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Structure Results for Multiple Tilings in 3D

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Abstract We study multiple tilings of 3-dimensional Euclidean space by a convex body. In a multiple tiling, a convex body P is translated with a discrete multiset Λ in such a way that each point of $\mathbb{R}^d$ gets covered exactly k times, except perhaps the translated copies of the boundary of P. It is known that all possible multiple tilers in $\mathbb{R}^3$ are zonotopes. In $\mathbb{R}^2$ it was known by the work of Kolountzakis (Discrete Comput Geom 23(4):537–553, 2000) that, unless P is a parallelogram, the multiset of translation vectors Λ must be a finite union of translated lattices (also known as quasi periodic sets). In that work (Kolountzakis, Discrete Comput Geom 23(4):537–553, 2000) the author asked whether the same quasi-periodic structure on the translation vectors would be true in $\mathbb{R}^3$. Here we prove that this conclusion is indeed true for $\mathbb{R}^3$. Namely, we show that if P is a convex multiple tiler in $\mathbb{R}^3$, with a discrete multiset Λ of translation vectors, then Λ has to be a finite union of translated lattices,

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unless P belongs to a special class of zonotopes. This exceptional class consists of two-flat zonotopes P, defined by the Minkowski sum of two 2-dimensional symmetric polygons in $\mathbb{R}^3$, one of which may degenerate into a single line segment. It turns out that rational two-flat zonotopes admit a multiple tiling with an aperiodic (nonquasi-periodic) set of translation vectors Λ. We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

Keywords Multiple tilings · Tilings · Lattices · Fourier transform · Zonotope · Quasi-periodicity

1 Introduction

The study of multiple tilings of Euclidean space began in 1936, when the famous Minkowski facet-to-facet conjecture [18] for classical tilings was extended to the setting of k-tilings with the unit cube, by Furtwängler [4]. Minkowski's facet-to-facet conjecture states that for any *lattice* tiling of $\mathbb{R}^d$ by translations of the unit cube, there exist at least two translated cubes that share a facet (face of co-dimension 1). This conjecture was strengthened by Keller [10] who conjectured the same conclusion for *any* cube tiling, not just lattice tilings. It was also strengthened in a different direction by Furtwängler [4] who, again, conjectured the same conclusion for any *multiple* lattice tiling.

To define a multiple tiling, suppose we translate a convex body P with a discrete multiset Λ, in such a way that each point of $\mathbb{R}^d$ gets covered exactly k times, except perhaps the translated copies of the boundary of P. We then call such a body a k-tiler, and such an action has been given the following names in the literature: a *k-tiling*, a *tiling at level k*, a *tiling with multiplicity k*, and sometimes simply a *multiple tiling*. We may use any of these synonyms here, and we immediately point out, for polytopes P, a trivial but useful algebraic equivalence for a tiling at level k:

$$\sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda}(x) = k, \tag{1}$$

for almost all $x \in \mathbb{R}^d$, where $\mathbf{1}_P$ is the indicator function of the polytope P.

Furtwängler's conjecture was disproved by Hajós [8] for dimension larger than 3 and for $k \geq 9$ while Furtwängler himself [4] proved it for dimension at most 3. Hajós [9] also proved Minkowski's conjecture in all dimensions. The ideas of Furtwängler were subsequently extended (but still restricted to cubes) by the important work of Perron [19], Robinson [20], Szabó [24], Gordon [5] and Lagarias and Shor [14]. These authors showed that for some levels k and dimensions d and under the lattice assumption as well as not, a facet-to-facet conclusion for k-tilings is true in $\mathbb{R}^d$, while for most values of k and d it is false.

There is a vast literature on the study of coverings of Euclidean space by a convex body, and an equally vast body of work on classical tilings by translations of one convex body, which must necessarily be a polytope (see for example [3,7]). On the one hand, when we consider a k-tiling polytope P, we obtain an exact covering of $\mathbb{R}^d$, in the sense that almost every point of $\mathbb{R}^d$ gets covered exactly k times. On the other hand, the family of k-tilers is much larger than the family of 1-tilers. Hence the study of k-tilings lies somewhere between coverings and 1-tilings.

