1] that Gromov cites works in the setting of geometric measure theory and is about integral currents and cycles; an older paper by Dold and Thom [6] works in the setting of simplicial sets, but only establishes a special case. It may well be that the theorem is well-known and clear to experts in the field, but not to us.

In order to show contractibility, we have the following simple *coning argument*: Suppose there is a vertex o in \mathcal{Z}^d such that we can inductively construct a *cone* $o * \mathcal{W}$ in \mathcal{Z}^d . That is, suppose we do the following, by induction on the dimension i : For each i -simplex τ that is a face of at least one d -face in \mathcal{W} , select an $(i + 1)$ -simplex $o * \tau$ in \mathcal{Z}^d that has τ as an i -face and o as the remaining vertex, in such a way that

$$\partial(o * \tau) = \tau + o * \partial\tau. \tag{4}$$

This condition means that our choices for higher-dimensional faces have to be consistent with what we have already committed to for lower-dimensional faces. We note that if \mathcal{Z}^d were just a simplicial complex, then for each τ there would be either a unique choice for $o * \tau$, or none at all, but for simplicial sets, there may be many choices. We also remark that $o * \tau$ may be a degenerate simplex, in the sense that o already appears as a vertex of τ .

Given a coning, one can contract \mathcal{W} to o , by continuously moving each simplex τ of \mathcal{W} towards o inside $o * \tau$.

To perform the coning for \mathcal{W} , we need to fix a vertex o in \mathcal{Z}^d and, for every k -simplex A of the triangulation \mathcal{T} , choose a $(k + 1)$ -simplex $o * \Delta_P(A)$ in such a way that the coning condition (4) is satisfied. That is, inductively, for every k -simplex A of \mathcal{T} , we have to choose a $(d - k - 1)$ -cochain $F_{Ao} \subseteq \binom{V}{d-k}$ such that

$$\delta F_{Ao} = F_A + \sum_{B \in \partial A} F_{Bo}.$$

Using the cofilling profile of Δ^{n-1} , we show that we can do this if all $\|F_y\|$ are small, thus obtaining a contradiction.

We choose o to be the vertex of \mathcal{Z}^d corresponding to the zero d -cocycle 0 in Δ^{n-1} . For every vertex y of the triangulation, let F_y be the corresponding vertex in \mathcal{Z}^d (d -cocycle in Δ^{n-1}). We pick an arbitrary *minimal* $(d - 1)$ -cochain F_{yo} with $\delta F_{yo} = F_y$ ($= F_y + 0$). By minimality, we have $\|F_y\| \geq \varphi_d(\|F_{yo}\|)$.

Next, consider an edge xy in the triangulation \mathcal{T} . The corresponding $(d - 1)$ -cochain F_{xy} satisfies $\delta F_{xy} = F_x + F_y$ (Lemma 9). It follows that

$$\delta(F_{xy} + F_{xo} + F_{yo}) = F_x + F_y + F_x + F_y = 0.$$

Now we pick a minimal $(d - 2)$ -cochain F_{xyo} such that $\delta F_{xyo} = F_{xy} + F_{xo} + F_{yo}$. It follows that $\|F_{xy} + F_{xo} + F_{yo}\| \geq \varphi_{d-1}(\|F_{xyo}\|)$. Moreover, by our choice of the triangulation, we have $\|F_{xy}\| = o(1)$. Thus, up to an $o(1)$ additive error, $\|F_{xo} + F_{yo}\| \geq \varphi_{d-1}(\|F_{xyo}\|)$, hence

$$\max\{\|F_{xo}\|, \|F_{yo}\|\} \geq \frac{1}{2}\varphi_{d-1}(\|F_{xyo}\|),$$

and so

$$\max\{\|F_x\|, \|F_y\|\} \geq \varphi_d(\frac{1}{2}\varphi_{d-1}(\|F_{xyo}\|))$$

(where we suppress $o(1)$ additive error terms in both formulas).

3.5 Gromov’s Method in the Topological Setting

As mentioned in the introduction, the setting of an n -point set $P \subseteq \mathbb{R}^d$ considered in the previous subsections corresponds to an affine map from the $(n - 1)$ -simplex Δ^n into \mathbb{R}^d .

Gromov’s method applies to much more general situations. More precisely, it allows for the setting to be generalized in several ways, as we now sketch.

1. The simplex Δ^{n-1} can be replaced by an arbitrary (finite) simplicial complex X . What is needed are lower bounds on the cofilling profile of X , which is defined as follows: For $k \geq 0$, let X_k be the set of k -dimensional faces of X . For each k , we have the coboundary operator of X which maps a subset $E \subseteq X_k$ to

$$\delta E := \{f \in X_{k+1} : f \text{ contains an odd number of } e \in E\}.$$

We can identify $E \subseteq X_k$ with a 0/1-vector indexed by X_k , i.e., with an element of the vector space $\mathbb{Z}_2^{X_k}$ over the 2-element field \mathbb{Z}_2 . In more usual (co)homological terminology, this latter vector space is denoted by $C^k(X; \mathbb{Z}_2)$ and called the space of k -dimensional cochains (with \mathbb{Z}_2 -coefficients), and δ is a linear map $C^k(X; \mathbb{Z}_2) \rightarrow C^{k+1}(X; \mathbb{Z}_2)$.

For a k -dimensional cochain E , we define $\|E\| := |E|/|X_k|$ as the density of E as before, and

$$\varphi_d^X(\alpha) := \min \{ \|\delta E\| : E \in C^*(X; \mathbb{Z}_2) \text{ minimal, } \|E\| \geq \alpha \},$$

where E is minimal if $\|E\| \leq \|E + \delta D\|$ for all $D \in C^*(X; \mathbb{Z}_2)$.

The space $Z^d(X)$ of d -dimensional cocycles of X is defined completely analogously to the case of the simplex.

