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Frankl–Füredi–Kalai Inequalities on the γ-Vectors of Flag Nestohedra

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Abstract For any flag nestohedron, we define a flag simplicial complex whose f-vector is the γ-vector of the nestohedron. This proves that the γ-vector of any flag nestohedron satisfies the Frankl–Füredi–Kalai inequalities, partially solving a conjecture by Nevo and Petersen (Discrete Comput. Geom. 45:503–521, 2010). We also compare these complexes to those defined by Nevo and Petersen (Discrete Comput. Geom. 45:503–521, 2010) for particular flag nestohedra.

Keywords Building set · Flag · f-Vector · Gamma-vector · Homology sphere · h-Vector · Nestohedron · Simplicial complex

1 Introduction

For any building set $\mathcal{B}$ there is an associated simple polytope $P_{\mathcal{B}}$ called the *nestohedron* (see Sect. 2, [10, Sect. 7] and [11, Sect. 6]). When $\mathcal{B} = \mathcal{B}(G)$ is the building set determined by a graph G, $P_{\mathcal{B}(G)}$ is the well-known graph-associahedron of G (see [1, Ex. 2.1], [11, Sects. 7 and 12], and [12]). The numbers of faces of $P_{\mathcal{B}}$ of each dimension are conveniently encapsulated in its γ-polynomial $\gamma(\mathcal{B}) = \gamma(P_{\mathcal{B}})$ defined below.

Recall that for a $(d-1)$-dimensional simplicial complex Δ, the f-polynomial is a polynomial in $\mathbb{Z}[t]$ defined as follows:

$$f(\Delta)(t) := f_0 + f_1 t + \cdots + f_d t^d,$$

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where $f_i = f_i(\Delta)$ is the number of $(i-1)$-dimensional faces of Δ, and $f_0(\Delta) = 1$. The h-polynomial is given by

$$h(\Delta)(t) := (t-1)^d f(\Delta)\left(\frac{1}{t-1}\right) = h_0 + h_1 t + \cdots + h_d t^d,$$

where $h_i = h_i(\Delta)$. When Δ is a homology sphere, $h(\Delta)$ is symmetric, i.e. $h_i(\Delta) = h_{d-i}(\Delta)$ for all i (this is known as the Dehn–Sommerville relations); hence it can be written

$$h(\Delta)(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i},$$

for some $\gamma_i \in \mathbb{Z}$. Then the γ-polynomial is given by

$$\gamma(\Delta)(t) := \gamma_0 + \gamma_1 t + \cdots + \gamma_{\lfloor \frac{d}{2} \rfloor} t^{\lfloor \frac{d}{2} \rfloor},$$

where $\gamma_i = \gamma_i(\Delta)$. The vectors of coefficients of the f-polynomial, h-polynomial and γ-polynomial are known respectively as the f-vector, h-vector and γ-vector. If P is a simple $(d+1)$-dimensional polytope then the dual simplicial complex Δ_P of P is the boundary complex (of dimension d) of the polytope that is polar dual to P. The f-vector, h-vector and γ-vector of P are defined via Δ_P as

$$f(P)(t) := t^d f(\Delta_P)\left(t^{-1}\right),$$

so that $f_i(P)$ is the number of i-dimensional faces of P, and

$$h(P)(t) := h(\Delta_P)(t),$$
$$\gamma(P)(t) := \gamma(\Delta_P)(t).$$

When $\mathcal{B}$ is a building set, we denote the γ-polynomial for $P_{\mathcal{B}}$ by $\gamma(\mathcal{B})$.

Recall that a simplicial complex Δ is *flag* if every set of pairwise adjacent vertices is a face. Gal [7] conjectured that:

Conjecture 1.1 [7, Conjecture 2.1.7] *If Δ is a flag homology sphere then $\gamma(\Delta)$ is nonnegative.*

This implies that the γ-vector of any flag polytope has nonnegative entries. Gal's conjecture was proven for flag nestohedra by Volodin in [12, Theorem 9].

In [6] Frankl, Füredi and Kalai characterize the f-vectors of balanced simplicial complexes, and their defining conditions are known as the Frankl–Füredi–Kalai inequalities. Frohmader [5] showed that the f-vector of any flag simplicial complex is the f-vector of a balanced complex. Nevo and Petersen conjectured the following strengthening of Gal's conjecture:

Conjecture 1.2 [8, Conjecture 6.3] *If Δ is a flag homology sphere then $\gamma(\Delta)$ satisfies the Frankl–Füredi–Kalai inequalities.*

They proved this in [8] for the following classes of flag spheres:

- Δ is a Coxeter complex (including the simplicial complex dual to $P_{\mathcal{B}(K_n)}$),
- Δ is the simplicial complex dual to an associahedron ($=P_{\mathcal{B}(\mathrm{Path}_n)}$),
- Δ is the simplicial complex dual to a cyclohedron ($=P_{\mathcal{B}(\mathrm{Cyc}_n)}$),
- Δ has $\gamma_1(\Delta) \le 3$,

by showing that the γ-vector of such Δ is the f-vector of a flag simplicial complex. In [9], Conjecture 1.2 is proven for the barycentric subdivision of a simplicial sphere, by showing that the γ-vector is the f-vector of a balanced simplicial complex.

In this paper we prove Conjecture 1.2 for all flag nestohedra:

Theorem 1.3 *If $P_{\mathcal{B}}$ is a flag nestohedron, there is a flag simplicial complex $\Gamma(\mathcal{B})$ such that $f(\Gamma(\mathcal{B})) = \gamma(P_{\mathcal{B}})$. In particular, $\gamma(P_{\mathcal{B}})$ satisfies the Frankl–Füredi–Kalai inequalities.*

Our construction for $\Gamma(\mathcal{B})$ depends on the choice of a "flag ordering" for $\mathcal{B}$ (see Sect. 3). In the special cases considered by Nevo and Petersen [8] our $\Gamma(\mathcal{B})$ does not always coincide with the complex they construct.

After completing this paper, the author proved Conjecture 1.2 in the more general context of edge subdivisions in [2]. This result was also proven independently by Volodin in [13] and [14], who had previously shown in [12] that flag nestohedra are a special case of polytopes obtainable from the cube by 2-truncations (see Theorems 2.5 and 2.6). The author and Volodin are currently working on amalgamating the two results. The result in [2] is shown to be equivalent to the result in this paper for flag nestohedra, where a flag ordering in this context corresponds to a subdivision sequence in [2].