It was known to Bolle [2] that in $\mathbb{R}^2$, every k-tiling convex polytope has to be a centrally symmetric polygon, and using combinatorial methods Bolle [2] gave a characterization for all polygons in $\mathbb{R}^2$ that admit a k-tiling with a *lattice* Λ of translation vectors. Kolountzakis [13] proved that if a convex polygon P tiles $\mathbb{R}^2$ multiply with *any* discrete multiset Λ, then Λ must be a *finite union of two-dimensional lattices*. The ingredients of Kolountzakis' proof include the idempotent theorem for the Fourier transform of a measure. Roughly speaking, the idempotent theorem of Meyer [17] tells us that if the square of the Fourier transform of a measure is itself, then the support of the measure is contained in a finite union of lattices. To put our main result into its proper context, we record here the precise result of Kolountzakis. A multiple tiling is called *quasi-periodic* if its multiset of discrete translation vectors Λ is a finite union of translated lattices, not necessarily all of the same dimension.

Theorem (Kolountzakis, 2002 [11]) *Suppose that K is a symmetric convex polygon which is not a parallelogram. Then K admits only quasi-periodic multiple tilings if any.*

Here we extend this result to $\mathbb{R}^3$, and we also find a fascinating class of polytopes analogous to the parallelogram of the theorem above. To describe this class, we first recall the definition of a *zonotope*, which is the Minkowski sum of a finite number of line segments. In other words, a zonotope equals a translate of $[-\mathbf{v}_1, \mathbf{v}_1] + \cdots + [-\mathbf{v}_N, \mathbf{v}_N]$, for some positive integer N and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N \in \mathbb{R}^d$. A zonotope may equivalently be defined as the projection of some l-dimensional cube. A third equivalent condition is that for a d-dimensional zonotope, all of its k-dimensional faces are centrally symmetric, for $1 \le k \le d$. For example, the zonotopes in $\mathbb{R}^2$ are the centrally symmetric polygons.

We shall say that a polytope $P \subseteq \mathbb{R}^3$ is a *two-flat zonotope* if P is the Minkowski sum of $n + m$ line segments which lie in the union of two different two-dimensional subspaces H_1 and H_2. In other words, H_1 contains n of the segments and H_2 contains m of the segments (if one of the segments belongs to both H_1 and H_2 we list it twice, once for each plane). Equivalently, P may be thought of as the Minkowski sum of two 2-dimensional symmetric polygons one of which may degenerate into a single line segment.

Recently, a structure theorem for convex k-tilers in $\mathbb{R}^d$ was found, and is as follows.

Theorem (Gravin, Robins, Shiryaev 2011 [6]) *If a convex polytope k-tiles $\mathbb{R}^d$ by translations, then it is centrally symmetric and its facets are centrally symmetric.*

In the present context of $\mathbb{R}^3$, it follows immediately from the latter theorem that a k-tiler $P \subset \mathbb{R}^3$ is necessarily a zonotope. In this paper we extend the result of Kolountzakis [11] from $\mathbb{R}^2$ to $\mathbb{R}^3$, providing a structure theorem for multiple tilings by polytopes in three dimensions.

Main Theorem *Suppose a polytope P k-tiles $\mathbb{R}^3$ with a discrete multiset Λ, and suppose that P is not a two-flat zonotope. Then Λ is a finite union of translated lattices.*

It turns out that if P is a rational two-flat zonotope, then P admits a k-tiling with a non-quasi-periodic set of translation vectors Λ, as we show in our Corollary 7.1. For some of the classical study of 1-tilings, and their interesting connections to zonotopes, the reader may refer to the work of [15,16,22,25], and [1]. Here we find it very useful to use the intuitive language of distributions [21,23] in order to think—and indeed discover—facts about k-tilings. To that end we introduce the distribution (which is locally a measure)

$$\delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda, \tag{2}$$

where δ_λ is the Dirac delta function at $\lambda \in \mathbb{R}^d$. To develop some intuition, we may check formally that

$$\delta_\Lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \delta_\lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda},$$

so that from the first definition (1) of k-tiling, we see that a polytope P is a k-tiler if and only if

$$\delta_\Lambda * \mathbf{1}_P = k. \tag{3}$$

The paper is modularized into short sections that highlight each step separately, and the organization runs as follows. In Sect. 3 we compute the Fourier transform of any 4-legged frame of a polytope, and show that its zeros form a certain countable union of hyperplanes. In Sect. 4 we find a sufficient condition, which we call the *intersection property*, for the Fourier transform of δ_Λ to have a discrete support. Then we show that if P is a k-tiler, and if the intersection property holds for all 4-legged frames of P, then supp $\widehat{\delta_\Lambda}$ (the support of the distribution $\widehat{\delta_\Lambda}$, the Fourier Transform of the distribution δ_Λ) is a discrete set in $\mathbb{R}^3$, of bounded density.

In Sect. 5, we prove that the intersection property implies the quasi-periodicity of Λ.

