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Ollivier's Ricci Curvature, Local Clustering and Curvature-Dimension Inequalities on Graphs

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Abstract In this paper, we explore the relationship between one of the most elementary and important properties of graphs, the presence and relative frequency of triangles, and a combinatorial notion of Ricci curvature. We employ a definition of generalized Ricci curvature proposed by Ollivier in a general framework of Markov processes and metric spaces and applied in graph theory by Lin–Yau. In analogy with curvature notions in Riemannian geometry, we interpret this Ricci curvature as a control on the amount of overlap between neighborhoods of two neighboring vertices. It is therefore naturally related to the presence of triangles containing those vertices, or more precisely, the local clustering coefficient, that is, the relative proportion of connected neighbors among all the neighbors of a vertex. This suggests to derive lower Ricci curvature bounds on graphs in terms of such local clustering coefficients. We also study curvature-dimension inequalities on graphs, building upon previous work of several authors.

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1 Introduction

When one studies empirical graphs, one of the most obvious and basic properties to investigate is the presence and number of triangles, that is, connected triples of vertices. In bipartite graphs, for instance, there are no triangles, whereas in a complete graph, every triple of vertices constitutes a triangle. A basic observation then is that when two neighboring vertices are contained in a triangle, their neighborhoods of radius 1 (let us assign to every edge the length 1 for the discussion in this introduction) share the third vertex of the triangle. That is, the more triangles those two neighboring vertices are contained in, the larger the overlap of their neighborhoods. This suggests an analogy with the notion of Ricci curvature in Riemannian geometry where a lower bound on the Ricci curvature also controls the amount of overlaps of distance balls from below. This is what we are going to explore in a quantitative manner in this paper.

In fact, Ricci curvature is a fundamental concept in Riemannian geometry, see e.g. [18]. It is a quantity computed from second derivatives of the metric tensor. It controls how fast geodesics starting at the same point diverge on average. Equivalently, it controls how fast the volume of distance balls grows as a function of the radius. As already indicated, it also controls the amount of overlap of two distance balls in terms of their radii and the distance between their centers. In fact, such lower bounds follow from a lower bound on the Ricci curvature. It was then natural to look for generalizations of such phenomena on metric spaces more general than Riemannian manifolds. That is, the question to find substitutes for the lower bounds on the above mentioned second derivative combinations of the metric tensor that yield the same geometric control on a general metric space. By now, there exist several insightful definitions of synthetic Ricci curvature on general metric measure spaces, see Sturm [29, 30], Lott–Villani [22], Ohta [23], Ollivier [24] etc.

As indicated, in this paper, we want to explore the implications of such ideas in graph theory. The geometric idea is that a lower Ricci curvature bound prevents geodesics from diverging too fast and balls from growing too fast in volume. On a graph, the analogue of geodesics starting in different directions, but eventually approaching each other again, would be a triangle. Therefore, it is natural that the Ricci curvature on a graph should be related to the relative abundance of triangles. The latter is captured by the local clustering coefficient introduced by Watts–Strogatz [33]. Thus, the intuition of Ricci curvature on a graph should play with the relative frequency of triangles a vertex shares with its neighbors. In fact, more precisely, since the local clustering coefficient averages over the neighbors of a vertex, this should really related to some notion of scalar curvature, as an average of Ricci curvatures in different directions, that is, for different neighbors of a given vertex.

Among the several definitions of generalized Ricci curvature in the literature mentioned above, the one of Ollivier works particularly well on discrete spaces like graphs. It is formulated in terms of the transportation distance between local measures:

$$\kappa(x, y) := 1 - W_1(m_x, m_y), \tag{1.1}$$

where x, y are vertices in our graph that are neighbors (written as $x \sim y$) and the measure $m_x = \frac{1}{d_x}$, where d_x is the degree of x, puts equal weight on all neighbors. $W_1(m_x, m_y)$ is the transportation distance between the two measures m_x and m_y (defined more precisely below). When two balls strongly overlap, as is the case in Riemannian geometry when the Ricci curvature has a large lower bound, then it is easier to transport the mass of one to the other. Analogously, in the graph case, when the two vertices share many triangles, then the transportation distance should be smaller, and the curvature therefore correspondingly larger. This is the idea of Ollivier's definition as we see it and explore in this paper. We shall obtain both upper and lower bounds for Ollivier's Ricci curvature on graphs in Sect. 3, which are optimal on many graphs.

Let us now formulate our main result (recalled and proved below as Theorem 3).

Theorem 1 *On a locally finite graph, we put for any pair of neighboring vertices* x, y,

$$\sharp(x, y) := \textit{number of triangles which include } x, y \textit{ as vertices} = \sum_{x_1, x_1 \sim x, x_1 \sim y} 1.$$

We then have

$$\kappa(x, y) \geq -\left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \wedge d_y}\right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \vee d_y}\right)_+ + \frac{\sharp(x, y)}{d_x \vee d_y}. \tag{1.2}$$

where $s_+ := \max(s, 0)$, $s \vee t := \max(s, t)$, $s \wedge t := \min(s, t)$.

This equality is sharp for instance for a complete graph of n vertices where the left and the right hand side both equal to $\frac{n-2}{n-1}$.

The local clustering coefficient introduced by Watts–Strogatz [33] is

$$c(x) := \frac{\text{number of edges between neighbors of } x}{\text{number of possible existing edges between neighbors of } x}, \tag{1.3}$$

which measures the extent to which neighbors of x are directly connected, i.e.,

$$c(x) = \frac{1}{d_x(d_x - 1)} \sum_{y, y \sim x} \sharp(x, y). \tag{1.4}$$

Thus, this local clustering coefficient is an average over the $\sharp(x, y)$ for the neighbors of x. Thus, we might also introduce some kind of scalar curvature (suggested in Problem Q in Ollivier [25]) as

$$\kappa(x) := \frac{1}{d_x} \sum_{y, y \sim x} \kappa(x, y). \tag{1.5}$$

For illustration, let us consider the case where our graph is d-regular, that is, $d_z = d$ for all vertices z. When $1 \geq \frac{2}{d} + \frac{\sharp(x,y)}{d}$ for all $y \sim x$, we would then get

$$\kappa(x) \geq -2 + \frac{4}{d} + \frac{3(d-1)}{d} c(x). \tag{1.6}$$

This example nicely illustrates the relation between Ollivier's curvature and the Watts–Strogatz clustering coefficient.

Without the triangle terms $\sharp(x, y)$ (which is the crucial term for our purposes), Theorem 1 is due to Lin–Yau [19, 21], and we take their proof as our starting point. Lin–Yau also obtain analogues of Bochner type inequalities and eigenvalue estimates as known from Riemannian geometry.