Here is a summary of the contents of this paper. Section 2 contains preliminary definitions and results relating to building sets and nestohedra. In Sect. 3 we define the flag simplicial complex $\Gamma(\mathcal{B})$ for a building set $\mathcal{B}$ and prove Theorem 1.3. In Sect. 4 we compare the simplicial complexes $\Gamma(\mathcal{B})$ to the flag simplicial complexes defined in [8].

2 Preliminaries

A *building set* $\mathcal{B}$ on a finite set S is a set of nonempty subsets of S such that:

- For any $I, J \in \mathcal{B}$ such that $I \cap J \neq \emptyset$, $I \cup J \in \mathcal{B}$.
- $\mathcal{B}$ contains the singletons $\{i\}$, for all $i \in S$.

$\mathcal{B}$ is *connected* if it contains S. For any building set $\mathcal{B}$, $\mathcal{B}_{\max}$ denotes the set of maximal elements of $\mathcal{B}$ with respect to inclusion. The elements of $\mathcal{B}_{\max}$ form a disjoint union of S, and if $\mathcal{B}$ is connected then $\mathcal{B}_{\max} = \{S\}$. Building sets $\mathcal{B}_1$, $\mathcal{B}_2$ on S are *equivalent*, denoted $\mathcal{B}_1 \cong \mathcal{B}_2$, if there is a permutation $\sigma : S \to S$ that induces a one to one correspondence $\mathcal{B}_1 \to \mathcal{B}_2$.

Example 2.1 Let G be a graph with no loops or multiple edges, with n vertices labelled distinctly from $[n]$. Then the graphical building set $\mathcal{B}(G)$ is the set of subsets of $[n]$ such that the induced subgraph of G is connected (see [3, 4], [11, Sects. 7 and 12] and [12]). $\mathcal{B}(G)_{\max}$ is the set of connected components of G.

Let $\mathcal{B}$ be a building set on S and $I \subseteq S$. The *restriction of $\mathcal{B}$ to I* is the building set

$$\mathcal{B}|_I := \{J \mid J \subseteq I, \text{and } J \in \mathcal{B}\} \quad \text{on } I.$$

The *contraction of $\mathcal{B}$ by I* is the building set

$$\mathcal{B}/I := \{J - (J \cap I) \mid J \in \mathcal{B}, J \nsubseteq I\} \quad \text{on } S - I.$$

We associate a polytope to a building set as follows. Let $e_1, \dots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. Given $I \subseteq [n]$, define the simplex $\Delta_I := \mathit{ConvexHull}(e_i \mid i \in I)$. Let $\mathcal{B}$ be a building set on $[n]$. The *nestohedron* $P_{\mathcal{B}}$ is a polytope defined in [10] and [11] as the Minkowski sum,

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} \Delta_I.$$

A $(d-1)$-dimensional face of a d-dimensional polytope is called a *facet*. A simple polytope P is *flag* if any collection of pairwise intersecting facets has nonempty intersection, i.e. its dual simplicial complex is flag. We use the abbreviation *flag complex* in place of flag simplicial complex. A building set $\mathcal{B}$ is *flag* if $P_{\mathcal{B}}$ is flag.

A *minimal flag building set* $\mathcal{D}$ on a set S is a connected building set on S that is flag, such that no proper subset of its elements forms a connected flag building set on S. Minimal flag building sets are described in detail in [11, Sect. 7.2]. They correspond to plane binary trees with leaf set S. Given such a tree, the leaves are labelled 1 to n, and the corresponding minimal flag building set is the union of the set of leaf descendants of each vertex of the tree. If $\mathcal{D}$ is a minimal flag building set then $\gamma(\mathcal{D}) = 1$ (see [11, Sect. 7.2]).

Let $\mathcal{B}$ be a building set. A *binary decomposition* or *decomposition* of a non-singleton element $B \in \mathcal{B}$ is a set $\mathcal{D} \subseteq \mathcal{B}$ that forms a minimal flag building set on B. Suppose that $B \in \mathcal{B}$ has a binary decomposition $\mathcal{D}$. The two maximal elements $D_1, D_2 \in \mathcal{D} - \{B\}$ with respect to inclusion are the *maximal components* of B in $\mathcal{D}$. Propositions 2.2 and 2.3 give alternative characterizations of when a building set is flag.

Proposition 2.2 [1, Lemma 7.2] *A building set $\mathcal{B}$ is flag if and only if every non-singleton $B \in \mathcal{B}$ has a binary decomposition.*

Proposition 2.3 [1, Corollary 2.6] *A building set $\mathcal{B}$ is flag if and only if for every non-singleton $B \in \mathcal{B}$, there exist two elements $D_1, D_2 \in \mathcal{B}$ such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = B$.*

It follows from Proposition 2.3 that a graphical building set is flag.

Lemma 2.4 [1, Lemma 2.7] *Suppose $\mathcal{B}$ is a flag building set. If $A, B \in \mathcal{B}$ and $A \subsetneq B$, then there is a decomposition of B in $\mathcal{B}$ that contains A.*

Recall the following theorems:

Theorem 2.5 [12, Lemma 6] *Let $\mathcal{B}$ and $\mathcal{B}'$ be connected flag building sets on S such that $\mathcal{B} \subseteq \mathcal{B}'$. Then $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by successively adding elements so that at each step the set is a flag building set.*

Theorem 2.6 [7, Proposition 2.4.3], [12, Proposition 3] *If $\mathcal{B}'$ is a flag building set on S obtained from a flag building set $\mathcal{B}$ on S by adding an element I, then*

$$\begin{aligned}\gamma\big(\mathcal{B}'\big) &= \gamma(\mathcal{B}) + t\gamma\big(\mathcal{B}'|_I\big)\gamma\big(\mathcal{B}'/I\big)\\ &= \gamma(\mathcal{B}) + t\gamma(\mathcal{B}|_I)\gamma(\mathcal{B}/I).\end{aligned}$$

3 The Flag Complex $\Gamma(\mathcal{B})$ of a Flag Building Set $\mathcal{B}$

In [12], Corollary 5 (which is attributed to Erokhovets [4]) states that any nestohedron $P_{\mathcal{B}}$ is combinatorially equivalent to a nestohedron $P_{\mathcal{B}_1}$ for a connected building set $\mathcal{B}_1$. Hence to prove Theorem 1.3 we need only consider connected building sets.

