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Diameter Graphs in $\mathbb{R}^4$

Andrey Kupavskii

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Abstract A *diameter graph in* $\mathbb{R}^d$ is a graph whose set of vertices is a finite subset of $\mathbb{R}^d$ and whose set of edges is formed by pairs of vertices that are at diameter apart. This paper is devoted to the study of different extremal properties of diameter graphs in $\mathbb{R}^4$ and on a three-dimensional sphere. We prove an analog of Vázsonyi's and Borsuk's conjecture for diameter graphs on a three-dimensional sphere with radius greater than $1/\sqrt{2}$. We prove Schur's conjecture for diameter graphs in $\mathbb{R}^4$. We also establish the maximum number of triangles a diameter graph in $\mathbb{R}^4$ can have, showing that the extremum is attained only on specific Lenz configurations.

Keywords Diameter graphs · Geometric graphs · Schur's conjecture · Number of cliques

1 Introduction

The following question was raised by Borsuk in 1933 [3]: is it true that any set of diameter 1 in $\mathbb{R}^d$ can be partitioned into $d+1$ parts of strictly smaller diameter? The positive answer to this question is called Borsuk's conjecture. Borsuk gave a positive answer to this question for $d = 2$, and later the same was proved for $d = 3$ (see [16,17]). Borsuk's conjecture was disproved by Kahn and Kalai in 1993 [13]. In that paper, they constructed a *finite* set of points in dimension 2016 such that it cannot be

A. Kupavskii
Moscow Institute of Physics and Technology, Moscow Region, Russia

Present address:
A. Kupavskii
École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
e-mail: kupavskii@ya.ru

partitioned into 2017 parts of smaller diameter. The bounds on the minimum dimension of the counterexample were obtained by several authors. Very recently, Bondarenko [2] disproved Borsuk's conjecture in dimensions $d \geq 65$.

An analog of Borsuk's conjecture for finite sets is well studied. A natural notion to work with in the finite case is that of *diameter graph*. A *diameter graph in* $\mathbb{R}^d$ is a graph whose set of vertices is a finite subset of $\mathbb{R}^d$ and whose set of edges is formed by the pairs of vertices that are at diameter apart. Next, we work only with sets of diameter 1. For a finite set X of unit diameter denote by $G(X)$ the diameter graph with the vertex set X. In terms of diameter graphs, Borsuk's problem for finite sets can be formulated as follows: is it true that for any $X \subset \mathbb{R}^d$ we have $\chi(G(X)) \leq d+1$? Here, $\chi(G)$ is the chromatic number of a graph.

In [12] Hopf and Pannwitz proved that the number of edges in any diameter graph in $\mathbb{R}^2$ is at most n, which easily implies Borsuk's conjecture for finite sets on the plane. Vázsonyi conjectured that any diameter graph in $\mathbb{R}^3$ on n vertices can have at most $2n-2$ edges. It is easy to see that Borsuk's conjecture for finite sets in $\mathbb{R}^3$ follows from this statement. This conjecture was proved independently by Grünbaum [9], Heppes [10] and Straszewicz [19].

In this paper, we prove Borsuk's and Vázsonyi's conjecture for finite sets on a three-dimensional sphere S^3_r of radius $r > 1/\sqrt{2}$ (note that we consider sets of Euclidean diameter 1). It is easy to see that Vázsonyi's conjecture fails for $S^3_{1/\sqrt{2}}$. Also, the analogue of Borsuk's conjecture fails for $S^3_{\sqrt{2/5}}$: there exists a set of unit diameter on this sphere that cannot be partitioned into *four* parts of smaller diameter. This set is just a regular unit simplex with five vertices. Diameter graphs on S^3_r are discussed in Sect. 2.

As we discussed above the study of the maximum number of edges in diameter graphs is related to Borsuk's conjecture. Surely, it has an independent interest. Extremal properties of diameter graphs and unit distance graphs were extensively studied. A *unit distance graph* in $\mathbb{R}^d$ is a graph whose set of vertices is a finite subset of $\mathbb{R}^d$ and whose set of edges is formed by pairs of vertices that are at unit distance apart (here we do not demand that the set of vertices is of diameter 1).

Denote by $D_d(l,n)$ ($U_d(l,n)$) the maximum number of cliques of size l in a diameter (unit distance) graph on n vertices in $\mathbb{R}^d$. Erdős [7,8] studied $U_d(2,n)$ and $D_d(2,n)$ for different d. He showed that for $d \geq 4$ we have $U_d(2,n)$, $D_d(2,n) = \frac{\lfloor d/2 \rfloor - 1}{2\lfloor d/2 \rfloor} n^2 + \bar{o}(n^2)$. Brass [4] and van Wamelen [22] determined $U_4(2,n)$ for all n. Swanepoel [20] determined $U_d(2,n)$ for even $d \geq 6$ and sufficiently large n and determined $D_d(2,n)$ for $d \geq 4$ and sufficiently large n. He also proved some results concerning the stability of the extremal configurations. We refine the result of Swanepoel concerning $D_4(2,n)$ by giving a reasonable bound on n: we show that his result holds for $n \geq 52$.

Functions $D_d(l,n)$, $U_d(l,n)$ and similar functions were studied in several papers. In particular, the following conjecture was raised in [18]:

Conjecture *(Schur et al.* [18]*) We have* $D_d(d,n) = n$ *for* $n \geq d+1$.

This was proved by Hopf and Pannwitz for $d=2$ in [12] and for $d=3$ by Schur et al. in [18]. They also proved that $D_d(d+1,n) = 1$. In [15] the authors proved that Schur's conjecture holds in some special case:

Theorem 1 (Morić and Pach [15]) *The number of d-cliques in a diameter graph on n vertices in $\mathbb{R}^d$ is at most n, provided that any two d-cliques share at least $d-2$ vertices.*

In this paper, we prove Schur's conjecture for $d=4$. Moreover, we determine the exact value of $D_4(3,n)$ for large n. This completes the full description of functions $D_4(l,n)$ for large n. We also improve the result from Theorem 1 (in Sect. 3).

In the next section, we discuss diameter graphs on three-dimensional spheres, and in Sect. 3 we discuss diameter graphs in $\mathbb{R}^4$.

2 Diameter Graphs on the Three-Dimensional Sphere

In this section, we prove the following theorem:

Theorem 2 *Let X be a finite subset of diameter 1 on S^3_r, $|X|=n$. If $r>1/\sqrt{2}$, then:*

1. *$G=G(X)$ has at most $2n-2$ edges.*
2. $\chi(G)\le 4$.
3. *Any two odd cycles in G have a common vertex.*

The proof is based on the approach which was suggested by Dol'nikov [6] and developed by Swanepoel [21]. The author is grateful to A.V. Akopyan, who suggested the key idea of reduction to the great sphere S (see the proof of the theorem). A.V. Akopyan proved Borsuk's and Vázsonyi's conjecture on the sphere before the author (private communication) but he has not written the proof. Moreover, he claims that the proof works also for the three-dimensional hyperbolic space.