In Riemannian geometry, the Bochner formula encodes deep analytic properties of Ricci curvature. It is a key ingredient in proving many results, e.g. the spectral gap of the Laplace–Beltrami operator. A lower bound of the Ricci curvature implies a curvature-dimension inequality involving the Laplace–Beltrami operator through the Bochner formula. In an important work, Bakry and Émery [2, 3] generalize this inequality to generators of Markov semigroups, which works on measure spaces. Their inequality contains plentiful information and implies a lot of functional inequalities including spectral gap inequalities, Sobolev inequalities, and logarithmic Sobolev inequalities and many celebrated geometric theorems (see [1] and the references therein). Lin–Yau [21] study such inequalities on locally finite graphs.

In the present paper, we also want to find relations on locally finite graphs between Ollivier's Ricci curvature and Bakry–Émery's curvature-dimension inequalities, which represent the geometric and analytic aspects of graphs, respectively. Again, this is inspired by Riemannian geometry where one may attach a Brownian motion with a drift to a Riemannian metric [24]. We also mention that the definitions given by Sturm and Lott–Villani are also consistent with that of Bakry–Émery [22, 29, 30]. So exploring the relations on nonsmooth spaces may provide a good point of view to connect Ollivier's definition to Sturm and Lott–Villani's (in this respect, see also Ollivier–Villani [26]). In Sect. 4, we use the local clustering coefficient again to establish more precise curvature-dimension inequalities than those of Lin–Yau [21]. And with this in hand, we prove curvature-dimension inequalities under the condition that Ollivier's Ricci curvature of the graph is positive.

Further analytical results following from curvature-dimension inequalities on finite graphs have been described in [19], and Lin–Lu–Yau [20] study a modified definition of Ollivier's Ricci curvature on graphs. Recently, Paeng [27] studied upper bounds for the diameter and volume of finite simple graphs in terms of Ollivier's Ricci curvature. For other works of synthetic Ricci curvatures on discrete spaces, see Dodziuk–Karp [14], Chung–Yau [9], Bonciocat–Sturm [7], and on cell complexes see Forman [17], Stone [28] etc.

We point out that, as in Riemannian geometry, both Ollivier's Ricci curvature and Bakry–Émery's curvature-dimension inequality can give lower bound estimates of the first eigenvalue λ_1 for the Laplace operator (see Ollivier [24], Bakry [1]). Therefore our results in fact relate λ_1 to the Watts–Strogatz local clustering coefficient, or the number of cycles with length 3. In [10], Diaconis and Stroock obtain several geometric bounds for eigenvalues of graphs, one of which is related to the number of odd length cycles. For more geometric quantities and methods concerning eigenvalue estimates in the study of Markov chains, see [11–13] and the references therein. We further explore the interaction between Ollivier's Ricci curvature and eigenvalues estimates in joint work with Frank Bauer, see [6].

In this paper, $G = (V, E)$ will denote an undirected connected simple graph without loops, where V is the set of vertices and E is the set of edges. V could be an infinite set. But we require that G is locally finite, i.e., for every $x \in V$, the number of edges connected to x is finite. For simplicity and in order to see more geometry, we mainly work on unweighted graphs. But we will also derive similar results on weighted graphs. In that case, we denote by w_{xy} the weight associated to $x, y \in V$, where $x \sim y$ (we may simply put $w_{xy} = 0$ if x and y are not neighbors, to simplify the notation). The unweighted case corresponds to $w_{xy} = 1$ whenever $x \sim y$. The degree of $x \in V$ is $d_x = \sum_{y, y\sim x} w_{xy}$.

2 Ollivier's Ricci Curvature and Bakry–Émery's Calculus

In this section, we present some basic facts about Ollivier's Ricci curvature and Bakry–Emery's Γ_2 calculus, in particular on graphs.

2.1 Ollivier's Ricci Curvature

Ollivier's Ricci curvature works on a general metric space (X, d), on which we attach to each point $x \in X$ a probability measure $m_x(\cdot)$. We denote this structure by (X, d, m).

For a locally finite graph $G = (V, E)$, we define the metric d as follows. For neighbors x, y, $d(x, y) = 1$. For general distinct vertices x, y, $d(x, y)$ is the length of the shortest path connecting x and y, i.e. the number of edges of the path. We attach to each vertices $x \in V$ a probability measure

$$m_x(y) = \begin{cases} \frac{w_{xy}}{d_x}, & \text{if } y \sim x; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

An intuitive illustration of this is a random walker that sits at x and then chooses amongst the neighbors of x with equal probability $\frac{1}{d_x}$.

Definition 1 (Ollivier) On (X, d, m), for any two distinct points $x, y \in X$, the (Ollivier–) Ricci curvature of (X, d, m) along (xy) is defined as

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}. \tag{2.2}$$

Here, $W_1(m_x, m_y)$ is the optimal transportation distance between the two probability measures m_x and m_y, defined as follows (cf. Villani [31, 32], Evans [16]).

Definition 2 For two probability measures μ_1, μ_2 on a metric space (X, d), the transportation distance between them is defined as

$$W_1(\mu_1, \mu_2) := \inf_{\xi \in \prod(\mu_1, \mu_2)} \int_{X\times X} d(x, y)\, d\xi(x, y), \tag{2.3}$$

where $\prod(\mu_1, \mu_2)$ is the set of probability measures on $X \times X$ projecting to μ_1 and μ_2.

In other words, ξ satisfies

$$\xi(A \times X) = \mu_1(A), \qquad \xi(X \times B) = \mu_2(B), \quad \forall A, B \subset X.$$

Remark 1 Intuitively, this distance measures the optimal cost to move one pile of sand to another one with the same mass. For case of a graph $G = (G, d, m)$, the supports of m_x and m_y are finite discrete sets, and thus, ξ is just a matrix with terms $\xi(x', y')$ representing the mass moving from $x' \in$ support of m_x to $y' \in$ support of m_y. That is, in this case,

$$W_1(m_x, m_y) = \inf_{\xi} \sum_{x', x' \sim x} \sum_{y', y' \sim y} d\big(x', y'\big)\xi\big(x', y'\big),$$

where the infimum is taken over all matrices ξ which satisfy

$$\sum_{x', x' \sim x} \xi\big(x', y'\big) = \frac{w_{yy'}}{d_y}, \qquad \sum_{y', y' \sim y} \xi\big(x', y'\big) = \frac{w_{xx'}}{d_x}.$$

We also call ξ a transfer plan. If we can find a particular transfer plan, we then get an upper bound for W_1 and therefore a lower bound for κ.

A very important property of transportation distance is the Kantorovich duality (see, e.g. Theorem 1.14 in Villani [31]). We state it here in our particular graph setting.

Proposition 1 (Kantorovich Duality)

$$W_1(m_x, m_y) = \sup_{f, 1-\mathrm{Lip}} \left[\sum_{z, z \sim x} f(z) m_x(z) - \sum_{z, z \sim y} f(z) d m_y(z) \right],$$

where the supremum is taken over all functions on G that satisfy

$$\big| f(x) - f(y) \big| \le d(x, y),$$

for any $x, y \in V$, $x \neq y$.