Suppose that $\mathcal{B}$ is a connected flag building set on $[n]$, $\mathcal{D}$ is a decomposition of $[n]$ in $\mathcal{B}$, and $I_1, I_2, \dots, I_k$ is an ordering of $\mathcal{B} - \mathcal{D}$, such that $\mathcal{B}_j = \mathcal{D} \cup \{I_1, I_2, \dots, I_j\}$ is a flag building set for all $0 \le j \le k$ (such an ordering exists by Theorem 2.5). We call the pair consisting of such a decomposition $\mathcal{D}$ and the ordering on $\mathcal{B} - \mathcal{D}$, a *flag ordering* of $\mathcal{B}$, denoted O, or $(\mathcal{D}, I_1, \dots, I_k)$. For any $I_j \in \mathcal{B} - \mathcal{D}$, we say an element in $\mathcal{B}_{j-1}$ is *earlier* in the flag ordering than I_j, and an element in $\mathcal{B} - \mathcal{B}_j$ is *later* in the flag ordering than I_j.

For any $j \in [k]$, define:

$$U_j := \{i \mid i < j, I_i \nsubseteq I_j, \text{there is no } I \in \mathcal{B}_{i-1} \text{ such that } I \backslash I_j = I_i \backslash I_j\},$$

and

$$V_j := \{i \mid i < j, I_i \subseteq I_j, \text{there exists } I \in \mathcal{B}_{i-1} \text{ such that } I_i \subsetneq I \subsetneq I_j\}.$$

If $i \in U_j \cup V_j$ then we say that I_i is *non-degenerate* with respect to I_j. If $I_i \in \mathcal{B}_{j-1}$ and $i \notin U_j$, then I_i is *U-degenerate* with respect to I_j, and if $I_i \notin \cup V_j$ then I_i is *V-degenerate* with respect to I_j.

Given a flag building set $\mathcal{B}$ with flag ordering $O = (\mathcal{D}, I_1, \dots, I_k)$ define a graph on the vertex set

$$V_O = \big\{v(I_1), \dots, v(I_k)\big\},$$

where for any $i < j$, $v(I_i)$ is adjacent to $v(I_j)$ if and only if $i \in U_j \cup V_j$. Then define a flag simplicial complex $\Gamma(O)$ whose faces are the cliques in this graph. If the flag ordering is clear then we denote $\Gamma(O)$ by $\Gamma(\mathcal{B})$. For any $S \subseteq [k]$, we let $\Gamma(O)|_S$ denote the induced subcomplex of $\Gamma(O)$ on the vertices $v(I_i)$ for all $i \in S$.

Example 3.1 Consider the flag building set $\mathcal{B}(\text{Path}_5)$ on [5]. It has a flag ordering O given by

$$\mathcal{D} = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, [2], [3], [4], [5]\big\},$$

and

$$I_1 = \{3, 4\}, \qquad I_2 = \{2, 3, 4\}, \qquad I_3 = \{2, 3\},$$
$$I_4 = \{2, 3, 4, 5\}, \qquad I_5 = \{3, 4, 5\}, \qquad I_6 = \{4, 5\}.$$

Then $\Gamma(O)$ has only two edges, namely

$$\big\{v(I_2), v(I_6)\big\} \quad \text{and} \quad \big\{v(I_3), v(I_4)\big\}.$$

These are edges because $I_2 = \{2, 3, 4\}$ is the earliest element which has image $\{2, 3\}$ in the contraction by I_6, and the element $I_3 = \{2, 3\}$ is a subset of $I_2 = \{2, 3, 4\}$ which is in turn a subset of I_4.

Suppose that $(\mathcal{D}, I_1, \dots, I_k)$ is a flag ordering. Then $\mathcal{D}/I_k$ is a decomposition of $[n] - I_k$, and we have an induced ordering of $(\mathcal{B}/I_k) - (\mathcal{D}/I_k)$, where the ith element is $I'_{u_i} := I_{u_i} \backslash I_k$ if u_i is the ith element of U_k (listed in increasing order). Then for all i, $\mathcal{D}/I_k \cup \{I'_{u_1}, \dots, I'_{u_i}\}$ is a flag building set. Hence we can also define a flag complex $\Gamma(\mathcal{B}/I_k)$. We label the vertices of $\Gamma(\mathcal{B}/I_k)$ by $v(I'_{u_1}), v(I'_{u_2}), \dots, v(I'_{u_{|U_k|}})$. Hence, we see that U-degenerate elements with respect to I_j are the elements that do not contribute to the building set $\mathcal{B}_j/I_j$.

Claim 3.2 *Let $\mathcal{B}$ be a connected flag building set with flag ordering $(\mathcal{D}, I_1, \dots, I_k)$. For all $I \in \mathcal{B}$ let $I' = I \backslash I_k$. Suppose $j \in U_k$ and $I \in \mathcal{B}_{j-1}$. Then $I \subseteq I_j$ if and only if $I' \subseteq I'_j$.*

Proof $\Rightarrow$: It is clear that $I \subseteq I_j$ implies $I' \subseteq I'_j$.

$\Leftarrow$: Suppose for a contradiction that $I' \subseteq I'_j$ and $I \nsubseteq I_j$. Then $I \cap I_j \neq \emptyset$ and $I \cup I_j \neq I_j$, which implies that (since $\mathcal{B}_j$ is a building set) $I \cup I_j \in \mathcal{B}_{j-1}$. We also have that $(I \cup I_j)' = I'_j$, which implies that I_j is U-degenerate with respect to I_k; a contradiction. □

Proposition 3.3 *Let $\mathcal{B}$ be a connected flag building set with flag ordering given by $(\mathcal{D}, I_1, \dots, I_k)$. Then $\Gamma(\mathcal{B}/I_k) \cong \Gamma(\mathcal{B})|_{U_k}$. The map on the vertices is given by $v(I'_i) \mapsto v(I_i)$.*

Proof $\Gamma(\mathcal{B})|_{U_k}$ is a flag complex with vertex set $v(I_{u_1}), v(I_{u_2}), \dots, v(I_{u_{|U_k|}})$ and $\Gamma(\mathcal{B}/I_k)$ is a flag complex with vertex set $v(I'_{u_1}), v(I'_{u_2}), \dots, v(I'_{u_{|U_k|}})$. Suppose that $i < j$ where $i, j \in U_k$. We need to show that $\{v(I'_j), v(I'_i)\} \in \Gamma(\mathcal{B}/I_k)$ if and only if $\{v(I_j), v(I_i)\} \in \Gamma(\mathcal{B})|_{U_k}$. Note that by Claim 3.2, $I_i \subseteq I_j$ if and only if $I'_i \subseteq I'_j$.