2. The target space \mathbb{R}^d can be replaced by an arbitrary triangulated d -dimensional manifold Y , or, even more generally, a \mathbb{Z}_2 -homology manifold. What we need is to be able to compute *intersection numbers* (modulo 2) between k -dimensional chains and $(d - k)$ -dimensional chains. Equivalently, we need that Poincaré duality (with \mathbb{Z}_2 coefficients) holds in Y .
3. Instead of affine maps, we can allow for arbitrary continuous maps $T : X \rightarrow Y$. Without loss of generality, one can think of T as a piecewise linear map in general position. This is not necessary for the argument, but may help the reader’s intuition. For instance, for such a map, the T -image of any $(d - k)$ -simplex σ of X intersects any k -simplex A of Y in a finite number of points in the relative interior of A , and the (algebraic, \mathbb{Z}_2) intersection number of $T(\sigma)$ and A is defined as the number of intersection points modulo 2.
4. Finally, instead of cohomology with \mathbb{Z}_2 -coefficients, one can work with other coefficient rings in the argument. Potentially, this might lead to stronger bounds. We will not discuss this generalization, since on the one hand, it is straightforward, and on the other hand, we would have to talk about orientations everywhere.

The basic structure of the proof remains the same, but we have to change the definition of the cochains F_A appropriately, as follows: If A is a k -dimensional simplex

in general position, one defines F_A as the set of $(d - k)$ -simplices of X whose images under T have odd intersection number with A .

Another way of interpreting this construction is as follows: Every k -simplex A defines, via Poincaré duality, a $(d - k)$ -cochain in Y , and this $(d - k)$ -cochain pulls back under T to a cochain F_A in X .

The basic identity

$$\delta F_A = F_{\partial A} = F_{A_0} + F_{A_1} + \dots + F_{A_k} \tag{5}$$

still holds. (This just says that on the level of chains and cochains, Poincaré duality exchanges boundary and coboundary operators).

In other words, every k -simplex A in general position defines a k -simplex $\Delta_T(A)$ in $\mathcal{Z}^d(X)$ via the intersection number construction. If Y is compact, i.e., has a finite triangulation, then the subspace $\mathcal{W} = \mathcal{W}(T) \subseteq \mathcal{Z}^d(X)$ is defined as the formal \mathbb{Z}_2 -linear combination of the d -simplices $\Delta_T(A)$, where A ranges over all d -simplices in the triangulation of Y . In other words, a d -simplex of $\mathcal{Z}^d(X)$ belongs to \mathcal{W} if it equals $\Delta_T(A)$ for an odd number of d -simplices A of the triangulation of Y .

An equivalent way of defining \mathcal{W} is as follows: The basic identity (5) implies that the map $A \mapsto \Delta_T(A)$ commutes with the boundary operator, i.e., it is a chain map (with \mathbb{Z}_2 -coefficients) from Y to $\mathcal{Z}^d(X)$. Thus, this map also induces a map in homology.

Let $[Y]$ denote the fundamental d -dimensional homology class (over \mathbb{Z}_2) of Y . If we fix a triangulation of Y , we can take a representative d -cycle for $[Y]$ that is the formal \mathbb{Z}_2 -linear combination of all the d -simplices of the triangulation. (If Y is not compact, as in the case $Y = \mathbb{R}^d$, then we have to work with homology with infinite supports.) Then $\mathcal{W} = \mathcal{W}(f)$ is defined as the image under the map Δ_T of $[Y]$, i.e., formally, it is a d -dimensional cycle (with \mathbb{Z}_2 -coefficients) in $\mathcal{Z}^d(X)$.

As before, the Almgren–Dold–Thom Theorem implies that this cycle is homologically nontrivial, so it cannot be contracted.

On the other hand, if every point of Y were covered by the T -images of “too few” d -simplices of X , then the same combinatorial coning as before would yield a contradiction (with the precise meaning of “too few” depending on the cofilling profile of X). Again, the combinatorial coning can also be viewed as an actual topological contraction of \mathcal{W} , considered as a subspace of $\mathcal{Z}^d(X)$, to the point o .

If Y is unbounded, or if we are guaranteed that there is a point of Y that is not covered by the image of T , then the same argument as before shows that

$$\max_y \|F_y\| \geq \varphi_d^X \left(\frac{1}{2} \varphi_{d-1}^X \left(\frac{1}{3} \varphi_{d-2}^X \left(\dots \frac{1}{d} \varphi_1^X \left(\frac{1}{d+1} \right) \dots \right) \right) \right),$$

where the maximum is over all vertices y of the triangulation of Y . (If Y is not unbounded and if every point in Y is covered by the T -images of some d -simplices of X , then we have to choose the apex o of the coning differently (not as the empty cocycle), which yields a weaker bound.)

In particular, if X is a \mathbb{Z}_2 -cohomological expander, i.e., if $\varphi_i^X(\alpha)/\alpha$ is bounded away from zero for all i , then there is some point of Y that is covered by a positive

fraction of all d -simplices of X (with a constant depending on the cofilling profile of X).

We remark that for the coning argument, we again choose the triangulation \mathcal{T} to be sufficiently fine with respect to the map T , i.e., we assume that for every simplex A of the triangulation of dimension $\dim A > 0$, we have $\|F_A\| = o(1)$. If the map T is very complicated (i.e., if we need a very fine subdivision of X to approximate T by a PL map), then the triangulation \mathcal{T} may require a huge number of simplices, so it is important that the whole argument is completely independent of the number of simplices in the triangulation \mathcal{T} .

4 The cofilling profile in the $d = 2$ case

Here we prove Theorem 5, which asserts that

$$\varphi_2(\alpha) \geq f(\alpha) := \frac{3}{4}(1 - \sqrt{1 - 4\alpha})(1 - 4\alpha).$$

Since $d = 2$, we deal with the size of the coboundary for an edge set E of a graph. The minimality of E means that *no edge cut has density more than $\frac{1}{2}$* in this graph; in other words, for every $S \subseteq V$, the number of edges of E going between S and $V \setminus S$ is at most $\frac{1}{2}|S| \cdot |V \setminus S|$.