We will need the following lemmas.

Lemma 1 *Fix some natural $d\ge 2$. Let X be a subset of S^{d-1}_r of unit diameter. If $r>\sqrt{d/(2d+2)}$, then X lies in an open hemisphere of S^{d-1}_r.*

Proof Since X is a subset in $\mathbb{R}^d$, by Jung's theorem, X can be covered by a ball B of radius $\sqrt{d/(2d+2)}$. The sphere that bounds the ball B and S^{d-1}_r intersect in a sphere S of radius not greater than $\sqrt{d/(2d+2)}$, and the intersection of B and S^{d-1}_r lies entirely in the open hemisphere bounded by the great sphere $S'\subset S^{d-1}_r$, which is parallel to S. □

The next lemma is a modification of Lemma 3 from [21].

Lemma 2 *Fix some natural $d\ge 2$. Let $x_1,\dots,x_k$ and $\sum_{i=1}^k\lambda_i x_i$ be distinct vectors of length $a>0$ in $\mathbb{R}^d$, where $\lambda_i\ge 0$. Fix some $b>0$. Suppose that for some vector $y\in\mathbb{R}^d$ we have $\|y-x_i\|\le b$ for each $i=1,\dots,k$. Then $\|y-\sum_{i=1}^k\lambda_i x_i\|<b$, if $\|y\|^2+a^2-b^2>0$.*

Proof Since none of the x_i are collinear, from the strict convexity of the Euclidean norm we get

$$1=\frac{\|\sum_{i=1}^k\lambda_i x_i\|}{a}<\sum_{i=1}^k\lambda_i.$$

For each i we have

$$b^2 \geq \langle y - x_i, y - x_i \rangle = a^2 + \|y\|^2 - 2\langle y, x_i \rangle,$$

and we obtain $\|y\|^2 + a^2 - b^2 \leq 2\langle y, x_i \rangle$.

Thus, we have

$$\|y - \sum_{i=1}^{k} \lambda_i x_i\|^2 = \|y\|^2 - 2\sum_{i=1}^{k} \lambda_i \langle y, x_i \rangle + a^2 \leq \|y\|^2 - (\|y\|^2 + a^2 - b^2) \sum_{i=1}^{k} \lambda_i + a^2$$
$$= (\|y\|^2 + a^2 - b^2)(1 - \sum_{i=1}^{k} \lambda_i) + b^2 < b^2,$$

since $\|y\|^2 + a^2 - b^2 > 0$. □

Definition 1 The spherical convex hull $\mathrm{conv}_S(x_1, \ldots, x_k)$ of points $x_1, \ldots, x_k$ that lie in a hemisphere on the sphere S' centered at the point O is the intersection of the sphere S' and a cone, formed by the vectors Ox_i (the cone consists of all vectors of the form $\sum_{i=1}^{k} \lambda_i Ox_i$, $\lambda_i \geq 0$). The vertices of $\mathrm{conv}_S(x_1, \ldots, x_k)$ are the points of $\mathrm{conv}_S(x_1, \ldots, x_k)$ that correspond to vectors that cannot be expressed as a non-trivial convex combination of the other vectors forming the cone. Alternatively, these are such points $y_1, \ldots, y_l$ of $\mathrm{conv}_S(x_1, \ldots, x_k)$ that $\mathrm{conv}_S(y_1, \ldots, y_l) = \mathrm{conv}_S(x_1, \ldots, x_k)$ and the set $\{y_1, \ldots, y_l\}$ is minimal.

It is fairly easy to show that the set of vertices of $\mathrm{conv}_S(x_1, \ldots, v_k)$ is a subset of $\{x_1, \ldots, x_k\}$. For two points x_1, x_2 on the sphere S we denote by $\widehat{x_1 x_2}$ the shorter arc of the great circle that contains these two points. By $\|x_1 - x_2\|_S$ we denote the length of the arc. For the points $x_1, \ldots, x_k$ on the sphere S we denote by $S(x_1, \ldots, x_k) \subseteq S$ the great sphere of minimal dimension that contains $x_1, \ldots, x_k$.

Lemma 3 *Let X be a subset of S_r^2 of diameter 1. If $r > \sqrt{3/8}$ then for any a_1, b_1, a_2, b_2 such that $(a_i, b_i) \in E(G(X))$, $i = 1, 2$, the arcs $\widehat{a_1 b_1}$ and $\widehat{a_2 b_2}$ intersect.*

Proof By Lemma 1, X lies in an open hemisphere of S_r^2. Suppose that the arcs do not intersect. Consider the spherical convex hull of the points a_1, a_2, b_1, b_2. We have the following two possibilities.

First, the spherical convex hull is a spherical triangle. Without loss of generality, assume that the vertices of the triangle are a_1, b_1, b_2. Then we can apply Lemma 2 for the points a_1, b_1, b_2 as x_i, a_2 as $\sum_{i=1}^{3} \lambda_i x_i$ and b_2 as y. We put $a = b = 1$ and obtain that, on the one hand, $\|a_2 - b_2\|$ should be strictly less than one, but on the other, these two vertices are connected by an edge, a contradiction.

Second, the convex hull is a spherical quadrilateral with $\widehat{a_1 b_1}$ and $\widehat{a_2 b_2}$ as two edges. Suppose that the other two edges of the quadrilateral are $\widehat{a_1 a_2}$ and $\widehat{b_1 b_2}$, so $\widehat{a_1 b_2}$ and $\widehat{a_2 b_1}$ are diagonals, and that they intersect at a point x. By the triangle inequality for the sphere we obtain that $\|a_1 - x\|_S + \|x - b_1\|_S > \|a_1 - b_1\|_S$, $\|a_2 - x\|_S + \|x - b_2\|_S > \|a_2 - b_2\|_S$. Consequently, at least one of the following two inequalities hold: $\|a_1 - b_2\|_S = \|a_1 - x\|_S + \|x - b_2\|_S > \|a_1 - b_1\|_S$ or

$\|a_2 - b_1\|_S = \|a_2 - x\|_S + \|x - b_1\|_S > \|a_1 - b_1\|_S$. Thus, either $\|a_1 - b_2\| > 1$ or $\|a_2 - b_1\| > 1$. □

Proof of Theorem 2 Consider a set X of diameter 1 on the sphere S_r^3 and the graph $G = G(X) = (V, E)$. By $N(v)$ we denote the set of neighbors of $v \in V$. Hereinafter $\text{conv}_S(N(v))$ is the set on the two-dimensional sphere $S^2(v)$, which is the intersection of S_r^3 and the sphere of unit radius with the center v. The convex hull is taken with respect to $S^2(v)$. □

Lemma 4 *For any two points $u, v \in V$ and any two points $x \in \text{conv}_S(N(v))$, $y \in \text{conv}_S(N(u))$ we have $\|x - y\|, \|x - u\| \leq 1$. Moreover, if x is not a vertex of $\text{conv}_S(N(v))$, then $\|x - y\|, \|x - u\| < 1$.*

Proof Consider an arbitrary point z on the sphere S_r^3 such that $\|z - v\| \leq 1$ and for any $w \in N(v)$ we have $\|z - w\| \leq 1$. We will prove that if x' is not a vertex of $\text{conv}_S(N(v))$, then $\|z - x'\| < 1$. Inequalities in the lemma follow from this. Indeed, first one have to apply this to u (as z) and x, $x \notin N(v)$, as x'. It is possible to do so since u is at less than unit distance apart from any vertex from V. We obtain that $\|x - u\| < 1$ for any $u \in V$. Then we apply the statement again to x, y (with x being z, and y being x').