(1) Suppose that $I_i \subseteq I_j$, and that $\{v(I'_i), v(I'_j)\} \in \Gamma(\mathcal{B}/I_k)$, so that there exists $I \in \mathcal{B}_{i-1}$ such that $I'_i \subsetneq I' \subsetneq I'_j$. By Claim 3.2, $I \subseteq I_j$ and since $I_i \subseteq I_j$ this implies

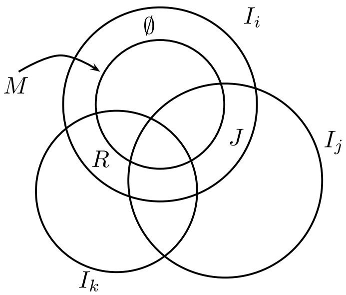

Fig. 1 A picture of the sets in case (2), assuming $M \subseteq I_i$. Note that $I_i \backslash (M \cup I_j \cup I_k) = \emptyset$ by the definition of M

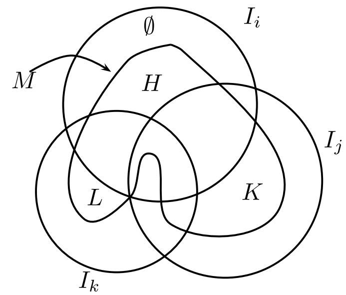

Fig. 2 A picture of the sets in case (2), assuming $M \nsubseteq I_i$. Note that $I_i \backslash (M \cup I_j \cup I_k) = \emptyset$ by the definition of M

$I \cup I_i \subseteq I_j$. Since $I \cap I_i \neq \emptyset$, we have $I \cup I_i \in \mathcal{B}_{i-1}$. Hence $I_i \subsetneq I \cup I_i \subsetneq I_j$ which implies $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{U_k}$.

Suppose that $I_i \subseteq I_j$ and that $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{U_k}$, so that there exists $I \in \mathcal{B}_{i-1}$ such that $I_i \subsetneq I \subsetneq I_j$. Then $I_i' \subseteq I' \subseteq I_j'$, and $I' \neq I_i'$ and $I' \neq I_j'$ since $i, j \in U_k$, so that $I_i' \subsetneq I' \subsetneq I_j'$. Hence $\{v(I_i'), v(I_j')\} \in \Gamma(\mathcal{B}/I_k)$.

(2) Suppose that $I_i \nsubseteq I_j$, and that $\{v(I_i'), v(I_j')\} \in \Gamma(\mathcal{B}/I_k)$, and suppose for a contradiction that $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B})|_{U_k}$, i.e. $i \notin U_j$. Then there exists $I \in \mathcal{B}_{i-1}$ such that $I \backslash I_j = I_i \backslash I_j$. Then $I' \backslash I_j' = I_i' \backslash I_j'$ which implies the contradiction that $\{v(I_i'), v(I_j')\} \notin \Gamma(\mathcal{B}/I_k)$.

Suppose that $I_i \nsubseteq I_j$, and that $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{I_k}$. We will prove the contrapositive that $\{v(I_i'), v(I_j')\} \notin \Gamma(\mathcal{B}/I_k)$ implies that $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B})|_{U_k}$. $\{v(I_i'), v(I_j')\} \notin \Gamma(\mathcal{B}/I_k)$ implies there exists $M \in \mathcal{B}_{i-1}$ such that $M' \backslash I_j' = I_i' \backslash I_j'$.

- Assume that $M \subseteq I_i$, and for this case refer to Fig. 1. Let $R := (I_i \backslash (M \cup I_j))$, and note that this is a subset of I_k since $I_i \backslash (M \cup I_i \cup I_k) = \emptyset$ by the definition of M. Also, let $J := I_i \backslash (M \cup I_k)$. Since $M \subseteq I_i$, by Lemma 2.4, there exists a decomposition of I_i in $\mathcal{B}_i$ that contains M. Hence M is contained in a maximal component D of this decomposition. Let D' be the other maximal component, and note that $D \cap D' = \emptyset$. If $D' \cap R = \emptyset$ then $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B})|_{U_k}$ since $D \backslash I_j = I_i \backslash I_j$, hence the desired condition holds. If $D' \cap J = \emptyset$ then $I_i \backslash I_k = D \backslash I_k$ which contradicts $i \in U_k$. If $D' \cap J \neq \emptyset$ and $D' \cap R \neq \emptyset$, then $(D' \cup I_j) \backslash I_k = I_j \backslash I_k$, which contradicts $j \in U_k$.
- Assume that $M \nsubseteq I_i$. For this case refer to Fig. 2. Let $H := I_i \backslash (I_j \cup I_k)$. In $(\mathcal{B}_j/I_k)/I_j'$ both I_i' and M' have the same image that is given by H, and $H \neq \emptyset$