As we have remarked at the end of Sect. 2, the minimality of E is a complicated property, computationally hard to test, for example. So we will only use it for singleton sets S . Thus, we will actually show that $\|\delta E\| \geq f(\alpha)$ (ignoring terms tending to 0 as $n \rightarrow \infty$) for every E with $\|E\| \geq \alpha$ and $\deg_E(v) \leq \frac{n}{2}$ for all $v \in V$ (where $\deg_E(v)$ denotes the number of neighbors of v in the graph (V, E)).

Before proceeding with the proof of Theorem 5, let us remark that this relaxation (i.e., ignoring all non-singleton S) already prevents us from obtaining a tight bound for φ_2 . For example, let us partition V into sets V_1, V_2, V_3 , where $|V_1| = \alpha n$ and $|V_2| = \frac{n}{2}$, and let E consist of all edges connecting V_1 to V_2 . This E is not minimal, but it does satisfy the degree condition. One easily checks that $\|E\| = \alpha$ and $\|\delta E\| = 3\alpha(\frac{1}{2} - \alpha) = \frac{3}{2}\alpha - 3\alpha^2$, which is smaller than the suspected tight bound for φ_2 from Proposition 7. However, at least the leading term is correct.

Now we proceed with the proof. Given E , let m_i denote the number of triples $f \in \binom{V}{3}$ that contain i edges of E , $i = 1, 2, 3$; we have $|\delta E| = m_1 + m_3$ by definition. An easy inclusion–exclusion consideration shows that

$$|\delta E| = (n - 2)|E| - \sum_{v \in V} \deg_E(v)(\deg_E(v) - 1) + 4t, \quad (6)$$

where t denotes the number of triangles in the graph (V, E) . Indeed, to check (6), it suffices to discuss how many times a triple $f \in \binom{V}{3}$ containing exactly i edges $e \in E$ contributes to the right-hand side, $i = 1, 2, 3$. For $i = 1$, such an f is only counted once in the term $(n - 2)|E|$, which counts the number of ordered pairs (e, v) , where $e \in E$ and $v \in V \setminus e$. A triple f with $i = 2$ is counted twice in the term $(n - 2)|E|$, but it is also counted twice in $\sum_{v \in V} \deg_E(v)(\deg_E(v) - 1)$, and thus its total contribution is zero. Finally, for $i = 3$, where f induces a triangle, it is counted

three times in $(n - 2)|E|$, six times in $\sum_{v \in V} \deg_E(v)(\deg_E(v) - 1)$, and four times in $4t$, so altogether it contributes $+1$ as it should.

As the next simplification, we will ignore the triangles, as well as the difference between $\deg_E(v)(\deg_E(v) - 1)$ and $\deg_E(v)^2$, and we will use (6) in the form

$$|\delta E| \geq (n - 2)|E| - \sum_{v \in V} \deg_E(v)^2. \tag{7}$$

Since $|E|$ is given, it remains to maximize $\sum_{v \in V} \deg_E(v)^2$, which is done in the next lemma.

Lemma 10 *Let $\alpha \leq \frac{1}{4}$. Let (V, E) be a graph on n vertices with $|E| = \alpha \binom{n}{2}$ and $\deg_E(v) \leq \frac{n}{2}$ for all $v \in V$. Then*

$$\sum_{v \in V} \deg_E(v)^2 \leq \left(\frac{\sigma}{4}(1 + 2\sigma - 4\sigma^2) + o(1)\right)n^3,$$

where $\sigma = (1 - \sqrt{1 - 4\alpha})/2$ is one of the two roots of $\sigma^2 - \sigma + \alpha = 0$.

Proof Let (V, E) be the given graph with n vertices and $|E| = \alpha \binom{n}{2}$ edges. By a sequence of transformations that do not change the number of edges and that do not decrease the sum of squared degrees, we convert it to a particular form.

Let us number the vertices v_1, \dots, v_n so that $d_1 \geq d_2 \geq \dots \geq d_n$, where $d_i := \deg_E(v_i)$. We note that for $d_i \geq d_j$, we have $(d_i + 1)^2 + (d_j - 1)^2 > d_i^2 + d_j^2$, and thus a transformation that changes d_i to $d_i + 1$ and d_j to $d_j - 1$ and leaves the rest of the degrees unchanged increases the sum of squared degrees.

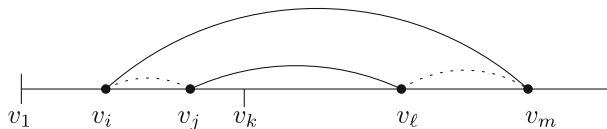
For an edge $e = \{v_i, v_j\}$ with $i < j$, we call v_i the *left end* of e and v_j the *right end*.

(i) *Let k be such that $d_1 = d_2 = \dots = d_k = \lfloor \frac{n}{2} \rfloor$, while $d_{k+1} < \lfloor \frac{n}{2} \rfloor$. Then we may assume that the left ends of all edges are among v_1, \dots, v_{k+1} .*

Indeed, if there is an edge $\{v_i, v_j\}$, $i > k + 1$, we can replace it with the edge $\{v_{k+1}, v_j\}$. This increases $\sum_{i=1}^{k+1} d_i$ (and possibly increases k), so after finitely many steps, we achieve the required condition.

(ii) *We may assume that every two vertices among v_1, \dots, v_k are connected.*

Proof: We may assume that (i) holds. Since we assume $\alpha \leq \frac{1}{4}$, we have $k \leq \lfloor \frac{n}{2} \rfloor$. Let us suppose $\{v_i, v_j\} \notin E$, $1 \leq i < j \leq k$. Since $d_i = d_j = \lfloor \frac{n}{2} \rfloor$, each of d_i, d_j is connected to at least two vertices among v_{k+1}, \dots, v_n . So we may assume $\{v_i, v_\ell\} \in E$, $\{v_j, v_m\} \in E$, $\ell, m \geq k + 1$, $\ell \neq m$. We also have $\{v_\ell, v_m\} \notin E$ (according to (i)). Thus, we can delete the edges $\{v_i, v_\ell\}$ and $\{v_j, v_m\}$ and add the edges $\{v_i, v_j\}$ and $\{v_\ell, v_m\}$.