From this property, a good choice of a 1-Lipschitz function f will yield a lower bound for W_1 and therefore an upper bound for κ.

Remark 2 We list some basic first observations about this curvature concept (see Ollivier [24]):

- $\kappa(x, y) \le 1$.
- Rewriting (2.2) gives $W_1(m_x, m_y) = d(x, y)(1 - \kappa(x, y))$, which is analogous to the expansion in the Riemannian case.
- A lower bound $\kappa(x, y) \ge k$ for any $x, y \in X$ implies

$$W_1(m_x, m_y) \le (1 - k) d(x, y), \tag{2.4}$$

  which can be seen as some kind of Lipschitz continuity of measures.

2.2 Bakry–Émery's Curvature-Dimension Inequality

2.2.1 Laplace Operator

We will study the following operator which is an analogue of the Laplace–Beltrami operator in Riemannian geometry.

Definition 3 The Laplace operator on (X, d, m) is defined as follows:

$$\Delta f(x) = \int_X f(y)\, dm_x(y) - f(x), \quad \text{for functions } f : X \longrightarrow R. \tag{2.5}$$

For our choice of $\{m_x(\cdot)\}$, this is the graph Laplacian studied by many authors, see e.g. [4, 5, 8, 14, 21].

2.2.2 Bochner Formula and Curvature-Dimension Inequality

In the Riemannian case, many analytical consequences of a lower bound of the Ricci curvature are obtained through the well-known Bochner formula,

$$\frac{1}{2}\Delta\big(|\nabla f|^2\big) = |\,\mathrm{Hess}\, f|^2 + \big\langle \nabla(\Delta f), \nabla f\big\rangle + \mathrm{Ric}(\nabla f, \nabla f).$$

Analytically, $|\,\mathrm{Hess}\, f|^2$ is difficult to define on a nonsmooth space. But using Schwarz's inequality, we have

$$|\,\mathrm{Hess}\, f|^2 \geq \frac{(\Delta f)^2}{m},$$

where m is the dimension constant. So we can use

$$\frac{1}{2}\Delta\big(|\nabla f|^2\big) \geq \frac{(\Delta f)^2}{m} + \big\langle \nabla(\Delta f), \nabla f\big\rangle + K|\nabla f|^2 \tag{2.6}$$

to characterize $\mathrm{Ric} \geq K$.

Bakry–Émery [1–3] take this inequality as the starting point and directly use the operators to define curvature bounds. Starting from an operator Δ, they define iteratively,

$$\begin{aligned}
\Gamma_0(f, g) &= fg, \\
\Gamma(f, g) &= \frac{1}{2}\big\{\Delta\Gamma_0(f, g) - \Gamma_0(f, \Delta g) - \Gamma_0(\Delta f, g)\big\}, \\
\Gamma_2(f, g) &= \frac{1}{2}\big\{\Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g)\big\}.
\end{aligned}$$

In fact, $\Gamma(f, f)$ is an analogue of $|\nabla f|^2$, and $\Gamma_2(f, f)$ is an analogue of $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla(\Delta f), \nabla f\rangle$ in (2.6).

Definition 4 We say an operator Δ satisfies a curvature-dimension inequality $\mathrm{CD}(m, K)$ if for all functions f in the domain of the operator

$$\Gamma_2(f, f)(x) \geq \frac{1}{m}\big(\Delta f(x)\big)^2 + K(x)\Gamma(f, f)(x), \quad \forall x \in X, \tag{2.7}$$

where $m \in [1, +\infty]$ is the dimension parameter, $K(x)$ is the curvature function.

As studied in Lin–Yau [21], applying this construction to the operator (2.5) gives

$$\Gamma(f, f)(x) = \frac{1}{2}\int_X \big(f(y) - f(x)\big)^2 \, dm_x(y). \tag{2.8}$$

In fact generally

$$\Gamma(f, g)(x) = \frac{1}{2}\int_X \big(f(y) - f(x)\big)\big(g(y) - g(x)\big)\, dm_x(y).$$

For the sake of convenience, we will denote

$$\mathrm{Hf}(x) := \frac{1}{4}\int_X\int_X \big(f(x) - 2f(y) + f(z)\big)^2 \, dm_y(z)\, dm_x(y).$$

By the calculation in Lin–Yau [21] we get

$$\begin{aligned}
\Delta\Gamma(f, f)(x) &= 2\,\mathrm{Hf}(x)\\
&\quad - \int_X\int_X \big(f(x) - 2f(y) + f(z)\big)\big(f(x) - f(y)\big)\, dm_y(z)\, dm_x(y),\\
2\Gamma(f, \Delta f)(x) &= -\big(\Delta f(x)\big)^2\\
&\quad - \int_X\int_X \big(f(z) - f(y)\big)\big(f(x) - f(y)\big)\, dm_y(z)\, dm_x(y),
\end{aligned}$$

and then,

$$\Gamma_2(f, f) = \mathrm{Hf}(x) - \Gamma(f, f)(x) + \frac{1}{2}\big(\Delta f(x)\big)^2. \tag{2.9}$$

3 Ollivier's Ricci Curvature and Triangles

In this section, we mainly prove lower bounds for Ollivier's Ricci curvauture on locally finite graphs. In particular we shall explore the implication between lower bounds of the curvature and the number of triangles including neighboring vertices; the latter is encoded in the local clustering coefficient. We remark that we only need to bound $\kappa(x, y)$ from below for neighboring x, y, since by the triangle inequality of W_1, this will also be a lower bound for $\kappa(x, y)$ of any pair of x, y. (See Proposition 19 in Ollivier [24].)

3.1 Unweighted Graphs

In this subsection, we only consider unweighted graphs.

In Lin–Yau [21], they prove a lower bound of Ollivier's Ricci curvature on locally finite graphs G. Here, for later purposes, we include the case where G may have vertices of degree 1 and get the following modified result.

Theorem 2 *On a locally finite graph $G=(V,E)$, we have for any pair of neighboring vertices x,y,*

$$\kappa(x,y)\geq -2\left(1-\frac{1}{d_x}-\frac{1}{d_y}\right)_+ = \begin{cases} -2+\frac{2}{d_x}+\frac{2}{d_y}, & \text{if } d_x>1 \text{ and } d_y>1;\\ 0, & \text{otherwise.}\end{cases}$$

Remark 3 Notice that if $d_x=1$, then we can calculate $\kappa(x,y)=0$ exactly. So, even though in this case $-2+\frac{2}{d_x}=0$, $\kappa(x,y)\geq\frac{2}{d_y}$ does not hold.

For completeness, we state the proof of Theorem 2 here. It is essentially the one in Lin–Yau [21] with a small modification.