since $H=\emptyset$ implies $I_i' \subseteq I_j'$, which contradicts Claim 3.2. Let $K := M \backslash (I_k \cup I_i)$. Then $K \neq \emptyset$ since $K = \emptyset$ implies $I_i \backslash I_k = M \backslash I_k$, which contradicts $i \in U_k$. Let $L := M \backslash (I_i \cup I_j)$. $L = \emptyset$ implies $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B})|_{U_k}$ since $(I_i \cup M) \backslash I_j = I_i \backslash I_j$, so the desired condition holds. Suppose now $L \neq \emptyset$. Then M intersects each of H, K and L. Let I be a minimal (for inclusion) element in $\mathcal{B}_{i-1}$ that intersects H, K and L. Then $|I| \geq 3$ and at least one of the maximal components of a decomposition of I (in $\mathcal{B}_{i-1}$) must intersect exactly two of K, H and L (since I is minimal with respect to intersecting H, K and L, and the components cannot both intersect exactly one set since their disjoint union is I). Denote such an element by $\widehat{D}$. Note that since $\widehat{D} \in \mathcal{B}_{i-1}$, and $\widehat{D} \cap I_i \neq \emptyset$, this implies by the definition of a building set that $\widehat{D} \cup I_i \in \mathcal{B}_{i-1}$. If $\widehat{D}$ intersects K and L then $(I_j \cup \widehat{D}) \backslash I_k = I_j \backslash I_k$ which contradicts $j \in U_k$. If $\widehat{D}$ intersects both K and H then $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B})|_{U_k}$ since $(I_i \cup \widehat{D}) \backslash I_j = I_i \backslash I_j$, so the desired condition holds. If $\widehat{D}$ intersects L and H, then $(I_i \cup \widehat{D}) \backslash I_k = I_i \backslash I_k$, which contradicts $i \in U_k$. □

We now consider the flag building set $\mathcal{B}|_{I_k}$. It is not necessarily true that $\mathcal{D}|_{I_k}$ is a decomposition of I_k. Let

$$\mathcal{D}_k := \mathcal{D}|_{I_k} \cup \{I_j \mid I_j \subseteq I_k, j \notin V_k\}.$$

The following claim holds for $\mathcal{D}_k$.

Claim 3.4 *Suppose $\mathcal{B}$ is a flag building set with flag ordering $(\mathcal{D}, I_1, \dots, I_k)$. Then $\mathcal{D}_k$ is a decomposition of I_k in $\mathcal{B}|_{I_k}$, and for any $i \leq k$, $\mathcal{D}_k \cup \{I_i \mid i \leq j \text{ and } i \in V_k\}$ is a flag building set on I_k.*

Proof We will first show that $\mathcal{D}_k$ is a decomposition of I_k in $\mathcal{B}|_{I_k}$. This can be seen by induction. We assume that for some $i < k$, the set of V-degenerate elements with respect to I_k in $\mathcal{B}_i$, that are a subset of I_k, together with $\mathcal{D}|_{I_k}$, are the union of a decomposition for each element in $(\mathcal{B}_i|_{I_k})_{\max}$. Then if $I_{i+1} \subseteq I_k$ and $i+1 \notin V_k$, then I_{i+1} is the union of two elements in $(\mathcal{B}_i|_{I_k})_{\max}$, so that the inductive hypothesis holds for $i+1$. It is also true that if $I_{i+1} \subseteq I_k$ and $i+1 \in V_k$, or if $I_{i+1} \nsubseteq I_k$, that the inductive hypothesis holds for $i+1$. The hypothesis clearly holds for $i=0$. Hence this statement holds by induction.

We will now show that for any $i \leq k$, $\mathcal{D}_k \cup \{I_i \mid i \leq j \text{ and } i \in V_k\}$ is a flag building set on I_k. This is true since $\mathcal{B}_i|_{I_k}$ is a flag building set, and each element in $\mathcal{B}_i|_{I_k}$ is a subset of, or disjoint to any element in $\mathcal{D}_k - \mathcal{B}_i|_{I_k}$. □

Since Claim 3.4 holds, we define $\Gamma(\mathcal{B}|_{I_k})$ to be the flag complex $\Gamma(O)$ with respect to the flag ordering O of $\mathcal{B}|_{I_k}$ with decomposition $\mathcal{D}_k$ and ordering of $\mathcal{B}|_{I_k} - \mathcal{D}_k$ given by $I_{v_1}, I_{v_2}, \dots, I_{v_{|V_k|}}$ where v_j is the jth element of V_k listed in increasing order. We label the vertices of $\Gamma(\mathcal{B}|_{I_k})$ by $v(I_{v_1}), \dots, v(I_{u_{|V_k|}})$ rather than by their index in V_k. In keeping with the notation that $\mathcal{B}_j$ is the flag building set obtained after adding elements indexed up to j, we let $(\mathcal{B}|_{I_k})_j$ denote the flag building set $\mathcal{D}_k \cup \{I_i \mid i \leq j \text{ and } i \in V_k\}$, so that $\Gamma((\mathcal{B}|_{I_k})_j)$ is defined. Note then that for any j, $\mathcal{B}_j|_{I_k} \subseteq (\mathcal{B}|_{I_k})_j$.

Proposition 3.5 *Let $\mathcal{B}$ be a connected flag building set with flag ordering given by $(\mathcal{D}, I_1, \ldots, I_k)$. Then $\Gamma(\mathcal{B}|_{I_k}) = \Gamma(\mathcal{B})|_{V_k}$.*

Proof Both $\Gamma(\mathcal{B}|_{I_k})$ and $\Gamma(\mathcal{B})|_{V_k}$ are both flag complexes with the vertex set $v(I_{v_1}), v(I_{v_2}), \ldots, v(I_{u_{|V_k|}})$. We need to show that for any $i, j \in V_k$ where $i < j$, $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$ if and only if $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B}|_{I_k})$.

$\Rightarrow$: Suppose that $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$. First assume that $I_i \subseteq I_j$. Then there is some $I \in \mathcal{B}_{i-1}$ such that $I_i \subsetneq b \subsetneq I_j$. Since $I \in \mathcal{B}_{i-1}|_{I_k}$ and $\mathcal{B}_{i-1}|_{I_k} \subseteq (\mathcal{B}|_{I_k})_{i-1}$ this implies that $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B}|_{I_k})$.

Now suppose that $I_i \nsubseteq I_j$. Suppose for a contradiction that $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B}|_{I_k})$. Then there exists some $D \in \mathcal{D}_k - \mathcal{D}|_{I_k}$, $D \in \mathcal{B}_{i-1}$, such that $D \cup I_j = I_i \cup I_j$. Since $i \in V_k$, there exists some $I \in \mathcal{B}_{i-1}$ such that $I_i \subsetneq I \subsetneq I_k$. Since $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$, we have that $I \backslash (I_i \cup I_j) \neq \emptyset$. Since the index of D is not in V_k, every element in the restriction to I_k that is earlier than D in the flag ordering is a subset of it or does not intersect it. This implies $I \subseteq D$, so $D \backslash (I_i \cup I_j) \neq \emptyset$, which contradicts $D \cup I_j = I_i \cup I_j$.