This increases the number of edges on $\{v_1, \dots, v_k\}$, which cannot decrease by the transformations in (i), so after finitely many steps, we achieve both (i) and (ii).

(iii) *We may assume that the right neighbors of each $v_i, 1 \leq i \leq k$, form a contiguous interval $v_{i+1}, v_{i+2}, \dots, v_{\lfloor n/2 \rfloor + 1}$.*

Indeed, if v_i is connected to some $v_{\ell+1}$ and not to $v_\ell, \ell > k$, we can replace the edge $\{v_i, v_{\ell+1}\}$ with $\{v_i, v_\ell\}$. This increases the sum of squared vertex degrees, and thus after finitely many steps, we can achieve (i)–(iii).

A graph satisfying (i)–(iii) is almost completely determined by its number of edges, except possibly for the neighbors of the vertex v_{k+1} . Each of v_1, \dots, v_k is connected to the first $\frac{n}{2}$ vertices, and there are no other edges, except possibly for those incident to v_{k+1} . Counting the left ends of edges, we have $\alpha\binom{n}{2} = |E| = kn/2 - \binom{k}{2} + O(n)$.

Writing $k = \sigma n$, we obtain $\sigma = (1 - \sqrt{1 - 4\alpha})/2 + o(1)$. The sum of the squared degrees is then $k\frac{n^2}{4} + (\frac{n}{2} - k)k^2 + O(n^2) = (\frac{\sigma}{4}(1 + 2\sigma - 4\sigma^2) + o(1))n^3$. Lemma 10 is proved. □

Proof of Theorem 5 Let $E \subseteq \binom{V}{2}$ be minimal with $|E| = \alpha\binom{n}{2}$. By (7) and Lemma 10,

$$\begin{aligned} |\delta E| &\geq (n - 2)|E| - \sum_{v \in V} \deg_E(v)^2 \\ &\geq 3\alpha\binom{n}{3} - \left(\frac{\sigma}{4}(1 + 2\sigma - 4\sigma^2) + o(1)\right)n^3 \\ &= \frac{3}{2}(2\sigma - 2\sigma^2 - \sigma(1 + 2\sigma - 4\sigma^2) + o(1))\binom{n}{3} \\ &= \frac{3}{2}(\sigma(1 - 2\sigma)^2 + o(1))\binom{n}{3} = \frac{3}{4}((1 - \sqrt{1 - 4\alpha})(1 - 4\alpha) + o(1))\binom{n}{3}, \end{aligned}$$

as claimed. □

A Promising Relaxation? As we have seen, in order to establish the tightness of the upper bound from Proposition 7, one has to use the minimality condition in a stronger way than we did in the above proof. On the other hand, it seems possible that the other relaxation we have made in that proof, namely, ignoring triangles, need not cost us anything. In other words, while in δE we count triples containing 1 or 3 edges, perhaps the example in Proposition 7 also minimizes, over all minimal E of a given size, the number of triples containing exactly one edge. This might be easier to prove, and triangles would be dealt with implicitly, since the example has no triangles.

On Gromov’s “ $\frac{2}{3}$ -Bound”. Sect. 3.7 of [9] claims the lower bound $\|(\partial^1)_{\text{fil}}^{-1}\|(\beta) \leq \frac{2}{3(1-\sqrt{\beta})}$ (which would yield $\varphi_2(\alpha) \geq \frac{3}{2}\alpha - (\frac{3}{2})^{3/2}\alpha^{3/2} + \frac{9}{8}\alpha^2 - O(\alpha^{5/2})$).

The argument as given does not seem to work, however (although it is also possible that we misunderstood something). It is supposed to be based on an inequality (the fifth displayed formula in Sect. 3.7, derived from the Loomis–Whitney inequality), which seems correct and is re-stated for $i = 1$ two lines below. In the language of graphs, the $i = 1$ case is equivalent to $\sum_{\{u,v\} \in \binom{V}{2}} \deg_E(u) \deg_E(v) \leq 2\frac{n-1}{n}|E|^2$.

However, the proof of the “ $\frac{2}{3}$ -bound” seems to employ a similar inequality for $i = 2$, which would claim that $\sum_{\{u,v\} \in \binom{V}{2}} \sqrt{\deg_E(u) \deg_E(v)} \leq 2|E|^{3/2}$. This is false, though (a graph as in the upper bound example, i.e., a complete bipartite graph with parts of very unequal size, is a counterexample). Probably this kind of proof can be saved, since it seems sufficient to take the last sum over $\{u, v\} \in E$, instead of all pairs, and then such an inequality is apparently true (but it does not seem to follow from Loomis–Whitney in a direct way).

5 The Cofilling Profile for $d > 2$

In this section we prove Theorem 6, a lower bound on φ_3 . We begin with an auxiliary fact concerning links of vertices.

Observation 11 *If $E \subseteq \binom{V}{d}$ is a minimal system, then $\text{lk}(v, E)$ is also minimal, for every vertex $v \in V$.*

Proof For a set system F on V , let us write $F_{\setminus v} := \{s \in F : v \notin s\}$.

We want to verify that for each $C \subseteq \binom{V}{d-2}$, $\text{lk}(v, E)$ contains at most half of δC . The sets of $\text{lk}(v, E)$ do not contain v , and so only sets of $(\delta C)_{\setminus v}$ may belong to $\text{lk}(v, E)$. But we have $(\delta C)_{\setminus v} = (\delta C_{\setminus v})_{\setminus v}$, and so we may restrict our attention to systems C whose sets all avoid v .