Proof of Theorem 2 Since $d(x,y)=1$ for $x\sim y$, we have

$$\kappa(x,y)=1-W_1(m_x,m_y). \tag{3.1}$$

Using Kantorovich duality, we get

$$\begin{aligned} W_1(m_x,m_y) &= \sup_{f,1-\text{Lip}}\left(\frac{1}{d_x}\sum_{z,z\sim x} f(z)-\frac{1}{d_y}\sum_{z',z'\sim y} f(z')\right)\\ &= \sup_{f,1-\text{Lip}}\left(\frac{1}{d_x}\sum_{z,z\sim x,z\neq y}\big(f(z)-f(x)\big)-\frac{1}{d_y}\sum_{z',z'\sim y,z'\neq x}\big(f(z')-f(y)\big)\right.\\ &\quad \left.+\frac{1}{d_x}\big(f(y)-f(x)\big)-\frac{1}{d_y}\big(f(x)-f(y)\big)+\big(f(x)-f(y)\big)\right)\\ &\leq \frac{d_x-1}{d_x}+\frac{d_y-1}{d_y}+\left|1-\frac{1}{d_x}-\frac{1}{d_y}\right|\\ &= 2-\frac{1}{d_x}-\frac{1}{d_y}+\left|1-\frac{1}{d_x}-\frac{1}{d_y}\right|\\ &= 1+2\left(1-\frac{1}{d_x}-\frac{1}{d_y}\right)_+. \end{aligned} \tag{3.2}$$

Inserting the above estimate into (3.1) gives

$$\kappa(x,y)\geq -2\left(1-\frac{1}{d_x}-\frac{1}{d_y}\right)_+. \qquad \square$$

Note that trees attain this lower bound. This coincides with the geometric intuition of curvature. Since trees have the fastest volume growth rate, it is plausible that they have the smallest curvature.

Proposition 2 *We consider a tree $T=(V,E)$. Then for any neighboring x, y, we have*

$$\kappa(x,y) = -2\left(1-\frac{1}{d_x}-\frac{1}{d_y}\right)_+. \tag{3.3}$$

Proof In fact with Theorem 2 in hand, we only need to prove that $1+2(1-\frac{1}{d_x}-\frac{1}{d_y})_+$ is also a lower bound of W_1. If one of x, y is a vertex of degree 1, say $d_x=1$, it is obvious that $W_1(m_x,m_y)=1$. So we only need to deal with the case $1-\frac{1}{d_x}-\frac{1}{d_y}\geq 0$.

We can find a 1-Lipschitz function f on a tree as follows.

$$f(z) = \begin{cases} 0, & \text{if } z\sim y, z\neq x; \\ 1, & \text{if } z=y; \\ 2, & \text{if } z=x; \\ 3, & \text{if } z\sim x, z\neq x. \end{cases} \tag{3.4}$$

Since on a tree, the path joining two vertices are unique, there is no further path between neighbors of x and y. So this can be easily extended to a 1-Lipschitz function on the whole graph. Then by Kantorovich duality, we have

$$\begin{aligned} W_1(m_x,m_y) &\geq \frac{1}{d_x}\big(3(d_x-1)+1\big) - \frac{1}{d_y}\cdot 2 \\ &= 3-\frac{2}{d_x}-\frac{2}{d_y}. \end{aligned} \tag{3.5}$$

This completes the proof. □

In order to make clear the geometric meaning of the term $(1-\frac{1}{d_x}-\frac{1}{d_y})_+$, and also to prepare the idea used in the next theorem, we give another method to get the upper bound of W_1. That works through a particular transfer plan. If

$$1-\frac{1}{d_x}-\frac{1}{d_y}\geq 0, \quad \text{or} \quad 1-\frac{1}{d_y}\geq\frac{1}{d_x},$$

then for m_y, the mass at all z such that $z\sim y$, $z\neq x$ is larger than that of m_x at y. So we can move the mass $\frac{1}{d_x}$ at y to z, $z\sim y$, $z\neq x$ for distance 1. Symmetrically, we can move a mass of $\frac{1}{d_y}$ at the vertices z which satisfy $z\sim x$, $z\neq y$ to x for distance 1. The remaining mass of $(1-\frac{1}{d_x}-\frac{1}{d_y})$ needs to be moved for distance 3. This gives

$$\begin{aligned} W_1(m_x,m_y) &\leq \left(\frac{1}{d_x}+\frac{1}{d_y}\right)\times 1 + \left(1-\frac{1}{d_x}-\frac{1}{d_y}\right)\times 3 \\ &= 3-\frac{2}{d_x}-\frac{2}{d_y}. \end{aligned} \tag{3.6}$$

If

$$1-\frac{1}{d_x}-\frac{1}{d_y}\le 0,$$

we only need to move the mass of m_x for distance 1 to the support of m_y. So we have in this case $W_1(m_x, m_y) = 1$. This gives the same upper bound as in (3.2).

From the view of transfer plans, the existence of triangles including neighboring vertices would save a lot of transport costs and therefore affect the curvature heavily. We denote for $x \sim y$,

$$\sharp(x,y) := \text{number of triangles which include } x, y \text{ as vertices} = \sum_{x_1, x_1\sim x, x_1\sim y} 1.$$

Remark 4 This quantity $\sharp(x, y)$ is related to the local clustering coefficient introduced by Watts–Strogatz [33],

$$c(x) := \frac{\text{number of edges between neighbors of } x}{\text{number of possible existing edges between neighbors of } x},$$

which measures the extent to which neighbors of x are directly connected. In fact, we have the relation

$$c(x) = \frac{1}{d_x(d_x-1)}\sum_{y, y\sim x}\sharp(x,y). \tag{3.7}$$

We will explore the relation between the curvature $\kappa(x, y)$ and the number of triangles $\sharp(x, y)$. A critical observation is that $\kappa(x, y)$ is symmetric w.r.t. x and y. So we try to express the curvature through symmetric quantities:

$$d_x \wedge d_y := \min\{d_x, d_y\}, \qquad d_x \vee d_y := \max\{d_x, d_y\}.$$

Theorem 3 *On a locally finite graph $G = (V, E)$, we have for any pair of neighboring vertices x, y,*

$$\kappa(x,y) \ge -\left(1-\frac{1}{d_x}-\frac{1}{d_y}-\frac{\sharp(x,y)}{d_x\wedge d_y}\right)_+ -\left(1-\frac{1}{d_x}-\frac{1}{d_y}-\frac{\sharp(x,y)}{d_x\vee d_y}\right)_+ +\frac{\sharp(x,y)}{d_x\vee d_y}.$$

Moreover, this inequality is sharp for certain graphs.

Remark 5 If $\sharp(x, y) = 0$, then this lower bound reduces to the one in Theorem 2.