$\Leftarrow$: Suppose that $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B}|_{I_k})$. First assume that $I_i \subseteq I_j$, so that there is some $D \in (\mathcal{B}|_{I_k})_{i-1}$ such that $I_i \subsetneq D \subsetneq I_j$. If $D \in \mathcal{B}_{i-1}|_{I_k}$ then clearly $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$, as desired. If $D \notin \mathcal{B}_{i-1}|_{I_k}$ then $D \in \mathcal{D}_k - \mathcal{D}|_{I_k}$. Since $i \in V_k$, there exists some $I \in \mathcal{B}_{i-1}$ such that $I_i \subsetneq I \subsetneq I_k$. Since the index of D is not in V_k, we have that $I_i \subsetneq I \subsetneq D$. This is because D either contains or does not intersect elements that are earlier in the flag ordering and contained in I_k. Then since $D \subsetneq I_j$ this implies $I \subsetneq I_j$ and since $I \in \mathcal{B}_{i-1}$ and $I_i \subsetneq I \subsetneq I_j$, this implies $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$.

Now assume that $I_i \nsubseteq I_j$. Suppose for a contradiction that $\{v(I_i), v(I_j)\} \notin \Gamma(\mathcal{B})|_{V_k}$. Then there exists $I \in \mathcal{B}_{i-1}|_{I_k}$ such that $I \cup I_j = I_i \cup I_j$. Since $\mathcal{B}_{i-1}|_{I_k} \subseteq (\mathcal{B}|_{I_k})_{i-1}$, this contradicts $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B}|_{I_k})$. □

Theorem 3.6 *Let $\mathcal{B}$ be a connected flag building set with flag ordering O. Then $\gamma(\mathcal{B}) = f(\Gamma(O))$.*

Proof This is a proof by induction on the number of elements of $\mathcal{B} - \mathcal{D}$, and on the size of the set S that $\mathcal{B}$ is on. The result holds for $k = 0$ since $f(\Gamma(\mathcal{D})) = 1 = \gamma(\mathcal{D})$, and when $|S| = 1$. So we assume $k \geq 1$ and that the result holds for all connected flag building sets with a smaller value of k.

By Propositions 3.3 and 3.5 and the inductive hypothesis we have $f(\Gamma(\mathcal{B})|_{U_k}) = f(\Gamma(\mathcal{B}/I_k)) = \gamma(\mathcal{B}/I_k)$, and $f(\Gamma(\mathcal{B})|_{V_k}) = f(\Gamma(\mathcal{B}|_{I_k})) = \gamma(\mathcal{B}|_{I_k})$.

Suppose that $u \in U_k$ and $w \in V_k$. Then $\{v(I_u), v(I_w)\} \in \Gamma(\mathcal{B})$, for suppose, by way of contradiction, that $\{v(I_u), v(I_w)\} \notin \Gamma(\mathcal{B})$, and suppose that $u < w$. Then there is some element $I \in \mathcal{B}_{u-1}$ such that $I \cup I_w = I_u \cup I_w$. This implies that $I \cup I_k = I_u \cup I_k$, which contradicts $u \in U_k$. Suppose that $w < u$. Then either $I_u \cap I_w = \emptyset$ or $I_w \subseteq I_u$ (otherwise $I_u \cup I_w$ makes I_u U-degenerate with respect to I_k). Suppose that $I_w \cap I_u = \emptyset$. Then since $\{v(I_u), v(I_w)\} \notin \Gamma(\mathcal{B})$, there exists $I \in \mathcal{B}_{w-1}$ such that $I \cup I_u = I_w \cup I_u$, and $I \cap I_u \neq \emptyset$. Then $I \cup I_u$ makes I_u U-degenerate with respect to I_k; a contradiction. Suppose that $I_w \subseteq I_u$. Now $w \in V_k$ implies there is some $I \in \mathcal{B}_{w-1}$ such that $I_w \subsetneq I \subsetneq I_k$. Also, $I \subseteq I_u$ else $I \cup I_u$ makes I_u U-degenerate with respect to I_k. However, this implies the contradiction that $\{v(I_u), v(I_w)\} \in \Gamma(\mathcal{B})$ since $I_w \subsetneq I \subsetneq I_u$.

Hence

$$\Gamma(\mathcal{B})|_{U_k \cup V_k} = \Gamma(\mathcal{B})|_{U_k} * \Gamma(\mathcal{B})|_{V_k},$$

and therefore

$$f\big(\Gamma(\mathcal{B})|_{U_k \cup V_k}\big) = f\big(\Gamma(\mathcal{B})|_{U_k}\big) f\big(\Gamma(\mathcal{B})|_{V_k}\big) = \gamma(\mathcal{B}/I_k)\gamma(\mathcal{B}|_{I_k}).$$

Since the vertex $v(I_k)$ is adjacent to the vertices indexed by elements in $U_k \cup V_k$, we have

$$f\big(\Gamma(\mathcal{B})\big) = f\big(\Gamma(\mathcal{B}_{k-1})\big) + t\gamma(\mathcal{B}/I_k)\gamma(\mathcal{B}|_{I_k}).$$

By the induction hypothesis this implies that

$$f\big(\Gamma(\mathcal{B})\big) = \gamma(\mathcal{B}_{k-1}) + t\gamma(\mathcal{B}|I_k)\gamma(\mathcal{B}/I_k),$$

which implies that $f(\Gamma(\mathcal{B})) = \gamma(\mathcal{B})$ by Theorem 2.6. □

For two flag orderings O_1, O_2 of a connected flag building set $\mathcal{B}$, it is not necessarily true that the flag complexes $\Gamma(O_1), \Gamma(O_2)$ are equivalent (up to change of labels on the vertices) even if they have the same decomposition. The following example provides a counterexample.