Let $D := C * v := \{c \cup \{v\} : c \in C\}$. We have $\delta D = ((\delta C)_{\setminus v}) * v$. Now E contains at most half of the sets of δD by minimality, so $\text{lk}(v, E)$ contains at most half of the sets of $(\delta C)_{\setminus v}$ as claimed. □

In the proof of Theorem 6, we consider a minimal system $E \subseteq \binom{V}{3}$, $\|E\| = \alpha$, and we want to show that it has a large coboundary. Conceptually, the proof splits into two cases: The first one deals with the situation where most of the triples $e \in E$ are incident to vertices of very large degrees. The second one concerns the situation where the maximum vertex degree is not much larger than the average vertex degree.

Dealing with High-Degree Vertices. We begin with the first case, with a significant share of high-degree vertices. Here we rely on the basic cofilling bound, which we are going to apply to the links of high-degree vertices (the links are minimal by Observation 11). From the sets in the coboundaries of the links we are going to obtain sets of δE ; some care is needed to avoid counting a single $f \in \delta E$ several times.

It is interesting to note that in this way we get a lower bound for φ_3 that has a correct limit behavior as $\alpha \rightarrow 0$, although for the links we employ the basic cofilling bound, which is far from correct for small α 's. This can be explained as follows: for those systems E that are near-extremal for φ_3 , the relevant vertex links are so large that the basic cofilling bound is almost tight for them.

We present the first part of the proof for d -tuples instead of triples, since specializing to triples would not make the argument any simpler.

Lemma 12 *Let E be a minimal system of d -tuples on $V = \{v_1, v_2, \dots, v_n\}$, let $\alpha := \|E\|$, let $r = \beta n$ be a parameter, and let $E_{\text{hi}} \subseteq E$ consist of those $e \in E$ that*

contain at least one vertex among v_1, \dots, v_r . Let F_{hi} be the set of those $f \in \delta E$ that contain at least one vertex among v_1, \dots, v_r . Then

$$\|F_{hi}\| \geq \frac{d+1}{d} \alpha_{hi} - \frac{(d+1)d}{2} \beta^2 - (d+1)\alpha\beta - O(n^{-1}),$$

where $\alpha_{hi} := \|E_{hi}\|$.

Proof Let $v \in V$ be a vertex, and let us write $L_v := \text{lk}(v, E)$. Then, by Observation 11, L_v is minimal, and thus $\|\delta L_v\| \geq \|L_v\|$ by the basic bound on φ_{d-1} . In terms of cardinalities, we can write this inequality as $|\delta L_v| \geq \frac{n-d+1}{d} |L_v|$.

Let us consider some $e \in \delta L_v$. We observe that if $v \notin e$ and $e \not\subseteq E$, then $e \cup \{v\} \in \delta E$. There are exactly $|L_v|$ sets $e \in \delta L_v$ that contain v , and so the number of $f \in \delta E$ that contain v is at least

$$\frac{n-d+1}{d} |L_v| - |L_v| - |E|. \tag{8}$$

To prove the lemma, we would like to sum this bound over $v = v_1, v_2, \dots, v_r$, but in this way, one $f \in \delta E$ might be counted several times. In order to avoid this, for each v_i we will count only those $f \in \delta E$ that contain v_i and avoid v_1, \dots, v_{i-1} .

In this way, from the term (8) for $v = v_i$ we need to subtract the number of $f \in \delta E$ that contain both v_i and some $v_j, j < i$. A trivial upper bound on this number is $(i-1) \binom{n-2}{d-1}$. Hence

$$\begin{aligned} |F_{hi}| &\geq \sum_{i=1}^r \left(\left(\frac{n-d+1}{d} - 1 \right) |L_{v_i}| - (i-1) \binom{n-2}{d-1} - |E| \right) \\ &\geq \frac{n}{d} \left(\sum_{i=1}^r |L_{v_i}| \right) - \frac{\beta^2}{2} n^2 \binom{n}{d-1} - \beta n |E| - O(n^d). \end{aligned}$$

(For the second inequality, $-2 \sum_{i=1}^r |L_{v_i}|$ is subsumed in the $O(n^d)$ -term.) Now $\sum_{i=1}^r |L_{v_i}| \geq |E_{hi}|$. We finally divide by $\binom{n}{d+1}$ in order to pass to the density (normalized size) measure $\|\cdot\|$, and we obtain the lemma. \square

Dealing with Low-Degree Vertices. Next, we will show that if the vertex degrees of E do not exceed the average vertex degree by too much, then δE is even significantly larger than in the upper bound example from Proposition 7. Here it is important for the argument that we deal with triples.

In this case, we are going to count the sets of the coboundary using two-term inclusion-exclusion, similar to the case $d = 2$ in the preceding section. This leads to bounding from above the sum of squares of the degrees of *pairs* of vertices. For this, by a suitable double counting, we use the assumption of low *vertex* degrees, and also the fact that the degrees of all pairs are bounded by $\frac{n}{2}$, which follows from the minimality of E .

Lemma 13 *Let $d = 3$ and let $E \subseteq \binom{V}{3}$. Suppose that $\deg_E(p) \leq \frac{n}{2}$ for each pair $p = \{u, v\}$ of vertices, and that $\deg_E(v) \leq \sigma \binom{n}{2}$ for each vertex v . Then*

$$\|\delta E\| \geq (2 - O(\sigma^{1/3})) \|E\|.$$

Proof Similar to the graph case, we will count only those 4-tuples in δE that contain exactly one $e \in E$. For each $e \in E$, we count $n - 3$ potential 4-tuples, and we subtract 1 for each $e' \in E$ sharing a pair with e . Thus,

$$|\delta E| \geq (n - 3)|E| - \sum_{p \in \binom{V}{2}} \deg_E(p)^2.$$

We need to estimate the second term.

We choose a threshold parameter τ , which we think of as being much larger than σ but still small, and we call a pair p *heavy* if $\deg_E(p) \geq \tau n$, and *light* otherwise.

Each $e \in E$ shares a light pair with at most $3\tau n$ other $e' \in E$, and so the contribution of light pairs is bounded as follows:

$$\sum_{p \text{ light}} \deg_E(p)^2 \leq 3\tau n|E|.$$

Let E_k be the number of $e \in E$ with exactly k heavy pairs, $k = 0, 1, 2, 3$. Since each heavy pair has degree at most $\frac{n}{2}$, reasoning as above, we can bound the contribution of the heavy pairs as

$$\sum_{p \text{ heavy}} \deg_E(p)^2 \leq \frac{n}{2}(|E_1| + 2|E_2| + 3|E_3|).$$

We aim at showing that $E_2 \cup E_3$ is small.