Example 1 On a complete graph $\mathcal{K}_n$ $(n \ge 2)$ with n vertices, $\sharp(x, y) = n - 2$ for any x, y. So Theorem 3 implies

$$\kappa(x,y) \ge \frac{n-2}{n-1}.$$

In fact, we can easily check that the above inequality is an equality. Also notice that on those graphs, the local clustering coefficient $c(x) = 1$ attains the largest value.

Before carrying out the proof of Theorem 3, we fix some notations. The vertices z that are adjacent to x or y, where $x \sim y$, are divided into three classes.

- common neighbors of x, y: $z \sim x$ and $z \sim y$:
- x's own neighbors: $z \sim x, z \nsim y, z \neq y$;
- y's own neighbors: $z \sim y, z \nsim x, z \neq x$.

Proof of Theorem 3 We suppose w.l.o.g.,

$$d_x = d_x \vee d_y, \qquad d_y = d_x \wedge d_y.$$

In principle, our transfer plan moving m_x to m_y should be as follows.

1. Move the mass of $\frac{1}{d_x}$ from y to y's own neighbors;
2. Move a mass of $\frac{1}{d_y}$ from x's own neighbors to x;
3. Fill gaps using the mass at x's own neighbors. Filling the gaps at common neighbors costs 2 and the one at y's own neighbors costs 3.

A critical point will be whether (1) and (2) can be realized or not. It is easy to see that we can realize step (1) if and only if

$$1 - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_y} \geq \frac{1}{d_x}, \quad \text{or} \quad A := 1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \wedge d_y} \geq 0. \tag{3.8}$$

That is, after taking off the mass at x and common neighbors, m_y still has at least a mass of $\frac{1}{d_x}$. Step (2) can be realized if and only if

$$1 - \frac{1}{d_x} - \frac{\sharp(x, y)}{d_x} \geq \frac{1}{d_y}, \quad \text{or} \quad B := 1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \vee d_y} \geq 0. \tag{3.9}$$

That is, after taking off the mass at y and common neighbors, m_x still has enough mass to fill $\frac{1}{d_y}$. Obviously, $A \leq B$.

We will divide the discussion into three cases according to whether the first two steps can be realized or not.

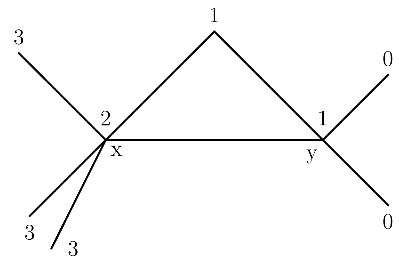
- $0 \leq A \leq B$. This means we can adopt the above transfer plan. By definition of $W_1(m_x, m_y)$, we get

$$\begin{aligned} W_1(m_x, m_y) &\leq \frac{1}{d_x} \times 1 + \frac{1}{d_y} \times 1 + \left(\frac{1}{d_y} - \frac{1}{d_x}\right) \times \sharp(x, y) \times 2 \\ &\quad + \left[1 - \frac{1}{d_x} - \frac{1}{d_y} - \left(\frac{1}{d_y} - \frac{1}{d_x}\right) \times \sharp(x, y) - \frac{1}{d_x}\sharp(x, y)\right] \times 3 \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y} - \frac{\sharp(x, y)}{d_y} - \frac{2\sharp(x, y)}{d_x}. \end{aligned}$$

Or in a symmetric way,

$$W_1(m_x, m_y) \leq 3 - \frac{2}{d_x \vee d_y} - \frac{2}{d_x \wedge d_y} - \frac{\sharp(x, y)}{d_x \wedge d_y} - \frac{2\sharp(x, y)}{d_x \vee d_y}. \tag{3.10}$$

Fig. 1 Mass moved from vertices with larger value to those with smaller ones



Moreover, in this case the following function f (as shown in Fig. 1) can be extended as a 1-Lipschitz function:

$$f(z)=\begin{cases}0, & \text{at } y\text{'s own neighbors;}\\ 1, & \text{at } y \text{ or common neighbors;}\\ 2, & \text{at } x;\\ 3, & \text{at } x\text{'s own neighbor,}\end{cases}$$

(that is, if there are no paths of length 1 between common neighbors and x's own neighbors, nor paths of length 1 or 2 between x's own neighbors and y's own ones) we have by Kantorovich duality

$$\begin{aligned}W_1(m_x,m_y)&\geq\frac{1}{d_x}\big[f(y)+3\big(d_x-1-\sharp(x,y)\big)+\sharp(x,y)\big]-\frac{1}{d_y}\big(f(x)+\sharp(x,y)\big)\\&=3-\frac{2}{d_x}-\frac{2}{d_y}-\frac{\sharp(x,y)}{d_y}-\frac{2\sharp(x,y)}{d_x}.\end{aligned}$$

That is, in this case, (3.10) should be an equality. In conclusion,

$$\kappa(x,y)\geq-2+\frac{2}{d_x}+\frac{2}{d_y}+\frac{\sharp(x,y)}{d_x\wedge d_y}+\frac{2\sharp(x,y)}{d_x\vee d_y},$$

and the "=" can be attained.

Remark 6 $A\geq 0$ is equivalent to

$$d_x\wedge d_y>1,\quad\text{and}\quad\sharp(x,y)\leq d_x\wedge d_y-1-\frac{d_x\wedge d_y}{d_x\vee d_y}.$$

Since $\sharp(x,y)\in\mathbf{Z}$, we know that $d_x\wedge d_y\geq 2$ and $\sharp(x,y)\leq d_x\wedge d_y-2$. This means both x and y have at least one own neighbor.

If $A<0$, we get

$$d_x\wedge d_y-1-\frac{d_x\wedge d_y}{d_x\vee d_y}<\sharp(x,y)\leq d_x\wedge d_y-1.$$

I.e., $\sharp(x,y)=d_x\wedge d_y-1$. This means the vertex with smaller degree has no own neighbors.

• $A < 0 \leq B$. In this case we cannot realize step (1) but step (2) can be realized. By the above remark, $A < 0$ implies that y has no own neighbors. Our transfer plan should be step (2) at first. Since $B \geq 0$ also implies

$$1 - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x} \geq \frac{1}{d_x}, \tag{3.11}$$

so we can move the mass of $\frac{1}{d_x}$ at y for distance 1 to common neighbors. Finally, we fill the gap at common neighbors for distance 2. In a formula,

$$\begin{aligned} W_1(m_x, m_y) &\leq \frac{1}{d_x} \times 1 + \frac{1}{d_y} \times 1 + \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x}\right) \times 2 \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{2\sharp(x,y)}{d_x}. \end{aligned}$$

Or in a symmetric manner,

$$W_1(m_x, m_y) \leq 2 - \frac{1}{d_x \vee d_y} - \frac{1}{d_x \wedge d_y} - \frac{2\sharp(x,y)}{d_x \vee d_y}. \tag{3.12}$$

Moreover, in case the following function f can be extended as a 1-Lipschitz one,

$$f(z) = \begin{cases} 0, & \text{at common neighbors;} \\ 1, & \text{at } x \text{ and } y; \\ 2, & \text{at } x\text{'s own neighbor,} \end{cases}$$

(that is, if there are no paths of length 1 between common neighbors and x's own neighbors) we have by Kantorovich duality

$$\begin{aligned} W_1(m_x, m_y) &\geq \frac{1}{d_x}\big[f(y) + 2\big(d_x - 1 - \sharp(x,y)\big)\big] - \frac{1}{d_y} f(x) \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{2\sharp(x,y)}{d_x}. \end{aligned}$$

In conclusion,

$$\kappa(x,y) \geq -1 + \frac{1}{d_x} + \frac{1}{d_y} + \frac{2\sharp(x,y)}{d_x \vee d_y},$$

and the "=" can be attained.