For some $w \in N(v)$ consider a vector Ow, where O is the center of $S^2(v)$, and the hyperplane π that is orthogonal to Ow and passes through O. The intersection of π and S_r^3 is the great sphere S' that contains v. The great circle $S(v, w)$ that contains v, w lies in the plane which is orthogonal to π, which means that the minimum of the distance between w and the points of S' is attained at one of the two points of $S(v, w) \cap S'$. Since $r > 1/\sqrt{2}$, the point O lies on the segment that connects the center of S_r^3 and v, and thus v is closer to w than to the other point from $S(v, w) \cap S'$. Consequently, for any point $s \neq v$ that lies on S' we have $\|s - w\| > 1$, so all points of $X \backslash \{v\}$ and z must lie on the side of π that contains w. Otherwise S' and $S^2(w)$ would intersect in at least two points, which is impossible. Therefore, $X \backslash \{v\}$ lies in the intersection of the *open* hemisphere of S_r^3, which is bounded by S' and contains w, and the spherical cap with the center at v bounded by $S^2(v)$.

Consider the projection z' of z on the hyperplane that contains $S^2(v)$. From the above considerations carried out for an arbitrary vertex of $\text{conv}_S(N(v))$ denoted by w we get that $\langle Oz', Ow\rangle > 0$. If $\|z' - w\| = b$ and $\|O - w\| = a$, then $\|O - z'\|^2 = b^2 - a^2 + 2\langle Oz', Ow\rangle > b^2 - a^2$. Thus, we can apply Lemma 2 and obtain that $\|z' - x\| < \max_{w \in N(v)} \|z' - w\|$. Consequently, $\|z - x\| < \max_{w \in N(v)} \|z - w\| \leq 1$. □

From Lemma 4, we obtain that the set $X' = \bigcup_{v \in X} \left(\{v\} \cup \text{conv}_S(N(v))\right)$ is a set of diameter 1. By Lemma 1 X' lies in an open hemisphere $H \subset S_r^3$. Denote by S the diametral sphere which bounds H.

For a vertex $v \in V$ we denote by $w_1, \dots w_s \in V$ the neighbors of v in G. For $i = 1, \dots, s$ let u_i, u_i' be the points of the intersection of the sphere S and the great circle $S(v, w_i)$ in S_r^3, where u_i is closer to w_i and u_i' is closer to v. Denote by $R(v)$ the set $\text{conv}_S(u_1, \dots, u_s)$ on the sphere S, and by $B(v)$ the set $\text{conv}_S(u_1', \dots, u_s')$, which is symmetric to $R(v)$ with respect to the center of S.

We note that the following important property of this "projection" holds: for any point u in $\mathrm{conv}_S(u_1, \ldots, u_s)$ the arc $\widehat{vu}$ intersects $\mathrm{conv}_S(N(v))$ at some point w. We argue in terms of the vectors that correspond to the points on the sphere S_r^3. By abuse of notation for the vectors in this paragraph, we use the same notation as for the points. Suppose the vector $u = \sum_{i=1}^k \lambda_i u_i$, where $\lambda_i \geq 0$. Then the great circle $S(v, u)$ is formed by vectors of the form $c_1 v + c_2(\sum_{i=1}^k \lambda_i u_i)$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary, with the only condition that one of them is non-zero. Recall that the points $w_1, \ldots, w_s$ lie on the sphere $S^2(v)$ with the center at O. For each point w in $\mathrm{conv}_S(N(v))$ the corresponding vector on S_r^3 may be expressed as a combination of vector v and of vectors Ow_i. On the other hand, for each $i = 1, \ldots, k$ vector Ow_i is a combination of v and u_i. Put w to be a point on $S^2(v)$ such that the corresponding vector on S_r^3 is $c'v + \sum_{i=1}^k \lambda_i Ow_i$. Then if instead of Ow_i we substitute a combination of u_i and v, we obtain a point on $S(v, u)$. Surely, this is a point of the arc $\widehat{vu}$. The property is justified.

Lemma 5

1. *For $u \neq v \in V$ the sets $R(v)$ and $R(w)$ do not intersect.*
2. *Suppose that for some $u, v \in V$ the sets $R(v)$ and $B(u)$ intersect. Then the intersection is a single point and in this case $(u, v) \in E$. Moreover, this point is a vertex of a spherical polyhedron $R(v)$, if $\deg u \geq 2$, and is a vertex of a spherical polyhedron $R(u)$, if $\deg v \geq 2$.*

Proof 1. Suppose that the sets $R(v)$ and $R(w)$ intersect at a point x. Consider the arcs $\widehat{vx}$ and $\widehat{wx}$. Suppose they do not lie on the same diametral circle. By the property discussed before lemma, the arcs $\widehat{vx}$ and $\widehat{wx}$ intersect $\mathrm{conv}_S(N(v))$ and $\mathrm{conv}_S(N(w))$ at points x_v and x_w respectively.

Consider the great two-dimensional sphere $S(x, v, w)$ in S_r^3. The arcs $\widehat{vx_v}$ and $\widehat{wx_w}$ do not intersect. Applying Lemma 3 we get that the distance between some of the points v, w, x_v, x_w is greater than 1. On the other hand, all these points belong to X', which is of diameter 1, a contradiction. Thus, the arcs lie on the same diametral circle, and v and w must coincide. Indeed, if not then v, w, x_v, x_w are four distinct points on one half-circle, and either $\|v - x_w\| > 1$ or $\|w - x_v\| > 1$.

2. Suppose that the sets $R(v)$ and $B(w)$ intersect at a point x. If the arcs $\widehat{vx}$ and $\widehat{wx}$ do not lie on the same diametral circle, then we can apply the considerations from the previous part.

If these two arcs lie on the same diametral circle, then $v \in N(w)$ and vice versa. Indeed, if $\|v - w\| < 1$, then $\|x_v - x_w\| > 1$, where $x_v = S(v, x) \cap \mathrm{conv}_S(N(v))$. On the other hand, according to Lemma 4, $\|x_v - x_w\| \leq 1$.

The second statement of point 2 of Lemma 5 follows easily from the second part of Lemma 4. □

We may assume that G does not have vertices of degree ≤ 1.