Let us consider a vertex $v \in V$ and see how many $e \in E_2 \cup E_3$ can be incident to it. More precisely, we want to estimate m_v , the number of $e \in E_2 \cup E_3$ that have two heavy pairs incident to v .

Let $\deg_E(v) = \sigma_v \binom{n}{2}$, where $\sigma_v \leq \sigma$, and let us consider the graph $G_v := (V, \text{lk}(v, E))$. The heavy pairs incident to v correspond to the *heavy vertices* of G_v , i.e., vertices of degree at least τn , and m_v is the number of pairs in G_v connecting two heavy vertices. By simple counting, G_v has at most $\frac{\sigma_v n}{\tau}$ heavy vertices, and thus $m_v \leq (\frac{\sigma_v n}{\tau})^2 / 2 \leq \frac{\sigma}{\tau^2} |\text{lk}(v, E)|$.

Summing over all vertices v , we have $|E_2| + |E_3| \leq \frac{3\sigma}{\tau^2} |E|$. Altogether we thus have

$$\sum_{p \in \binom{V}{2}} \deg_E(p)^2 \leq 3\tau n|E| + \frac{n}{2}|E| + O\left(\frac{\sigma}{\tau^2}\right)n|E|.$$

Finally, setting $\tau := \sigma^{1/3}$ and normalizing by $\binom{n}{3}$, we obtain the claim of the lemma. □

Proof of Theorem 6 This is a straightforward consequence of Lemmas 12 and 13. We consider a minimal $E \subseteq \binom{V}{3}$ with $\|E\| = \alpha$. We enumerate the vertices of V as v_1, \dots, v_n in the order of decreasing degrees. For a suitable parameter $\beta > \alpha$ (depending on α), we set $r := \beta n$, we let E_{hi} be those $e \in E$ that contain a vertex among v_1, \dots, v_r , and let $E_{\text{lo}} := E \setminus E_{\text{hi}}$.

We have $\sum_{i=1}^n \deg_E(v_i) = 3|E|$, and so the degrees of the vertices v_r, v_{r+1}, \dots, v_n are bounded from above by $\frac{3|E|}{r}$. Hence Lemma 13 with $\sigma := \alpha/\beta$ gives $\|\delta E_{10}\| \geq (2 - (\alpha/\beta)^{1/3})\alpha_{10}$, where $\alpha_{10} := \|E_{10}\|$.

Let F_{hi} be the set of those $f \in \delta E$ that contain a vertex among v_1, \dots, v_r ; then Lemma 12 yields $\|F_{hi}\| \geq \frac{4}{3}\alpha_{hi} - O(\beta^2)$ (the $\alpha\beta$ term in the lemma is insignificant since we assume $\alpha \leq \beta$).

We now observe that if some $f \in \delta E_{10}$ does not contain any of v_1, \dots, v_r , then it belongs to δE (since it cannot contain any $e \in E_{hi}$). The number of $f \in \delta E_{10}$ that do contain some vertex among v_1, \dots, v_r is bounded by $r|E_{10}|$. Altogether we thus have

$$\begin{aligned} \|\delta E\| &\geq \|F_{hi}\| + \|\delta E_{10}\| - O(\beta\alpha_{10}) \\ &\geq \frac{4}{3}\alpha_{hi} - O(\beta^2) + (2 - (\alpha/\beta)^{1/3})\alpha_{10} \\ &\geq \frac{4}{3}\alpha - O(\beta^2) + \left(\frac{2}{3} - O((\alpha/\beta)^{1/3})\right)\alpha_{10}. \end{aligned}$$

If we set $\beta := C\alpha$ for a sufficiently large constant C , then the term $O((\alpha/\beta)^{1/3})$ becomes smaller than $\frac{2}{3}$, and the whole term involving α_{10} is nonnegative. Thus, we are left with $\|\delta E\| \geq \frac{4}{3}\alpha - O(\alpha^2)$ as claimed. □

6 Pagodas and a Better Bound on c_3

We recall the lower bound for the Bárány constant c_d from Proposition 4:

$$c_d \geq \varphi_d\left(\frac{1}{2}\varphi_{d-1}\left(\frac{1}{3}\varphi_{d-2}\left(\dots \frac{1}{d}\varphi_1\left(\frac{1}{d+1}\right)\dots\right)\right)\right), \tag{9}$$

As a case study, we will concentrate on c_3 , the first open case. The various numerical bounds are as follows.

- Gromov’s bound, obtained from (9) via the basic cofilling bound and the precise value of φ_1 , is

$$c_3 \geq \frac{1}{16} = 0.0625.$$

- For comparison, the best lower achieved by other methods, due to Basit et al. [5], is

$$c_3 \geq 0.05448.$$

- With *maximal optimism*, assuming that the upper bounds of Proposition 7 are tight for φ_2 and φ_3 , (9) would give

$$c_3 \stackrel{??}{\geq} 0.0877695.$$

- However, the best upper bound, which we suspect to be the truth, only gives

$$c_3 \leq 4!/4^4 = \frac{3}{32} = 0.09375.$$

Here we will show how the lower bound for c_3 can be improved *beyond* (9). The specific number we achieve is not very impressive: $c_3 \geq 0.06332$. However, it is important that the proof relies only on the basic cofilling bounds on φ_2 and φ_3 . If better lower bounds on φ_2 or φ_3 could be proved in suitable ranges of α , which would improve the lower bound (9), we would automatically get a further (slight) improvement from the proof below; in this sense, the method is “orthogonal” to bounds on the cofilling profiles.