Remark 7 Noting that if $\sharp(x,y) = d_x \wedge d_y - 1$ then $B \geq 0$ is equivalent to

$$d_x \vee d_y \geq \frac{d_x \wedge d_y}{d_x \wedge d_y - 1} d_x \wedge d_y. \tag{3.13}$$

In this case, one of d_x, d_y has no own neighbors, and if the other one has sufficiently many own neighbors, $B \geq 0$ will be satisfied.

• $A \le B < 0$. In this case, neither step (1) nor (2) is applicable. Also, y has no own neighbor, and $B < 0$ implies that we can move all the mass at x's own neighbors to x at first. And then we move the mass of $\frac{1}{d_x}$ at y for distance 1 to fill the gaps at x and the common neighbors. In a formula,

$$W_1(m_x, m_y) \le \left(1 - \frac{\sharp(x,y)}{d_x}\right) \times 1 = 1 - \frac{\sharp(x,y)}{d_x}.$$

Or in a symmetric way,

$$W_1(m_x, m_y) \le 1 - \frac{\sharp(x,y)}{d_x \vee d_y}. \tag{3.14}$$

We can find a 1-Lipschitz function

$$f(z) = \begin{cases} 0, & \text{at } x \text{ and common neighbors;} \\ 1, & \text{at } y \text{ and } x\text{'s own neighbors.} \end{cases}$$

Then by Kantorovich duality

$$\begin{aligned} W_1(m_x, m_y) &\ge \frac{1}{d_x}\big(f(y) + d_x - 1 - \sharp(x,y)\big) - \frac{1}{d_y} \times 0 \\ &= 1 - \frac{\sharp(x,y)}{d_x}. \end{aligned}$$

In this case f can be extended to a 1-Lipschitz function on the graph, so we get finally

$$\kappa(x,y) = \frac{\sharp(x,y)}{d_x}.$$

Luckily, we can write those three cases in a uniform formula. □

Remark 8 From extending f to a 1-Lipschitz function, we see that the paths of length 1 or 2 between neighbors of x and y have an important effect on the curvature. That is, in addition to triangles, quadrangles and pentagons are also related to Ollivier's Ricci curvature. But polygons with more than five edges do not impact it.

Remark 9 If we see the graph $G = (V, E)$ as a metric measure space (G, d, m), then the term $\sharp(x,y)/d_x \vee d_y$ is exactly $m_x \wedge m_y(G) := m_x(G) - (m_x - m_y)_+(G)$, i.e. the intersection measure of m_x and m_y. From a metric view, the vertices x_1 that satisfy $x_1 \sim x$, $x_1 \sim y$ constitute the intersection of the unit metric spheres $S_x(1)$ and $S_y(1)$.

From Theorem 3, we can force the curvature $\kappa(x,y)$ to be positive by increasing the number $\sharp(x,y)$.

Theorem 4 *On a locally finite graph $G = (V, E)$, for any neighboring x, y, we have*

$$\kappa(x,y) \le \frac{\sharp(x,y)}{d_x \vee d_y}. \tag{3.15}$$

Proof Since except for the mass at common neighbors which we need not move, the others have to be moved for a distance at least 1, we have

$$W_1(m_x, m_y) \geq \left(1 - \frac{\sharp(x,y)}{d_x \vee d_y}\right) \times 1. \qquad \square$$

So if $\kappa(x,y) > 0$, then $\sharp(x,y)$ is at least 1. Moreover, if $\kappa(x,y) \geq k > 0$, we have

$$\sharp(x,y) \geq \lceil k d_x \vee d_y \rceil, \tag{3.16}$$

where $\lceil a \rceil := \min\{A \in \mathbf{Z} | A \geq a\}$, for $a \in R$.

We will denote $D(x) := \max_{y, y \sim x} d_y$. By the relation (3.7), we can get immediately

Corollary 1 *The scalar curvature at x can be controlled by the local clustering coefficient at x,*

$$\frac{d_x - 1}{d_x} c(x) \geq \kappa(x) \geq -2 + \frac{d_x - 1}{d_x \vee D(x)} c(x).$$

Remark 10 In fact in some special cases, we can get more precise lower bounds:

$$\kappa(x) \geq \begin{cases} -2 + \frac{2}{d_x} + \frac{2}{D(x)} + [\frac{(d_x-1)}{d_x} + \frac{2(d_x-1)}{d_x \vee D(x)}]c(x), & \text{if } A \geq 0 \text{ for all } y \sim x; \\ -1 + \frac{1}{d_x} + \frac{1}{D(x)} + \frac{2(d_x-1)}{d_x \vee D(x)} c(x), & \text{if } A < 0 \leq B \text{ for all } y \sim x; \\ \frac{d_x - 1}{d_x \vee D(x)} c(x), & \text{if } B < 0 \text{ for all } y \sim x. \end{cases}$$

3.2 Weighted Graphs

The preceding considerations readily extend to weighted graphs.

Theorem 5 *On a weighted locally finite graph $G = (V, E)$, we have*

$$\kappa(x,y) \geq -2\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y}\right)_+. \tag{3.17}$$

Moreover, weighted trees attain this lower bound.

Theorem 6 *On a weighted locally finite graph $G = (V, E)$, we have*

$$\begin{aligned} \kappa(x,y) \geq & -\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \vee \frac{w_{x_1 y}}{d_y}\right)_+ \\ & - \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}\right)_+ \\ & + \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}. \end{aligned}$$

The inequality is sharp for certain graphs.

Remark 11 Notice that the term replacing the number of triangles here satisfies

$$\sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y} = m_x \wedge m_y(G).$$

Proof Similar to the proof of Theorem 3, we need to understand the following two terms:

$$A_w := 1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1x}}{d_x} \vee \frac{w_{x_1y}}{d_y},$$
$$B_w := 1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}.$$

Only the transfer plan in the case $A_w < 0 \leq B_w$ needs a more careful discussion. □

Theorem 7 *On a weighted locally finite graph $G = (V, E)$, we have for any neighboring x, y*

$$\kappa(x, y) \leq \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}. \tag{3.18}$$

4 Curvature-Dimension Inequalities

In this section, we establish curvature-dimension inequalities on locally finite graphs. A very interesting one is the inequality under the condition $\kappa \geq k > 0$. Curvature-dimension inequalities on locally finite graphs are studied in Lin–Yau [21]. We first state a detailed version of their results. Let us denote $D_w(x) := \max_{y, y \sim x} \frac{d_y}{w_{yx}}$. Notice that on an unweighted graph, this is the $D(x)$ we used in Sect. 3.