Example 3.7 Let $\mathcal{B} = \mathcal{B}(\mathrm{Cyc}_5)$, and let

$$\mathcal{D} = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, [2], [3], [4], [5]\big\}.$$

Let O_1 be the flag ordering with decomposition $\mathcal{D}$ and the following ordering of $\mathcal{B} - \mathcal{D}$:

$$\{2,3\},\ \{2,3,4\},\ \{2,3,4,5\},\ \{4,5\},\ \{3,4,5\},\ \{3,4\},$$
$$\{3,4,5,1\},\ \{4,5,1,2\},\ \{5,1,2,3\},\ \{4,5,1\},\ \{5,1,2\},\ \{1,5\}.$$

Let O_2 be the flag ordering with decomposition $\mathcal{D}$ and the following ordering of $\mathcal{B} - \mathcal{D}$:

$$\{2,3\},\ \{2,3,4\},\ \{2,3,4,5\},\ \{3,4\},\ \{3,4,5\},\ \{4,5\},\ \{3,4,5\},\ \{3,4\},$$
$$\{3,4,5,1\},\ \{4,5,1,2\},\ \{5,1,2,3\},\ \{4,5,1\},\ \{5,1,2\},\ \{1,5\}.$$

Then $\Gamma(O_1)$ and $\Gamma(O_2)$ are depicted in Fig. 3.

4 The Flag Complexes of Nevo and Petersen

In this section we compare the flag complexes that we have defined to those defined for certain graph-associahedra by Nevo and Petersen [8]. They define flag complexes $\Gamma(\widehat{\mathfrak{S}}_n)$, $\Gamma(\widehat{\mathfrak{S}}_n(312))$ and $\Gamma(P_n)$ such that:

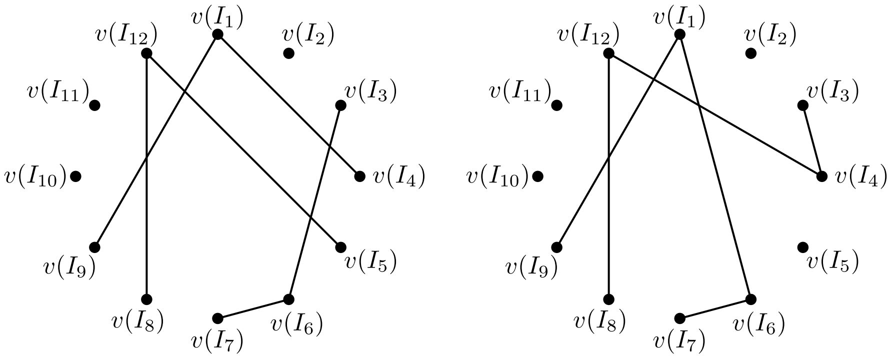

Fig. 3 $\Gamma(O_1)$ is on the *left*, and $\Gamma(O_2)$ is on the *right*

- $\gamma(\mathcal{B}(K_n)) = f(\Gamma(\widehat{\mathfrak{S}}_n))$,
- $\gamma(\mathcal{B}(\text{Path}_n)) = f(\Gamma(\widehat{\mathfrak{S}}_n(312)))$,
- $\gamma(\mathcal{B}(\text{Cyc}_n)) = f(\Gamma(P_n))$.

In Proposition 4.3, we show that for all n, there is a flag ordering for $\mathcal{B}(\text{Path}_n)$ so that

$$\Gamma\bigl(\mathcal{B}(\text{Path}_n)\bigr) \cong \Gamma\bigl(\widehat{\mathfrak{S}}_n(312)\bigr).$$

We also show, namely in Propositions 4.2 and 4.5, that the analogous statement is not true for $\mathcal{B}(K_n)$ and $\mathcal{B}(\text{Cyc}_n)$, although we have omitted the proofs, which were done by a manual case analysis.

4.1 The Flag Complexes $\Gamma(\mathcal{B}(K_n))$ and $\Gamma(\widehat{\mathfrak{S}}_n)$

The permutohedron is the nestohedron $P_{\mathcal{B}(K_n)}$. Note that $\mathcal{B}(K_n)$ consists of all nonempty subsets of $[n]$. The γ-polynomial of $P_{\mathcal{B}(K_n)}$ is the descent generating function of $\widehat{\mathfrak{S}}_n$, which denotes the set of permutations with no double descents or final descent (see [11, Theorem 11.1]). First we recall the definition of $\Gamma(\widehat{\mathfrak{S}}_n)$ given by Nevo and Petersen [8, Sect. 4.1].

A *peak* of a permutation $w = w_1 \cdots w_n$ in $\mathfrak{S}_n$ is a position $i \in [1, n-1]$ such that $w_{i-1} < w_i > w_{i+1}$, (where $w_0 := 0$). We denote a peak at position i with a bar $w_1 \cdots w_i | w_{i+1} \cdots w_n$. A *descent* of a permutation $w = w_1 \cdots w_n$ is a position $i \in [n-1]$ such that $w_{i+1} < w_i$. Let $\widehat{\mathfrak{S}}_n$ denote the set of permutations in $\mathfrak{S}_n$ with no double (i.e. consecutive) descents or final descent, and let $\widetilde{\mathfrak{S}}_n$ denote the set of permutations in $\mathfrak{S}_n$ with one peak. Then $\widehat{\mathfrak{S}}_n \cap \widetilde{\mathfrak{S}}_n$ consists of all permutations of the form

$$w_1 \cdots w_i | w_{i+1} \cdots w_n,$$

where $1 \le i \le n-2$, $w_1 < \cdots < w_i$, $w_i > w_{i+1}$, $w_{i+1} < \cdots < w_n$.

Define the flag complex $\Gamma(\widehat{\mathfrak{S}}_n)$ on the vertex set $\widehat{\mathfrak{S}}_n \cap \widetilde{\mathfrak{S}}_n$ where two vertices

$$u = u_1 | u_2$$

and

$$v = v_1|v_2$$

with $|u_1| < |v_1|$ are adjacent if there is a permutation $w \in \mathfrak{S}_n$ of the form

$$w = u_1|a|v_2.$$

Equivalently, if $v_2 \subseteq u_2$, $|u_2 - v_2| \geq 2$, $\min(u_2 - v_2) < \max(u_1)$ and $\max(u_2 - v_2) > \min(v_2)$. (Since there must be two peaks in w this implies $|a| \geq 2$.) The faces of $\Gamma(\widehat{\mathfrak{S}}_n)$ are the cliques in this graph.