We begin by returning to the argument in Sect. 3.4 that proves (9), and for simplicity, we specialize to the $d = 3$ case. In that argument, we considered a tetrahedron $wxyz$ in the triangulation \mathcal{T} , with the corresponding set systems (cochains) $F_A \subseteq \binom{V}{5-|A|}$ for all nonempty sets $A \subseteq S := \{w, x, y, z\}$. We have $\|F_A\| = o(1)$ unless $|A| = 1$. We also produced the set systems $F_{Ao} \subseteq \binom{V}{4-|A|}$ satisfying the relations

$$\delta F_{Ao} = F_A + \sum_B F_{Bo}, \tag{10}$$

where the sum is over all $B \subset A$ of size $|A| - 1$. Moreover, the crux of the argument was the existence of a tetrahedron $wxyz$ for which we also have

$$F_S + \sum_{B \in \binom{S}{3}} F_{Bo} = V. \tag{11}$$

We introduce the notation $X \approx Y$ for sets X, Y of k -tuples, meaning that $|X + Y| = o(n^k)$, or in other words, $\|X + Y\| = o(1)$. We eliminate the sets F_A with $|A| \neq 1$ from our considerations, since they are all small; then (10) and (11) become \approx relations among the various F_{Ao} .

We introduce a definition reflecting these relations; we chose to call the resulting object, a structure made of cochains of various dimensions, a *pagoda*. In order to make the notation more intuitive, we distinguish sets of various cardinalities by different letters (with indices), writing V for sets of vertices (0-cochains), E for sets of edges (1-cochains), F for sets of triples (2-cochains), and G for sets of four-tuples. We also change the indexing of the sets into an (isomorphic but) more convenient one. This leads to the following definition.

Definition 14 A (3-dimensional) *pagoda* over V consists of vertex sets $V_1, V_2, V_3, V_4 \subseteq V$, edge sets $E_{12}, E_{13}, \dots, E_{34} \subseteq \binom{V}{2}$, sets $F_{123}, F_{124}, F_{134}, F_{234} \subseteq \binom{V}{3}$ of triples, and a set $G = G_{1234} \subseteq \binom{V}{4}$ of 4-tuples (the *top* of the pagoda). The sets V_i, E_{ij} and F_{ijk} are *minimal* and they satisfy the following relations (here i, j, k denote mutually distinct indices):

$$V_1 + V_2 + V_3 + V_4 \approx V, \quad \delta V_i \approx \sum_j E_{ij}, \quad \delta E_{ij} \approx \sum_k F_{ijk}, \quad \delta F_{ijk} \approx G.$$

Thus, the argument of Proposition 4 shows that $c_3 \geq c_3^{\text{top}} \geq \liminf_{|V| \rightarrow \infty} \min \|G\|$, where the minimum is over tops G of pagodas over V .

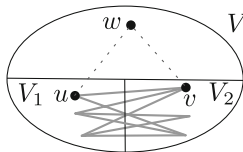
We know of *no example* of a pagoda whose top is smaller than the best known upper bound for c_3 , i.e., $\|G\| = \frac{3}{32}$. So it is possible that the value of c_3 can be determined precisely using a combinatorial analysis of pagodas. Unfortunately, we have only a much weaker result.

Proposition 15 *The top G of every pagoda satisfies, for all n sufficiently large, $\|G\| \geq \frac{1}{16} + \varepsilon_0$, for a positive constant $\varepsilon_0 > 0.00082$. Consequently, $c_3 \geq c_3^{\text{top}} \geq \frac{1}{16} + \varepsilon_0 > 0.06332$.*

A similar, but more complicated, argument probably also works for higher-dimensional pagodas.

Remark Developing the ideas from the forthcoming proof of Proposition 15, one can set up a rather complicated optimization problem, whose optimum provides a lower bound for c_3^{top} . However, solving this (non-convex) optimization problem rigorously seems rather difficult. Numerical computations indicate that the optimum is approximately 0.0703125. This would be an improvement much more significant than the one in Proposition 15, but still far from the suspected true value of c_3 .

Proof of Proposition 15 We begin by an outline of the argument. We consider a pagoda with $\|G\| \leq \frac{1}{16} + \varepsilon_0$, where $\varepsilon_0 \geq 0$ is a yet unspecified (small) parameter. Reasoning essentially as in the derivation of (9) and using the basic bound for φ_3 and φ_2 and the true value of φ_1 , we find that no $\|E_{ij}\|$ and no $\|\delta E_{ij}\|$ may be significantly larger than $\frac{1}{8}$, and no V_i may occupy much more than $\frac{1}{4}$ of the vertex set. Hence the V_i are almost disjoint and have sizes close to $\frac{1}{4}$. Then one can argue that almost all edges of E_{12} , say, have to go between V_1 and V_2 , as in the following picture:



Most of E_{12} goes between V_1 and V_2 .
Most triples like $\{u, v, w\}$ are in δE_{12} .

Then, however, the coboundary δE_{12} contains almost all triples of the form indicated in the picture, and thus $\|\delta E_{12}\|$ is close to $\frac{3}{16}$, rather than to $\frac{1}{8}$, which is a contradiction showing that ε_0 cannot be taken arbitrarily small.

Now we proceed with a more detailed and more quantitative argument. Since $\|G\| \leq \frac{1}{16} + \varepsilon_0$ and $G \approx \delta F_{ijk}$, we have $\|F_{ijk}\| \leq \frac{1}{16} + \varepsilon_0$ according to the basic bound for φ_3 (ignoring, for simplicity, the $o(1)$ terms coming from \approx). Further, since each δE_{ij} is a sum of two F_{ijk} 's, the basic bound for φ_2 gives

$$\|E_{ij}\| \leq \frac{1}{8} + 2\varepsilon_0, \quad 1 \leq i < j \leq 4.$$

Similarly, since each δV_i is the sum of three E_{ij} 's, and since $\varphi_1(\alpha) = 2\alpha(1 - \alpha)$, we get

$$\|V_i\| \leq \frac{1}{4} + \varepsilon_1, \tag{12}$$

where $\varepsilon_1 = \frac{1}{4}(1 - \sqrt{1 - 48\varepsilon_0}) = 6\varepsilon_0 + 72\varepsilon_0^2 + O(\varepsilon_0^3)$ is given by the equation $\varphi_1(\frac{1}{4} + \varepsilon_1) = 3(\frac{1}{8} + 2\varepsilon_0)$.