In Sect. 6 we discover a fascinating family of k-tilers, associated to a non-discrete supp $\widehat{\delta_\Lambda}$. We prove that if P tiles $\mathbb{R}^3$ with multiplicity, by translations with a discrete multiset Λ, and the intersection property fails to hold, then P must be a two-flat zonotope.

The proof of the Main Theorem is also given in Sect. 6; this proof is quite short since it just strings together all of the results of the previous sections. In the final section, we show that each rational two-flat zonotope admits a very peculiar non-quasi-periodic k-tiling. We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

2 Preliminaries

Suppose the polytope P tiles multiply with the translates $\Lambda \subseteq \mathbb{R}^d$. We will need to understand some basic facts about how the Λ points are distributed, for example in Theorem 5.1 below.

Definition 2.1 (Uniform density) A multiset $\Lambda \subseteq \mathbb{R}^d$ has asymptotic density ρ if

$$\lim_{R\to\infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \to \rho$$

uniformly in $x \in \mathbb{R}^d$. In this case we write $\rho = \operatorname{dens} \Lambda$. Another (weaker) notion is that of *bounded* density—we say that Λ has (uniformly) bounded density if $\#(\Lambda \cap B_R(x))/|B_R(x)| \le M$ for $x \in \mathbb{R}^d$, and $R > 1$. We say then that Λ has density (uniformly) bounded by M.

Since we intend to speak of the Fourier Transform of δ_Λ it is important to us that if Λ has bounded density then δ_Λ is a tempered distribution [21,23], and, therefore, its Fourier Transform $\widehat{\delta_\Lambda}$ is well defined. And, it is almost obvious by comparing volumes that, if a polytope P k-tiles $\mathbb{R}^d$ with translates Λ then Λ has density $k/|P|$.

For any symmetric polytope P, and any face $F \subset P$, we define F^- to be the face of P symmetric to F with respect to P's center of symmetry. We call F^- the *opposite face* of F.

Throughout the paper, we use the notation $\mathbf{x}^\perp$ to denote the perpendicular subspace, of codimension 1, to the vector $\mathbf{x}$. We also use the standard convention of boldfacing all vectors, to differentiate between $\mathbf{v}$ and v, for example. We furthermore use the convention that $[\mathbf{e}]$ denotes the 1-dimensional line segment from 0 to the endpoint of the vector $\mathbf{e}$. Whenever it is clear from context, we will also write $[e]$ to denote the same line segment—for example, in the case that e denotes an edge of a polytope.

We let $\mathsf{Z}(f)$ be the zero set of the function f.

Definition 2.2 [4-legged-frame of a polytope]

(a) Suppose $P \in \mathbb{R}^3$ is a zonotope (symmetric polytope with symmetric facets). A collection of four (one-dimensional) edges of P is called a 4-*legged-frame* if whenever e is one of the edges then there exist two vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ such that the four edges are

$$[e],\ [e] + \boldsymbol{\tau}_1, [e] + \boldsymbol{\tau}_2 \text{ and } [e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2,$$

and such that the edges $[e]$ and $[e] + \boldsymbol{\tau}_1$ belong to the same face of P and the edges $[e] + \boldsymbol{\tau}_2$ and $[e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ belong to the opposite face.

(b) For a set of four legs as above the *leg measure* is the measure supported on the legs and is equal to arc-length on the two legs $[e]$ and $[e] + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ and minus arc-length on the two legs $[e] + \boldsymbol{\tau}_1$ and $[e] + \boldsymbol{\tau}_2$. We denote this measure by μ_{e,τ_1,τ_2}. The leg measure is defined up to sign.

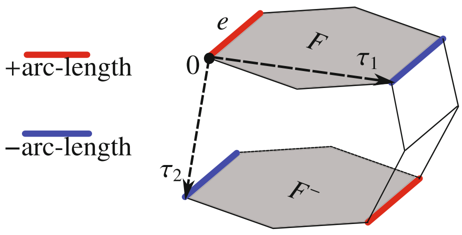


Fig. 1 Four legs of a convex polytope and leg-measure. The *colored* segments are edges of the polytope. Translating an edge by $\pm\tau_1$ gives you the opposite edge on a face. Translating by $\pm\tau_2$ takes you to the opposite face (Color figure online)

Remark 2.1 A set of four legs of a symmetric polytope with symmetric faces is determined uniquely if we know two opposite edges on a face (these edges are the first two legs). The other two legs are then the corresponding opposite edges on the opposite face (Fig. 1).