We construct a bipartite double cover $C = (V(C), E(C))$ of G, which has a symmetric drawing on S. We choose a point $c(v)$ in the interior of $R(v)$, and the antipodal point $c'(v)$ in the interior of $B(v)$. We connect all vertices of $R(v)$ with $c(v)$ by great arcs (since all the vertices in G have degree ≥ 2, by Lemma 5 each neighbor of v corresponds to some vertex of $R(v)$). We also draw antipodal arcs from vertices of $B(v)$ to $c'(v)$. The set $V(C)$ consists of $c(v), c'(v)$, where $v \in V$; the set of edges $E(C)$ consists of all pairs $c(v), c'(w), v, w \in V$ that are joined by curves that consist of two great arcs (one in R(v), the other in B(w)) that share a point. What we described before is thus the drawing of C on S. It is easy to see that if for any $v \in V$ we correspond $c(v), c'(v)$ to v, then we indeed get a double covering of G. Moreover, C is bipartite, since we can color $c(v), v \in V$, in red and $c'(v), v \in V$ in blue. This is a proper coloring according to Lemma 5.

The graph C is a planar bipartite graph on $2n$ vertices, so it has at most $4n - 4$ edges. Consequently, graph G has at most $2n-2$ edges and the first point of Theorem 2 is proved.

For any graph G such that any subgraph $H = (V(H), E(H))$ of G satisfies $|E(H)| \leq 2|E(H)| - 2$ it is easy to show that $\chi(G) \leq 4$. Indeed, assume that n_0 is the minimal n such that there is a graph G of order n satisfying the above described property and such that $\chi(G) \geq 5$. G contains a vertex v of degree ≤ 3. By minimality of n_0, $\chi(G\backslash\{x\}) \leq 4$. But then we can color v in the color that differs from colors of its neighbors and obtain a proper coloring of G in four colors.

To prove the last point of Theorem 2 we note that each odd cycle in G corresponds in the drawing of C described above to a closed self-symmetric curve on the sphere without self-intersections. Any two such curves must intersect. But they can intersect only in $c(v)$ (and $c'(v)$) for some $v \in V$. This means that the corresponding odd cycles in G share vertex v. The proof of the theorem is complete.

It is worth noting that in the proof the analogous statement for diameter graphs in $\mathbb{R}^3$ given in the paper [21] there is a slight inaccuracy related to the intersections of sets $R(x), B(y)$. In [21] Swanepoel used the following lemma, which is an analog of point 2 of Lemma 5:

Lemma 6 [21, Lemma 2] *If $R(x)$ and $B(y)$ intersect, then xy is a diameter and $R(x) \cap B(y) = \{y - x\}$.*

Then Swanepoel constructed a bipartite double cover using the same considerations as above. However, this lemma is not enough to construct a bipartite double cover which is a planar graph, so the final conclusion from [21], "By Lemmas 1 and 2 no edges cross, and the theorem follows," is wrong. The important thing missing is that after deleting all the vertices of degree 1, each point in $R(x), B(x)$ that correspond to diameters in the graph must be a vertex of the spherical polygon $R(x), B(x)$. The problem is that Lemma 2 does not exclude the following configuration: $R(x)$ and $B(y)$ are arcs $\widehat{u_x v_x}$ and $\widehat{u_y v_y}$ that intersect at the interior point $z = y - x$. This corresponds to the situation when x is connected by an edge to $y, x + u_x, x + v_x$ (see [21]), and y is connected to x, u_y, v_y. The conditions of Lemma 2 from [21] are satisfied in this situation, but if one tries to construct a drawing of C as described above, he ends up with a drawing that has self-intersections.

Fortunately, this configuration is impossible to get in $\mathbb{R}^3$, since the statement, analogous to the second part of the point 2 of Lemma 5 holds for diameter graphs in $\mathbb{R}^3$ (and it is in fact easy to deduce from Lemma 3 from [21]).

Nevertheless, if we consider the sphere S_r^3 with $r = 1/\sqrt{2}$, then we indeed can get the configuration described above, if we try to carry out the proof of Theorem 2 in this case. The graph G we need to consider is a complete bipartite graph on $2n$ vertices with equal part sizes. It has a standard realization on S_r^3, with two parts placed on two orthogonal diametral circles. The statement of the theorem indeed does not hold for such a graph since it has n^2 edges. Besides, this example shows that the bound on r in Theorem 2 is sharp.

3 Diameter Graphs in $\mathbb{R}^4$

As we already mentioned, Brass [4] and Van Wamelen [22] determined $U_4(2, n)$:

Theorem 3 *For $n \geq 5$,*

$$U_4(2, n) = \begin{cases} \lfloor n^2/4 \rfloor + n & \textit{if } n \textit{ is divisible by } 8 \textit{ or } 10, \\ \lfloor n^2/4 \rfloor + n - 1 & \textit{otherwise.} \end{cases}$$

Thus, we have $U_4(2, n) \leq n^2/4 + n$ for any $n \geq 1$. In [20] Swanepoel established the maximum number of edges in a diameter graph in $\mathbb{R}^4$, if n is sufficiently large:

Theorem 4 *For all sufficiently large n, $D_4(2, n) = F_2(n)$, where*

$$F_2(n) = \begin{cases} t_2(n) + \lceil n/2 \rceil + 1 & \textit{if } n \not\equiv 3 \bmod 4, \\ t_2(n) + \lceil n/2 \rceil & \textit{if } n \equiv 3 \bmod 4, \end{cases}$$

where $t_2(n) = \lfloor n/2 \rfloor \lceil n/2 \rceil$ is the number of edges in a complete bipartite graph on n vertices with almost equal part sizes.

In this section, we prove the following theorem:

Theorem 5

1. *The statement of Theorem 4 holds for $n \geq 52$.*
2. *For all sufficiently large n we have $D_4(3, n) = F_3(n)$, where*

$$F_3(n) = \begin{cases} (n-1)^2/4 + n & \textit{if } n \equiv 1 \bmod 4, \\ (n-1)^2/4 + n - 1 & \textit{if } n \equiv 3 \bmod 4, \\ n(n-2)/4 + n & \textit{if } n \equiv 0 \bmod 2. \end{cases}$$

3. *(Schur's conjecture in $\mathbb{R}^4$) For all $n \geq 5$ we have $D_4(4, n) = n$.*

Remark 1 It seems hard to derive any reasonable bound on n from the proof of Theorem 4 by Swanepoel. It is due to the fact that the proof relies on the stability theorem due to Simonovits [1, Theorem 4.2, Sect. 6].