Theorem 8 *On a weighted locally finite graph $G = (V, E)$, the Laplace operator Δ satisfies*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}\big(\Delta f(x)\big)^2 + \left(\frac{2}{D_w(x)} - 1\right)\Gamma(f, f)(x). \tag{4.1}$$

Remark 12 Since in this case we attach the measure (2.1), we get

$$\mathrm{Hf}(x) = \frac{1}{4}\frac{1}{d_x} \sum_{y, y \sim x} \frac{w_{xy}}{d_y} \sum_{z, z \sim y} w_{yz}\big(f(x) - 2f(y) + f(z)\big)^2.$$

We only need to choose special $z = x$ in the second sum and then (2.8) and (2.9) imply the theorem.

4.1 Unweighted Graphs

We again restrict ourselves to unweighted graphs.

We observe that the existence of triangles causes cancelations in calculating the term $\mathrm{Hf}(x)$. This gives

Theorem 9 *On a locally finite graph $G=(V,E)$, the Laplace operator satisfies*

$$\Gamma_2(f,f)(x) \geq \frac{1}{2}\big(\Delta f(x)\big)^2 + \bigg(\frac{1}{2}t(x)-1\bigg)\Gamma(f,f)(x), \tag{4.2}$$

where

$$t(x) := \min_{y,y\sim x}\bigg(\frac{4}{d_y} + \frac{1}{D(x)}\sharp(x,y)\bigg).$$

Remark 13 Notice that if there is a vertex y, $y\sim x$, such that $\sharp(x,y)=0$, this will reduce to (4.1).

Proof Starting from (2.9), the main work is to compare $\mathrm{Hf}(x)$ with

$$\Gamma(f,f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y,y\sim x}\big(f(y)-f(x)\big)^2.$$

First we try to write out $\mathrm{Hf}(x)$ as

$$\mathrm{Hf}(x) = \frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}\bigg[\frac{4}{d_y}\big(f(x)-f(y)\big)^2 + \frac{1}{d_y}\sum_{z,z\sim y,z\neq x}\big(f(x)-2f(y)+f(z)\big)^2\bigg].$$

If there is a vertex x_1 which satisfies $x_1\sim x$, $x_1\sim y$, we have

$$\begin{aligned}
&\frac{1}{d_y}\big(f(x)-2f(y)+f(x_1)\big)^2 + \frac{1}{d_{x_1}}\big(f(x)-2f(x_1)+f(y)\big)^2\\
&\quad\geq \frac{1}{D(x)}\Big[\big(f(x)-f(y)\big)^2 + \big(f(y)-f(x_1)\big)^2 + 2\big(f(x)-f(y)\big)\big(f(x_1)-f(y)\big)\\
&\qquad + \big(f(x)-f(x_1)\big)^2 + \big(f(y)-f(x_1)\big)^2 + 2\big(f(y)-f(x_1)\big)\big(f(x)-f(x_1)\big)\Big]\\
&\quad= \frac{1}{D(x)}\Big[\big(f(x)-f(y)\big)^2 + 4\big(f(y)-f(x_1)\big)^2 + \big(f(x)-f(x_1)\big)^2\Big]\\
&\quad\geq \frac{1}{D(x)}\big(f(x)-f(y)\big)^2.
\end{aligned}\tag{4.3}$$

So the existence of a triangle which includes x and y will give another term

$$\frac{1}{D(x)}\big(f(y)-f(x)\big)^2$$

to the sum in $\mathrm{Hf}(x)$. Since this effect is symmetric w.r.t. y and x_1, we can get

$$\begin{aligned}\mathrm{Hf}(x) &\geq \frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}\left(\frac{4}{d_y}+\frac{1}{D(x)}\sharp(x,y)\right)\big(f(y)-f(x)\big)^2\\ &\geq t(x)\frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}\big(f(y)-f(x)\big)^2\\ &= t(x)\cdot\frac{1}{2}\Gamma(f,f)(x).\end{aligned}$$

Inserting this into (2.9) completes the proof. □

Recalling Theorem 4 and the subsequent discussion, we get the following curvature-dimension inequalities on graphs with positive Ollivier–Ricci curvature.

Corollary 2 *On a locally finite graph $G=(V,E)$, if $\kappa(x,y)>0$, then we have*

$$\Gamma_2(f,f)(x)\geq\frac{1}{2}\big(\Delta f(x)\big)^2+\left(\frac{5}{2D(x)}-1\right)\Gamma(f,f)(x). \tag{4.4}$$

Corollary 3 *On a locally finite graph $G=(V,E)$, if $\kappa(x,y)\geq k>0$, then we have*

$$\begin{aligned}\Gamma_2(f,f)(x) \geq{}& \frac{1}{2}\big(\Delta f(x)\big)^2\\ &+\left(\frac{1}{2}\min_{y,y\sim x}\left\{\frac{4}{d_y}+\frac{\lceil kd_x\vee d_y\rceil}{D(x)}\right\}-1\right)\Gamma(f,f)(x).\end{aligned} \tag{4.5}$$

Remark 14 Observe that a rough inequality in this case is

$$\Gamma_2(f,f)(x)\geq\frac{1}{2}\big(\Delta f(x)\big)^2+\left(\frac{2}{D(x)}+\frac{kd_x}{2D(x)}-1\right)\Gamma(f,f)(x).$$

Comparing this one with (4.1), we see that positive κ increases the curvature function here.

Remark 15 We point out that the condition $\kappa(x,y)\geq k>0$ implies that the diameter of the graph is bounded by $\frac{2}{k}$ (see Proposition 23 in Ollivier [24]). So in this case the graph is a finite one.