3 The Fourier Transform of a 4-Legged Frame

Lemma 3.1 *Suppose* $\mathbf{e}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{R}^3$ *are linearly independent and consider the measure*
$\mu = \mu_{e,\tau_1,\tau_2}$ *(see Definition 2.2). Then the zero-set of the Fourier Transform* $\hat{\mu}$*, is*

$$\mathsf{Z}(\hat{\mu}) = \mathbb{H}_{-0}(\mathbf{e}) \cup \mathbb{H}(\boldsymbol{\tau}_1) \cup \mathbb{H}(\boldsymbol{\tau}_2), \tag{4}$$

where, if $\mathbf{x}$ *is a non-zero vector and* $\mathbf{x}^* = \frac{\mathbf{x}}{|\mathbf{x}|^2}$ *is its geometric inverse,*

$$\mathbb{H}(\mathbf{x}) = \mathbb{Z}\mathbf{x}^* + \mathbf{x}^\perp \tag{5}$$

and

$$\mathbb{H}_{-0}(\mathbf{x}) = \big(\mathbb{Z} \setminus \{0\}\big)\mathbf{x}^* + \mathbf{x}^\perp. \tag{6}$$

Here $\mathbf{x}^\perp$ *is the hyperplane orthogonal to the vector* $\mathbf{x}$*, so that* $\mathbb{H}(\mathbf{x}) = \mathbb{Z}\mathbf{x}^* + \mathbf{x}^\perp$ *is a collection of parallel hyperplanes, orthogonal to* $\mathbf{x}$ *spaced by* $1/|\mathbf{x}|$.

Proof Translating a measure does not alter the zero-set of its FT so we may translate μ so that 0 is the midpoint of the first line segment, which now runs from $-\mathbf{e}/2$ to $\mathbf{e}/2$. Denoting by ν the arc-length measure on this line segment and writing $\alpha = \delta_0 - \delta_{\tau_1}$ and $\beta = \delta_0 - \delta_{\tau_2}$ we obtain μ as a convolution:

$$\mu = \nu * \alpha * \beta.$$

Taking the FT we get that

$$\mathsf{Z}(\hat{\mu}) = \mathsf{Z}(\hat{\nu}) \cup \mathsf{Z}(\hat{\alpha}) \cup \mathsf{Z}(\hat{\beta}).$$

Based on the calculation of the FT of the indicator function of $[-\frac{1}{2}, \frac{1}{2}]$

$$\int_{-1/2}^{1/2} \mathrm{e}^{-2\pi i \xi x}\, \mathrm{d}x = \frac{\sin \pi \xi}{\pi \xi}$$

we conclude that

$$\widehat{\nu}(\mathbf{u}) = |\mathbf{e}| \frac{\sin \pi \langle \mathbf{u}, \mathbf{e} \rangle}{\pi \langle \mathbf{u}, \mathbf{e} \rangle}.$$

One also immediately obtains the formulas

$$\widehat{\alpha}(\mathbf{u}) = 2i \mathrm{e}^{-\pi i \langle \boldsymbol{\tau}_1, \mathbf{u} \rangle} \sin \pi \langle \boldsymbol{\tau}_1, \mathbf{u} \rangle$$

and

$$\widehat{\beta}(\mathbf{u}) = 2i \mathrm{e}^{-\pi i \langle \boldsymbol{\tau}_2, \mathbf{u} \rangle} \sin \pi \langle \boldsymbol{\tau}_2, \mathbf{u} \rangle.$$

Since $\widehat{\nu}$, $\widehat{\alpha}$ and $\widehat{\beta}$ vanish precisely on $\mathbb{H}_{-0}(\mathbf{e})$, $\mathbb{H}(\boldsymbol{\tau}_1)$ and $\mathbb{H}(\boldsymbol{\tau}_2)$ respectively, the proof of Lemma 3.1 is complete. □

4 A Sufficient Condition for $\widehat{\delta_\Lambda}$ to have Discrete Support

Theorem 4.1 *Suppose P is a symmetric polytope in $\mathbb{R}^3$ with symmetric faces and Λ is a multiset of points in $\mathbb{R}^3$ such that P tiles at level k, a positive integer, when translated at the locations $\lambda \in \Lambda$. Then we have*

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup \bigcap_{e, \tau_1, \tau_2} \big(\mathbb{H}_{-0}(\mathbf{e}) \cup \mathbb{H}(\boldsymbol{\tau}_1) \cup \mathbb{H}(\boldsymbol{\tau}_2)\big), \tag{7}$$

where δ_Λ is the measure corresponding to Λ defined in (2) and the intersection in (7) is taken over all 4-legged frames (e, τ_1, τ_2) of P.