To prove the third part of Theorem 5 we will need the following theorem, which is derived easily from Theorems 1 and 2:

Theorem 6 *Two d-cliques in a diameter graph G in $\mathbb{R}^d$ cannot share exactly $d-3$ vertices. In particular, if any two d-cliques in G share at least $d-3$ vertices, then the number of d-cliques in G is at most the number of vertices of G.*

Proof Consider two d-cliques K_1, K_2 in G that share $d-3$ vertices $v_1, \ldots, v_{d-3}$. The vertices $w_1, w_2, w_3 \in K_1$ and $u_1, u_2, u_3 \in K_2$ that are different from $v_1, \ldots, v_{d-3}$ lie on a 3-dimensional sphere S_r^3 of radius

$$r = \sqrt{1 - \frac{d-4}{2d-6}} > 1/\sqrt{2}.$$

Thus, we can apply part 3 of Theorem 2 to the points of G that lie on S^3_r and obtain that any two triangles on S^3_r must share a vertex. So, some of the vertices of the triangles $u_1u_2u_3$, $w_1w_2w_3$ must coincide. We obtain that K_1, K_2 must share at least $d-2$ vertices. To finish the proof we apply Theorem 1. □

In Sects. 3.1, 3.2, 3.3 we prove the first, the second and the third part of Theorem 5 respectively.

3.1 Number of Edges

The configuration that gives the lower bound in Theorem 4 is called a *Lenz configuration* (see [20]). Consider two circles C_1 and C_2 with a common center of radius r_1 and r_2, respectively. Suppose that the circles lie in two orthogonal planes and that $r_1^2 + r_2^2 = 1$. A finite set S is a *Lenz configuration*, if $S \subset C_1 \cup C_2$ for some circles C_1, C_2 that satisfy the above described conditions.

Note that if a diameter graph in $\mathbb{R}^4$ contains a complete bipartite graph with at least three vertices in each part as a spanning subgraph, then its vertices form a Lenz configuration.

Thus, we need to prove only the upper bound. As in [20], we prove that, indeed, the maximum number of edges is attained only on the Lenz configurations.

We will need the lemma which is a version of the famous Kővári–Sós–Turán theorem [14]:

Lemma 7 *Let $s, n \in \mathbb{N}, 0 < c < 1/2$. If $G = (V, E)$ is a graph on n vertices, $e = |E| \geq cn^2$, and if $2cn(2cn-1)(2cn-2) > (s-1)(n-1)(n-2)$, then G contains a copy of $K_{s,3}$ as a subgraph.*

Proof Suppose $V = \{v_1, \dots, v_n\}$ and d_i is the degree of v_i. If

$$\sum_{i=1}^{n} \binom{d_i}{3} > (s-1)\binom{n}{3}, \tag{1}$$

then, by the pigeonhole principle, some s vertices from V have three common neighbors. These s vertices together with their three common neighbors form a copy of $K_{s,3}$. Applying Jensen's inequality, one can check that the left-hand side is minimized when all d_i are equal, so (1) follows from the inequality:

$$n\frac{2e}{n}\left(\frac{2e}{n}-1\right)\left(\frac{2e}{n}-2\right) > (s-1)n(n-1)(n-2) \iff$$
$$2cn(2cn-1)(2cn-2) > (s-1)(n-1)(n-2).$$

□

From Lemma 7 we obtain the following corollary:

Corollary 1 *If $G = (V, E)$ is a graph on $n \geq 52$ vertices, $e = |E| \geq n^2/4$, then G contains a copy of $K_{7,3}$ as a subgraph.*

Let G be a diameter graph in $\mathbb{R}^4$ on n vertices with $D_4(2,n) \geq F_2(n)$ edges. Since $n \geq 52$, from Corollary 1 we obtain that G contains a copy of $K_{7,3}$. Suppose the set $V_1 \subset V$ is a maximal subset such that $G[V_1]$ contains $K_{l,m}, l \geq 7, m \geq 3$, as a spanning subgraph.

The number of edges between V_1 and $V \backslash V_1$ is at most $4(|V| - |V_1|)$. Indeed, if some vertex v from $V \backslash V_1$ is connected to five vertices in V_1, then it is connected to at least three vertices from one part of $K_{l,m}$ and it must be cocircular with the vertices of the other part. Thus, we can add v to the bipartite graph and obtain a contradiction with the maximality of V_1.

Denote $x = |V_1| \geq 10$. We obtain the following inequality on $D_4(2,n)$:

$$\begin{aligned} D_4(2,n) &\leq F_2(x) + 4(n-x) + |E(G[V \backslash V_1])| \\ &\leq F_2(x) + 4(n-x) + (n-x)^2/4 + (n-x), \end{aligned}$$

where the last inequality follows from the fact that $D_4(2,n) \leq U_4(2,n) \leq n^2/4 + n$. We use that $n^2/4 + n/2 \leq F_2(n) \leq n^2/4 + n/2 + 5/4$:

$$\begin{aligned} D_4(2,n) &\leq x^2/4 + x/2 + 5/4 + 5(n-x) + (n-x)^2/4 \\ &= n^2/4 + n/2 - x(n-x)/2 + 5/4 + 9(n-x)/2 \\ &\leq F_2(n) - (x-9)(n-x)/2 + 5/4. \end{aligned}$$

Thus, if $n - x \geq 3$, then by the inequality above the graph G cannot have the maximum number of edges. If $n - x = 1$ or 2, then we can use the improved bound $|E(G[V \backslash V_1])| \leq (n-x)^2/4 + (n-x) - 5/4$ and obtain that G cannot have the maximum number of edges in this case either. Thus, $n - x = 0$ and the vertices of the graph G form a Lenz configuration. The first part of Theorem 5 is proved.

3.2 Number of Triangles

First, we show that there is a Lenz configuration on n vertices with $F_3(n)$ triangles and that $F_3(n)$ is indeed the maximum number of triangles among n-vertex Lenz configurations. The following lemma was stated in [20]:

Lemma 8 *Let S be an n-vertex subset of a circle, $G = (S, E)$ is the diameter graph of S.*

1. *If the radius of the circle $> 1/\sqrt{3}$, then we have $|E| \leq 1$.*
2. $|E| \leq \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$

Consider a Lenz configuration V, $|V| = n \geq 5$, that lies on two orthogonal circles C_1 and C_2, where C_2 has radius $\geq 1/\sqrt{2}$. Put $V_1 = V \cap C_1, |V_1| = a$, $V_2 = V \cap C_2, |V_2| = n - a$. The number of diameters in V_2 is at most one, while the number of diameters in V_1 is at most $2\lfloor (a-1)/2 \rfloor + 1$. Thus, the number of triangles in $G(V)$ is at most

$$a + (n-a)\big(2\lfloor (a-1)/2 \rfloor + 1\big) = n + 2(n-a)\lfloor (a-1)/2 \rfloor,$$

and for each $n - 2 \geq a \geq 2, n \geq 5$ there is a Lenz configuration with that exact number of triangles. It is not difficult to show that the maximum over a of the number of triangles is exactly $F_3(n)$.