Let us revisit the example of a complete graph $\mathcal{K}_n$ $(n\geq 2)$ with n vertices. Recall in Example 1, we know

$$\kappa(x,y)=\frac{n-2}{n-1},\quad \forall x,y.$$

For the curvature-dimension inequality on $\mathcal{K}_n$, Theorem 9 or Corollary 3 using the above κ implies

$$\begin{aligned}\Gamma_2(f,f) &\geq \frac{1}{2}(\Delta f)^2 + \left(\frac{2}{n-1} - 1 + \frac{1}{2}\frac{n-2}{n-1}\right)\Gamma(f,f) \\ &= \frac{1}{2}(\Delta f)^2 + \frac{4-n}{2(n-1)}\Gamma(f,f). \end{aligned} \tag{4.6}$$

Moreover, the curvature term in the above inequality cannot be larger. To see this, we calculate, using the same trick as in (4.3),

$$\begin{aligned}\mathrm{Hf}(x) &= \frac{1}{4(n-1)^2}\sum_{y,y\sim x}\sum_{z,z\sim x}\big(f(x)-2f(y)+f(z)\big)^2 \\ &= \frac{n+2}{2(n-1)}\Gamma(f,f)(x) + \frac{1}{(n-1)^2}\sum_{(x_1,x_2)}\big(f(x_1)-f(x_2)\big)^2,\end{aligned}$$

where $\sum_{(x_1,x_2)}$ means the sum over all unordered pairs of neighbors of x. Recalling (2.9), we get

$$\begin{aligned}\Gamma_2(f,f)(x) = \frac{1}{2}(\Delta f)^2(x) + \frac{4-n}{2(n-1)}\Gamma(f,f)(x) \\ + \frac{1}{(n-1)^2}\sum_{(x_1,x_2)}\big(f(x_1)-f(x_2)\big)^2.\end{aligned} \tag{4.7}$$

For any vertex x, we can find a particular function $\overline{f}$,

$$\overline{f}(z) = \begin{cases} 2, & \text{when } z = x; \\ 1, & \text{when } z \sim x, \end{cases} \tag{4.8}$$

such that the last term in (4.7) vanishes, and $\Gamma(\overline{f},\overline{f}) \neq 0$. This means the curvature term in (4.6) is optimal for dimension parameter 2.

But the curvature term $\frac{4-n}{2(n-1)}$ behaves very differently from κ. In fact as $n \to +\infty$,

$$\frac{4-n}{2(n-1)} \searrow -\frac{1}{2} \quad \text{whereas } \kappa \nearrow 1.$$

To get a curvature-dimension inequality with a curvature term which behaves like κ, it seems that we should adjust the dimension parameter. In fact, we have

Proposition 3 *On a complete graph $\mathcal{K}_n$ $(n \geq 2)$ with n vertices, the Laplace operator Δ satisfies for $m \in [1, +\infty]$,*

$$\Gamma_2(f,f)(x) \geq \frac{1}{m}\big(\Delta f(x)\big)^2 + \left(\frac{4-n}{2(n-1)} + \frac{m-2}{m}\right)\Gamma(f,f)(x). \tag{4.9}$$

Moreover, for every fixed dimension parameter m, the curvature term is optimal.

Proof We have from (4.7)

$$\Gamma_2(f,f)(x) = \frac{1}{m}(\Delta f)^2(x) + \frac{4-n}{2(n-1)}\Gamma(f,f)(x) + \frac{1}{(n-1)^2}\sum_{(x_1,x_2)}\big(f(x_1)-f(x_2)\big)^2 + \left(\frac{1}{2}-\frac{1}{m}\right)(\Delta f)^2.$$

Let us denote the sum of the last two terms by I. Then we have

$$\begin{aligned} I &= \frac{1}{(n-1)^2}\Bigg\{\left(\frac{1}{2}-\frac{1}{m}\right)\sum_{y,y\sim x}\big(f(y)-f(x)\big)^2 \\ &\quad + \sum_{(x_1,x_2)}\Bigg[\big(f(x_1)-f(x)\big)^2 + \big(f(x_2)-f(x)\big)^2 \\ &\quad + \left(2\left(\frac{1}{2}-\frac{1}{m}\right)-2\right)\big(f(x_1)-f(x)\big)\big(f(x_2)-f(x)\big)\Bigg]\Bigg\} \\ &= \frac{1}{(n-1)^2}\Bigg[\left(\frac{1}{2}-\frac{1}{m}\right)\sum_{y,y\sim x}\big(f(y)-f(x)\big)^2 \\ &\quad + \left(1-\frac{m+2}{2m}\right)(n-2)\sum_{y,y\sim x}\big(f(y)-f(x)\big)^2 \\ &\quad + \sum_{(x_1,x_2)}\frac{m+2}{2m}\big(f(x_1)-f(x_2)\big)^2\Bigg] \\ &= \frac{m-2}{m}\Gamma(f,f)(x) + \frac{m+2}{2m(n-1)^2}\sum_{(x_1,x_2)}\big(f(x_1)-f(x_2)\big)^2. \end{aligned}$$

This finishes the proof. □

An interesting point appears when we choose the dimension parameter m of $\mathcal{K}_n$ as $n-1$. Then we have

$$\Gamma_2(f,f) \geq \frac{1}{n-1}(\Delta f)^2 + \frac{1}{2}\frac{n-2}{n-1}\Gamma(f,f),$$

where the curvature term is exactly $\frac{1}{2}\kappa$. From the fact that $\mathcal{K}_n$ could be considered as the boundary of a $(n-1)$ dimensional simplex, the m we choose here seems also natural.

Remark 16 We point out another similar fact here. On a locally finite graph with maximal degree D and minimal degree larger than 1, Theorem 2 and Theorem 8 imply that

$$\kappa(x,y) \geq 2\left(\frac{2}{D}-1\right), \quad \forall x, y, \tag{4.10}$$

and

$$\Gamma_2(f,f) \geq \frac{1}{2}(\Delta f)^2 + \left(\frac{2}{D} - 1\right)\Gamma(f,f), \tag{4.11}$$

respectively. It is not difficult to see that for regular trees with degree larger than 1, the curvature term in (4.11) is optimal. (Just consider the extension of the function (4.8), taking values 0 on vertices which are not x and neighbors of x there.) So on regular trees, the curvature term is also exactly $\frac{1}{2}\kappa$.

Remark 17 In Erdös–Harary–Tutte [15], they define the dimension of a graph G as the minimum number n such that G can be embedded into a n dimensional Euclidean space with every edge of G having length 1. It is interesting that by their definition, the dimension of $\mathcal{K}_n$ is also $n-1$ and the dimension of any tree is at most 2.

From the above observations, it seems natural to expect stronger relations between the lower bound of κ and the curvature term in the curvature-dimension inequality if one chooses proper dimension parameters.

4.2 Weighted Graphs

We have similar results on weighted graphs here, with similar proofs.

Theorem 10 *On a weighted locally finite graph $G=(V,E)$, the Laplace operator satisfies*

$$\Gamma_2(f,f)(x) \geq \frac{1}{2}\big(\Delta f(x)\big)^2 + \left(\frac{1}{2}t_w(x) - 1\right)\Gamma(f,f)(x), \tag{4.12}$$

where

$$t_w(x) := \min_{y, y\sim x}\left\{\frac{4w_{xy}}{d_y} + \sum_{x_1, x_1\sim x, x_1\sim y}\left(\frac{w_{xy}}{d_y} \wedge \frac{w_{xx_1}}{d_{x_1}}\right)\frac{w_{x_1y}}{w_{xy}}\right\}.$$

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