Example 4.1 Taking only the part after the peak, $\widehat{\mathfrak{S}}_5 \cap \widetilde{\mathfrak{S}}_5$ can be identified with the set of subsets of [5] of sizes 2,3 and 4 which are not $\{4,5\}$, $\{3,4,5\}$, or $\{2,3,4,5\}$. Then the edges of $\Gamma(\widehat{\mathfrak{S}}_5)$ are given by:
$\{1,2,3,4\}$ is adjacent to each of $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$,
$\{1,2,3,5\}$ is adjacent to each of $\{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}$,
$\{1,2,4,5\}$ is adjacent to each of $\{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}$, and
$\{1,3,4,5\}$ is adjacent to each of $\{3,4\}, \{3,5\}$.

Proposition 4.2 *There is no flag ordering of $\mathcal{B}(K_5)$ so that*

$$\Gamma\big(\mathcal{B}(K_5)\big) \cong \Gamma(\widehat{\mathfrak{S}}_5).$$

The proof of Proposition 4.2, which is a manual case analysis, has been omitted.

4.2 The Flag Complexes $\Gamma(\mathcal{B}(\mathrm{Path}_n))$ and $\Gamma(\widehat{\mathfrak{S}}_n(312))$

The associahedron is the nestohedron $P_{\mathcal{B}(\mathrm{Path}_n)}$. Note that $\mathcal{B}(\mathrm{Path}_n)$ consists of all intervals $[j,k]$ with $1 \leq j \leq k \leq n$. The γ-polynomial of the associahedron is the descent generating function of $\widehat{\mathfrak{S}}_n(312)$, which denotes the set of 312-avoiding permutations with no double or final descents (see [11, Sect. 10.2]). We now describe the flag complex $\Gamma(\widehat{\mathfrak{S}}_n(312))$ defined by Nevo and Petersen [8, Sect. 4.2].

Given distinct integers a, b, c, d such that $a < b$ and $c < d$, the pairs (a,b), (c,d) are *non-crossing* if either:

- $a < c < d < b$ (or $c < a < b < d$), or
- $a < b < c < d$ (or $c < d < a < b$).

Define $\Gamma(\widehat{\mathfrak{S}}_n(312))$ to be the flag complex on the vertex set

$$V_n := \big\{(a,b) \mid 1 \leq a < b \leq n-1\big\},$$

with faces the sets S of V_n such that if $(a,b) \in S$ and $(c,d) \in S$ then (a,b) and (c,d) are non-crossing.

Let O denote the flag ordering of $\mathcal{B} = \mathcal{B}(\mathrm{Path}_n)$ with decomposition $\mathcal{D} = \{\{1\},\{2\},\{3\},\{4\},\{n\},[2],[3],[4],[n]\}$, where elements $A, B \in \mathcal{B} - \mathcal{D}$ are ordered so that A is earlier than B if:

- $\max(A) < \max(B)$, or
- $\max(A) = \max(B)$ and $|A| > |B|$.

Proposition 4.3 *For the flag ordering O of $\mathcal{B} = \mathcal{B}(\mathrm{Path}_n)$ described above, $\Gamma(O) \cong \Gamma(\widehat{\mathfrak{S}}_n(312))$ where the bijection on the vertices is given by $v([a+1, b+1]) \mapsto (a, b)$.*

Proof Since $\mathcal{B} - \mathcal{D} = \{[j, k] \mid 2 \le j < k \le n\}$, it is clear that the stated map on vertices is a bijection. Let $[l, m], [j, k]$ be distinct elements of $\mathcal{B} - \mathcal{D}$ with $[l, m]$ occurring before $[j, k]$. Then $m \le k$, and if $m = k$ we have $l < j$. If $[l, m] \nsubseteq [j, k]$ then $v([l, m])$ is adjacent to $v([j, k])$ if and only if $m < j$. If $[l, m] \subseteq [j, k]$ (which entails $m < k$), then $v([l, m])$ is adjacent to $v([j, k])$ if and only if $j < l$. So in either case $v([l, m])$ is adjacent to $v([j, k])$ if and only if $(l - 1, m - 1)$ and $(j - 1, k - 1)$ are non-crossing. □

4.3 The Flag Complexes $\Gamma(\mathcal{B}(\mathrm{Cyc}_n))$ and $\Gamma(P_n)$

The cyclohedron is the nestohedron $P_{\mathcal{B}(\mathrm{Cyc}_n)}$. Note that $\mathcal{B}(\mathrm{Cyc}_n)$ consists of all sets $\{i, i + 1, i + 2, \dots, i + s\}$ where $i \in [n]$, $s \in \{0, 1, \dots, n - 1\}$, and the elements are taken mod n. By [11, Proposition 11.15], $\gamma_r(\mathcal{B}(\mathrm{Cyc}_n)) = \binom{n}{r, r, n-2r}$. We now describe the flag complex $\Gamma(P_n)$ defined by Nevo and Petersen [8, Sect. 4.3].

Define the vertex set

$$V_{P_n} := \{(l, r) \in [n - 1] \times [n - 1] \mid l \neq r\}.$$

$\Gamma(P_n)$ is the flag complex on the vertex set V_{P_n} where vertices $(l_1, r_1), (l_2, r_2)$ are adjacent in $\Gamma(P_n)$ if and only if l_1, l_2, r_1, r_2 are all distinct and either $l_1 < l_2$ and $r_1 < r_2$, or $l_2 < l_1$ and $r_2 < r_1$.

Example 4.4 $\Gamma(P_5)$ is the flag complex on vertices

$$V_{P_5} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$$

with edges

$$\{(1, 3), (2, 4)\}, \{(3, 1), (4, 2)\}, \{(1, 2), (3, 4)\}, \{(1, 2), (4, 3)\}, \{(2, 1), (4, 3)\}, \{(2, 1), (3, 4)\}.$$

Note that $\Gamma(P_5)$ has exactly two vertices of degree two, and has six connected components, four of which contain more than one vertex.

Proposition 4.5 *There is no flag ordering of $\mathcal{B}(\mathrm{Cyc}_5)$ so that $\Gamma(\mathcal{B}(\mathrm{Cyc}_5)) \cong \Gamma(P_5)$.*

The proof of Proposition 4.5, which is a manual case analysis, has been omitted.

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