Next, we prove the following auxiliary statement concerning the number of triangles in a diameter graph:

Lemma 9 *Any diameter graph $G = (V, E)$ in $\mathbb{R}^4$ on n vertices has at most $4|E|/3 - 2n/3$ triangles. In particular, this quantity is at most $n^2/3 + 2n/3$.*

Proof Suppose $V = \{v_1, \ldots, v_n\}$, and v_i has degree d_i. All neighbors of v_i lie on a three-dimensional unit sphere, thus, by Theorem 2, there are at most $2d_i - 2$ edges among the neighbors of v_i. So the vertex v_i is contained in at most $2d_i - 2$ triangles. This gives the first bound on the number of triangles $t(G)$ in

$$G : t(G) \leq \sum_{i=1}^{n} \frac{2d_i - 2}{3} = 4|E|/3 - 2n/3.$$

As for the second bound, we know that $|E| \leq n^2/4 + n$ for all n. One only has to combine these two bounds. □

Now, we go on to the proof of the second part of Theorem 5. Consider a graph $G = (V, E)$ with at least $F_3(n)$ triangles. We will show that, if n is sufficiently large, then G has exactly $F_3(n)$ triangles, and V forms a Lenz configuration. By Lemma 9, $|E| \geq 3n^2/16$. Choose n large enough, so that

$$(\sqrt{n}-8/3)(\sqrt{n}-16/3) > \frac{2^4}{3^3}(\sqrt{n}-1)(\sqrt{n} - 2). \tag{2}$$

This choice will be explained later. We apply Lemma 7 to the graph G with $s = n/32$. Simple calculations show that, since $|E| \geq 3n^2/16$, the conditions of Lemma 7 are satisfied. Thus, G contains a subgraph $K_{s,3}$ on a set V'. Next, as in the proof of the previous part of the theorem, we choose a maximal set $V_1 \supset V'$, such that V_1 contains a copy of $K_{s_1,t_1}, s_1 \geq s, t_1 \geq 3$, as a spanning subgraph. We run the following inductive procedure. Denote by $V(i)$ the set of available vertices at the moment i. At the initial moment the set of available vertices is equal to V. The procedure at the step i is as follows:

1. We choose $V_i \subset V(i - 1)$ to be a maximal set in $V(i - 1)$ that contains a copy of $K_{s_1,t_1}, s_1 \geq s, t_1 \geq 3$, as a spanning subgraph. We require that $|V_i| \geq |V(i - 1)|/32$.
2. We set $V(i) = V(i - 1)\backslash V_i$.
3. If $|V(i)| \leq \sqrt{n}$, we stop, otherwise we go on to the step $i + 1$.

Note that at each step we have $E(G[V(i)]) \geq 3|V(i)|^2/16$. This can be checked similar to the end of the proof of the first part of Theorem 5. We again rely on the fact that each vertex in $V(i - 1)\backslash V_i$ has at most 4 neighbors in V_i.

We need to prove that it is always possible to execute step 1. For that we need to verify that we can apply Lemma 7 with $c = 3/16$. The inequality from Lemma 7 we need to check looks almost exactly like inequality (2), but with $|V(i)|$ instead of $\sqrt{n}$. If we are to apply the step 1, then by step 3 we have $|V(i)| > \sqrt{n}$, and the inequality (2) with $|V(i)|$ instead of $\sqrt{n}$ also holds.

It is easy to see that procedure terminates in $k \le 20 \ln n$ steps, since $|V(i)| \le (1 - 1/32)^i n = e^{\ln n - i \ln(32/31)}$. For convenience put $V_{k+1} = V(k)$.

Now, we can estimate the number $t(G)$ of triangles in G. Denote by e_i (t_i) the number of edges (triangles) in V_i. We obtain the following estimate:

$$t(G) \le \sum_{i=1}^{k+1} t_i + \binom{k}{2}(8(2n-2)+6n) + 4ke_{k+1} + (4k)^2 n = \sum_{i=1}^{k} t_i + O(n \ln^2 n). \quad (3)$$

Let us explain the inequality. The first sum counts triangles that lie entirely in one of the parts of the vertex set partition.

The second summand bounds from above the number of triangles that have one vertex in some V_i and and two vertices in some V_j, $k \ge j > i$. First, we choose i and j. Next, the vertices of V_i, V_j lie on two pairs of circles. There are at most 8 vertices of V_i that lie on the circles that contain V_j, since otherwise we could find three vertices from V_i that lie on the same circle in V_i and that fall onto the same circle of V_j. Consequently, these two circles would coincide, and V_i and V_j would have to lie on the same pair of circles. This contradicts the maximality of V_i. The number of triangles with these 8 vertices is at most $8(2n-2)$. All vertices that do not lie on the circles that contain V_j have at most four neighbors in V_j, thus, each is contained in at most 6 triangles. We bound the number of such triangles by $6n$.

The third term counts the number of triangles that have exactly two vertices in V_{k+1}. We bound their number from above as follows. First, we choose an edge in V_{k+1}, and then for one of its endpoints we choose a neighbor from some V_i (there are at most $4k$ possibilities for this choice).

The fourth summand bounds from above the number of triangles that we did not count in the first three summands. For each triangle of this type, there is a part V_i of the partition that contains exactly one vertex v of the triangle, and two other vertices lie in the parts $V_j, V_l,\ j, l < i$. There are n choices for the vertex v. Next, there are less than k^2 choices to choose two parts of the partition in which two other vertices of the triangle lie. Finally, for each j, each vertex from $V_i, i > j$, is connected to at most four vertices from V_j.

The equality in (3) is due to the following. First, $k = O(\ln n)$. Second, $|V(k)| \le \sqrt{n}$, thus $e_{k+1}, t_{k+1} = O(n)$ by Lemma 9.

Suppose $|V_1| \le n - n^{0.2}$. One can verify that for given $a, b > 0$ we have $F_3(a+b) \ge F_3(a) + F_3(b)$ for $a, b \in \mathbb{N}$. Besides, if $a, b \in \mathbb{N}, a > 2b$ and $a + b$ is sufficiently large, then $F_3(a+1) + F_3(b-1) \ge F_3(a) + F_3(b)$. Therefore, we have the following bound:

$$\sum_{i=1}^{k+1} t_i \le F_3(n - n^{0.2}) + F_3(n^{0.2}) \le n + n^2/4 - n^{0.2}(n - n^{0.2})/2 = F_3(n) - \Omega(n^{1.2}).$$

It follows that if $|V_1| \le n - n^{0.2}$, then for sufficiently large n we have $t(G) < F_3(n)$. Consider the case when $|V_1| > n - n^{0.2}$. Recall that $V(1) = V \setminus V_1$. We have

$|V(1)| < n^{0.2}$, and for a given vertex v in $V(1)$ the degree of v in $G[V(1)]$ is at most $n^{0.2}$. The vertex v is connected to at most four vertices from V_1. Thus $\deg v = O(n^{0.2})$, and, following the considerations in Lemma 9, we can easily show that the number of triangles that contain v is $O(n^{0.2})$. On the other hand, if we remove the vertex v from the graph and add a vertex to a Lenz configuration formed by V_1, then from the behavior of the function $F_3(n)$ we can see that the number of triangles formed by the points of V_1 will increase by $\Omega(n)$, and the total number of triangles in G will surely increase, if n is large enough.