Next, let E_{ij}^{dbl} be the set of edges $e \in E_{ij}$ that belong to both δV_i and δV_j , and let E_{ij}^{sgl} consist of those $e \in E_{ij}$ that belong to exactly one of $\delta V_i, \delta V_j$.

Using $\delta V_i \approx \sum_j E_{ij}$ and summing over $i = 1, \dots, 4$, we have

$$\sum_{i=1}^4 \|\delta V_i\| \leq 2 \sum_{i < j} \|E_{ij}^{dbl}\| + \sum_{i < j} \|E_{ij}^{sgl}\|. \tag{13}$$

For every i , we have, using (12),

$$\|V_i\| \geq 1 - \sum_{j \neq i} \|V_j\| \geq 1 - 3(\frac{1}{4} + \varepsilon_1) \geq \frac{1}{4} - 3\varepsilon_1.$$

Hence the left-hand side of (13) is at least $4\varphi_1(\frac{1}{4} - 3\varepsilon_1) = 4(\frac{3}{8} - \varepsilon_2)$, where ε_2 is defined by the last equality (we get $\varepsilon_2 = 18\varepsilon_0 + 864\varepsilon_0^2 + O(\varepsilon_0^3)$).

Further, for every $i < j$ we have $\|E_{ij}^{dbl}\| + \|E_{ij}^{sgl}\| \leq \|E_{ij}\| \leq \frac{1}{8} + 2\varepsilon_0$. Summing this inequality over the six pairs i, j and subtracting the result from (13), we arrive at

$$4(\frac{3}{8} - \varepsilon_2) - \frac{6}{8} - 6 \cdot 2\varepsilon_0 \leq \sum_{i < j} \|E_{ij}^{dbl}\|.$$

Hence there are i and j with

$$\|E_{ij}^{dbl}\| \geq \frac{1}{8} - \frac{2}{3}\varepsilon_2 - 2\varepsilon_0.$$

Let us fix the notation so that $i = 1$ and $j = 2$ have this property.

Using $\|E_{12}^{dbl}\| + \|E_{12}^{sgl}\| \leq \frac{1}{8} + 2\varepsilon_0$ again, we further obtain

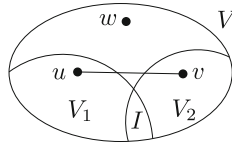
$$\|E_{12}^{sgl}\| \leq 4\varepsilon_0 + \frac{2}{3}\varepsilon_2. \tag{14}$$

Let us divide the edges in E_{12}^{dbl} into two subsets A and B , where A consists of the edges $e \in E_{12}^{dbl}$ that have one endpoint in $V_1 \setminus V_2$ and the other in $V_2 \setminus V_1$. Then each edge in $B := E_{12}^{dbl} \setminus A$ necessarily connects a vertex of $I := V_1 \cap V_2$ to a vertex in $V \setminus (V_1 \cup V_2)$.

The plan is now to show that, for ε_0 sufficiently small, the edges of A “contribute” many triples to δE_{12} . First we check that B is small, and we begin by bounding the size of I : We have $V = V_1 \cup \dots \cup V_4$, by inclusion–exclusion we get $1 \leq \|V_1\| + \|V_2\| - \|I\| + \|V_3\| + \|V_4\|$, and thus $\|I\| \leq \sum_{i=1}^4 \|V_i\| - 1 \leq 4\varepsilon_1$. By the minimality of E_{12} , each vertex in I has degree at most $\frac{2}{2}$, and so $\|B\| \leq \|I\| \leq 4\varepsilon_1$. Therefore,

$$\|A\| \geq \|E_{12}^{dbl}\| - \|B\| \geq \frac{1}{8} - \frac{2}{3}\varepsilon_2 - 2\varepsilon_0 - 4\varepsilon_1. \tag{15}$$

Now let us consider an edge $e = \{u, v\} \in A$ and a vertex $w \in V \setminus (V_1 \cup V_2)$. The triple $\{u, v, w\}$ belongs to δE_{12} unless one of the edges $\{u, w\}$ and $\{v, w\}$ lies in E_{12} ; see the following illustration:



The (normalized) number of “candidate triples” $\{u, v, w\}$ with $\{u, v\} \in A$ and $w \in V \setminus (V_1 \cup V_2)$ is $3\|A\| \cdot \|V \setminus (V_1 \cup V_2)\|$. Moreover, if one of the edges $\{u, w\}, \{v, w\}$ lies in E_{12} , then it lies in E_{12}^{sgl} . At the same time, a given edge $e' \in E_{12}^{\text{sgl}}$ may “kill” at most $\max(|V_1 \setminus V_2|, |V_2 \setminus V_1|) \leq (\frac{1}{4} + \varepsilon_1)n$ candidate triples $\{u, v, w\}$.

Hence

$$\|\delta E_{12}\| \geq 3\|A\|(\frac{1}{2} - 2\varepsilon_1) - 3\|E_{12}^{\text{sgl}}\|(\frac{1}{4} + \varepsilon_1) \geq \frac{3}{16} - f(\varepsilon_0),$$

where, employing (14) and (15), we calculate that $f(\varepsilon_0) = 6\varepsilon_0 + \frac{27}{4}\varepsilon_1 - 24\varepsilon_1^2 + \frac{3}{2}\varepsilon_2 - 2\varepsilon_1\varepsilon_2 = \frac{147}{2}\varepsilon_0 + 702\varepsilon_0^2 + O(\varepsilon_0^3)$. Since we have assumed $\|\delta E_{12}\| \leq \frac{1}{8} + 2\varepsilon_0$, we finally obtain $f(\varepsilon_0) + 2\varepsilon_0 \geq \frac{1}{16}$. The numerical lower bound in the proposition follows from this. \square

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