Thus "moving" all vertices from $V(1)$ to V_1 will increase the total number of triangles. At the end, we obtain that the vertices of G form a Lenz configuration, which concludes the proof of this part of the theorem.

3.3 Schur's Conjecture in $\mathbb{R}^4$

Consider a diameter graph $G = (V, E)$.

By Theorem 6 any two 4-cliques in G either have at least two common vertices, or do not have any. We show that V can be decomposed into disjoint sets of vertices $V_1, \ldots, V_k$ with the following properties. First, any 4-clique lies entirely in one of the sets $V_1, \ldots, V_k$. Second, inside any of V_i any pair of 4-cliques intersect in at least two vertices. In other words, we want to split the set of all 4-cliques into equivalence classes, in which we consider cliques equivalent if they intersect. Next, we put V_i to be the union of all vertices of the cliques from the i-th equivalence class.

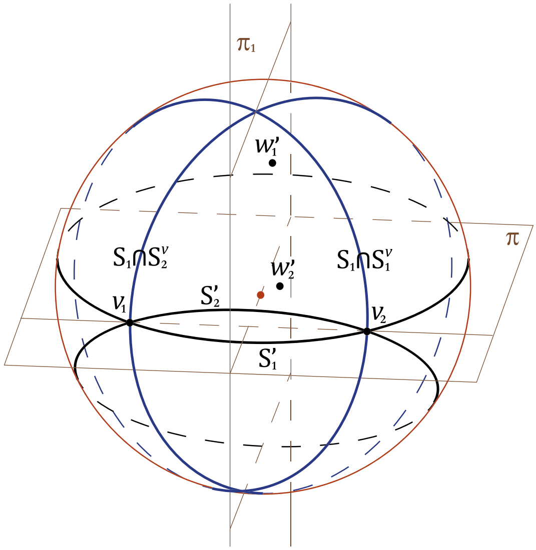
To prove that such a partition exists, we need to show that this is indeed an equivalence relation. All we need to check is transitivity, i.e., that there is no such triple of 4-cliques K^1, K^2, andK^3, such that $|K^1 \cap K^2| = |K^1 \cap K^3| = 2$, $|K^2 \cap K^3| = 0$. Note that if the cardinality of the intersection of K^1 with one of the rest is greater than 2, then, by the pigeonhole principle, the other two also have to intersect.

Denote by v_1, v_2, v_3, v_4 the vertices of K^1, where $v_1, v_2 \in K^2$, $v_3, v_4 \in K^3$. The other vertices are $w_1, w_2 \in K^2$, $w_3, w_4 \in K^3$. The hyperplane that passes through v_1, v_2, v_3, v_4 we denote by π. The points v_1, v_2, w_3, w_4 lie on a two-dimensional sphere S_1 of radius $\sqrt{3}/2$. Its center is the middle of the segment that connects v_3, v_4, while the sphere itself lies in the hyperplane γ that is orthogonal to the segment. Analogously, the points v_3, v_4, w_1, w_2 lie on a two-dimensional sphere S_2 of radius $\sqrt{3}/2$, whose center is the midpoint of the segment v_1v_2.

According to Lemma 3, the arcs $\widehat{v_3v_4}$ and $\widehat{w_1w_2}$ (as well as $\widehat{v_1v_2}$ and $\widehat{w_3w_4}$) intersect, which implies that w_1 and w_2 (as well as w_3 and w_4) lie in different closed halfspaces bounded by π. Indeed, $\pi \cap S_1$ is the great circle that passes through v_1, v_2, and w_3, w_4 have to be on the opposite sides of this great circle. Moreover, it is easy to derive from the proof of Lemma 3 that none of the w_i lie in the plane π. Otherwise it would be either an interior point of the arc v_1v_2 (or v_3v_4), or it would coincide with one of the v_j. In the first case, based on Lemma 3, we would obtain a contradiction with the fact that $\|v_i - w_j\| \le 1$, while $\|w_1 - w_2\| = \|w_3 - w_4\| = 1$. In the second case the intersection of some two of the cliques K_i would be greater than 2.

Denote by π^+, π^- two open halfspaces bounded by π. W.l.o.g., $w_1, w_3 \in \pi^+$, $w_2, w_4 \in \pi^-$. Consider three-dimensional spheres S_1^w, S_2^w of unit radius with centers

Fig. 1 3-dimensional configuration of spheres for the proof of part 3 of Theorem 5



in w_1, w_2. They intersect with S_1 in the points v_1, v_2, and none of the two spheres S_i^w contain S_1. Otherwise the distance from w_1 (or w_2) to any point of S_1 would be the same, and, by the law of cosines, the vector that connects the center o of S_1 with w_1 (w_2) would be orthogonal to γ, which is not true. Indeed, since w_1, w_2 do not lie in π but o lies in π, both $\overline{ow_1}$ and $\overline{ow_2}$ have a non-zero component that is orthogonal to π. On the other hand, since $v_3, v_4 \in \pi$, the vector $\bar{u}$ that is orthogonal to π is also orthogonal to v_3v_4 and, consequently, lies in the hyperplane γ. As we already established, the scalar product of $\overline{ow_1}$ ($\overline{ow_2}$) and $\bar{u}$ is non-zero, which means that $\overline{ow_1}$ and $\overline{ow_2}$ are not orthogonal to γ. Therefore, the intersections of S_1 and S_i^w are *circles* S_i' on S_1 which pass through the points v_1, v_2 (see Fig. 1).

Our goal is to show that there is no room for the points w_3, w_4 such that all the conditions based on the fact that G is a diameter graph are satisfied. The points w_3, w_4 lie on the sphere S_1. At the same time $w_3, w_4 \in B_1^w \cap B_2^w \cap B_1^v \cap B_2^v$, where B_i^w are unit balls with centers at w_i, while B_i^v are unit balls of unit radius with centers at v_i. By S_i^v we denote the boundary sphere of B_v^i. We prove that the intersection $S_1 \cap B_1^w \cap B_2^w \cap B_1^v \cap B_2^v$ cannot contain a pair of points at unit distance apart except for v_1, v_2.

Henceforth, all the considerations are limited to the hyperplane γ, and, not willing to introduce excessive notations, we modify all the notations of balls, spheres, hyperplanes, and halfspaces so that the notations now correspond to these objects intersected with γ (instead of the objects in $\mathbb{R}^4$). In particular, we will denote by S_1^w, S_2^w two-dimensional spheres (with centers in w_1', w_2'), which are the intersections of the three-dimensional spheres S_1^w, S_2^w with γ; by π we denote the plane $\pi \cap \gamma$. Note that w_1' lies in π^+ and w_2' lies in π^-.

Let π_1 be the two-dimensional plane which is orthogonal to the segment v_1v_2 and passes through the midpoint of the segment. The center of S_1 and the points w_1', w_2' all lie in π_1. We denote by π_1^+ the open halfspace containing v_1, and by π_1^- the open halfspace containing v_2. Let π_2 be the two-dimensional plane which is orthogonal to both π and π_1 and passes through the center of S_1. It is not difficult to see that the set $S_1 \cap B_1^v \cap B_2^v$ lies entirely in the open halfspace π_2^+ that is bounded by π_2 and contains v_1, v_2. For $u' \in S_1$ denote by $H_{u'}$ an open hemisphere with center in u'. We intend to show that $S_1 \cap B_1^v \cap B_2^v \subset H_u = S_1 \cap \pi_2^+$, where u is the midpoint of the arc $\widehat{v_1 v_2}$. Since the radius of S_1 is greater than $1/\sqrt{2}$, we have $S_1 \cap B_i^v \subset H_{v_i}$ for $i = 1, 2$. On the other hand, since $u \in \widehat{v_1 v_2}$, we surely have $H_{v_1} \cap H_{v_2} \subset H_u$. Therefore, we have the following chain of inclusions: $S_1 \cap B_1^v \cap B_2^v \subset H_{v_1} \cap H_{v_2} \subset H_u = S_1 \cap \pi_2^+$.

Next, we prove that in the halfspace π_1^- the circles S_1' and $S_1 \cap S_1^v$ intersect only in v_2. Surely, there are at most two intersection points in total. Due to the fact that S_1' lies on S_1, the intersection of these two circles coincides with the intersection of the sphere S_1^v and the circle S_1'. Further, since the center of S_1^v lies on the circle S_1', the intersection points of these two spheres should be symmetric in the plane that contains S_1' with respect to the line that contains their centers. But since one center lies in π_1^+ and the other lies in π_1, one of the intersection points must lie in π_1^+, and v_2 is indeed their only intersection point in π_1^-. An analogous fact holds for the circles S_2' and $S_1 \cap S_1^v$, and also in the symmetric halfspace π_1^+ for the circles $S_1 \cap S_2^v$ and S_i' and a point v_1.

Recall that $w_1' \in \pi^+, w_2' \in \pi^-$. The set $S_1 \cap B_1^w$ is situated above the plane containing the circle S_1' (in the direction of the normal vector to the plane π that points to π^+). Analogously, the set $S_1 \cap B_2^w$ is situated below the plane containing the circle S_2'. We show this for $S_1 \cap B_1^w$. For this consider a reflection $\mathcal{R} : S_1 \to S_1$ with respect to the plane π. Then for any point $u \in S_1 \cap \pi^+$ we have $\|w_1' - u\| < \|w_1' - \mathcal{R}(u)\|$, because $w_1' \in \pi^+$. The circle S_1' bounds the set $S_1 \cap B_1^w$, and the point on S_1 that is above the center of $S_1 \cap B_1^w$ is closer to w_1' than the point that is below the center. Note that the planes of the circles S_i' cannot be orthogonal to π, since otherwise the point w_i' would lie in the plane π.

The circles S_1', S_2' split the sphere S_1 into four parts, and one of them is the set $S_1 \cap B_1^w \cap B_2^w$. From the above considerations, we get that depending on the positions of the points w_1', w_2', the set $S_1 \cap B_1^w \cap B_2^w$ has two possible locations out of four. The reason is that it is impossible, the set $S_1 \cap B_1^w \cap B_2^w$ is bounded by a shorter arc $v_1 v_2$ of S_1' and a greater arc $v_1 v_2$ of S_2' (or vice versa), because in this case $v_1 v_2$ of S_1' is either below or above both circles S_1', S_2'. To prove this, we first note that from a parity argument follows that if move along the sphere S_1 and cross one of the circles (not in v_1, v_2), then, if we were in an admissible region, we arrive to a not admissible region, and vice versa. Thus, it suffices to show that the region between two shorter arcs is admissible. We already know that $v_1, v_2 \in \pi_2^+$. Consider the plane π', which is parallel to π_2 and passes through v_1, v_2. Any circle on S_1 that contains v_1, v_2 must have its shorter arc $v_1 v_2$ in the halfspace with respect to π' in which the point u lies, which shows that the region between the two shorter arcs $v_1 v_2$ of S_1', S_2' is above one of the two circles and below the other. We are left with the following two cases.

Case 1: The set $S_1 \cap B_1^w \cap B_2^w$ on S_1 is bounded by the greater arcs of the circles S_i' with the endpoints v_1, v_2. We specify the set $S_1 \cap B_1^v \cap B_2^v$ in the following way:

$$S_1 \cap B_1^v \cap B_2^v = (S_1 \cap B_1^v \cap \pi_1^-) \cup (S_1 \cap B_2^v \cap \pi_1^+).$$

Several paragraphs before we proved that the sets $S_1 \cap B_1^w \cap B_2^w$ and $S_1 \cap B_2^v \cap \pi_1^+$ intersect only in the vertex v_1, while the sets $S_1 \cap B_1^w \cap B_2^w$ and $S_1 \cap B_1^v \cap \pi_1^-$ intersect only in the vertex v_2. Thus, we obtain that

$$\left(S_1 \cap B_1^v \cap B_2^v\right) \cap \left(S_1 \cap B_1^w \cap B_2^w\right) = \{v_1, v_2\},$$

and there is no room for w_3, w_4 at all.

Case 2: The set $S_1 \cap B_1^w \cap B_2^w$ on the sphere is bounded by the shorter arcs of the circles S_i' with the endpoints in v_1, v_2. In that case the set $S_1 \cap B_1^w \cap B_2^w$ lies entirely in the spherical cap H, which is cut off by the plane π', which is parallel to π_2 and passes through v_1, v_2. Moreover, only the points v_1, v_2 lie in the intersection of $S_1 \cap B_1^w \cap B_2^w$ and π'. Indeed, the set $S_1 \cap B_1^w$ does not intersect with $\pi' \cap S_1 \cap \pi^-$ due to the description of the position of the set $S_1 \cap B_1^w$ given before the case 1 (recall that the halfspace π^+, π^- are open). Analogously, $S_1 \cap B_2^w$ does not intersect with $\pi' \cap S_1 \cap \pi^+$.

On the other hand, the shorter arcs of the circles S_i' must lie inside H.

The circle $S_1 \cap \pi'$ has diameter 1, and the points that lie on the sphere S_1 in the interior of H, cannot be at unit distance apart. Thus, the distance between a pair of points in $S_1 \cap B_1^w \cap B_2^w$ cannot be equal to one, if these points do not coincide with v_1, v_2. It means that inside $S_1 \cap B_1^w \cap B_2^w$ there is no room for the points w_3, w_4.

We proved that the above described partition of the vertex set V into sets $V_1, \dots V_k$ indeed exists. We apply Theorem 1 to each V_i and obtain that the number of 4-cliques on each set V_i does not exceed $|V_i|$, thus the total number of cliques does not exceed $\sum_i |V_i| = n$. The proof of Schur's conjecture in $\mathbb{R}^4$ is complete.

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