Proof We know from [6] (see Lemmas 3.1 and 3.2 in [6]) that if P tiles with Λ and μ is a leg measure on P then μ also tiles with Λ, at level 0. In other words $\mu * \delta_\Lambda = 0$. Since P tiles when translated by Λ it follows that $|\Lambda \cap [-R, R]^3| = O(R^3)$, hence δ_Λ is a tempered distribution and we may take its FT which gives us $\widehat{\mu}\widehat{\delta_\Lambda} = 0$. This implies (see the details in [12, Sect. 1.2])

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup \mathsf{Z}(\hat{\mu}).$$

But the measure μ is exactly the one described in Lemma 3.1 and since this must be true for all sets of four legs of P we conclude (7). □

Corollary 4.1 *Suppose P is a k-tiler with a discrete multiset Λ, in $\mathbb{R}^3$. Let the following* intersection property *hold:*

$$\bigcap_{e,\tau_1.\tau_2} \left(\mathbf{e}^\perp \cup \boldsymbol{\tau}_1^\perp \cup \boldsymbol{\tau}_2^\perp\right) = \{0\}, \tag{8}$$

where the intersection above is taken over all sets of 4*-legged frames of P.*

Then $\operatorname{supp}\widehat{\delta_\Lambda}$ *is a discrete set in $\mathbb{R}^3$, of bounded density.*

Proof The sets which are being intersected in (7) are all unions of planes. For this set to be non-discrete it must be the case that it contains an entire line of direction, say $\mathbf{u} \in \mathbb{R}^3 \setminus \{0\}$.

This in turn implies that there is a selection X_ℓ of $\mathbf{e}$, $\boldsymbol{\tau}_1$ or $\boldsymbol{\tau}_2$ for each set ℓ of four legs such that $\mathbf{u} \in X_\ell^\perp$. This contradicts condition (8).

Having established that the intersection in the right hand side of (7) is a discrete point set we observe that the larger set

$$\bigcap_{\mathbf{e},\tau_1,\tau_2} \mathbb{H}(\mathbf{e}) \cup \mathbb{H}(\boldsymbol{\tau}_1) \cup \mathbb{H}(\boldsymbol{\tau}_2) \tag{9}$$

is a finite union of discrete *groups*, each of them of the form

$$\bigcap_{\ell} \mathbb{H}_\ell,$$

where ℓ runs through all possible sets of four legs of P and for each $\ell = \{e, \tau_1, \tau_2\}$ the set $\mathbb{H}_\ell$ is one of $\mathbb{H}(\mathbf{e}), \mathbb{H}(\boldsymbol{\tau}_1), \mathbb{H}(\boldsymbol{\tau}_2)$. Since each discrete group has bounded density so has the set (9) and $\operatorname{supp}\widehat{\delta_\Lambda}$ as its subset. □

5 The Intersection Property Implies Quasi-periodicity of Λ

In this section we show how the discreteness of $\operatorname{supp}\widehat{\delta_\Lambda}$ implies a rather rigid structure for Λ. Below we quote the result from [11], where the multidimensional statements had been proved in general, despite the fact that the final conclusions in [11] are given only for dimension 2.

Theorem 5.1 (Kolountzakis, 2002) *Suppose that for the multiset $\Lambda \subseteq \mathbb{R}^d$*

(1) Λ *has uniformly bounded density;*
(2) $S := \operatorname{supp}\widehat{\delta_\Lambda}$ *is discrete;*
(3) $|S \cap B_R(0)| \le C \cdot R^d$, *for some positive constant C.*

Then Λ is a finite union of translated d-dimensional lattices.

Next, we verify the conditions of the theorem above, for our 3-dimensional k-tilers with a multiset Λ.

Claim 5.1 *Suppose that convex polytope P k-tiles $\mathbb{R}^d$ with Λ and the intersection property* (8) *of Corollary* 4.1 *is true. Then Λ is a finite union of translated d-dimensional lattices.*

Proof We just need to verify conditions (1), (2) and (3) of Theorem 5.1.

Hypothesis (1) simply follows from the fact that in each sufficiently large ball $B_R(x)$ of $\mathbb{R}^d$ every point is covered exactly k times by the translations of P with the set $\Lambda \cap B_{R'}(x)$, where $R' = R + \operatorname{diam} P$.

Hypotheses (2) and (3) follow from Corollary 4.1. □

6 k-Tilers Associated to a Non-discrete supp $\widehat{\delta_\Lambda}$

In this section we study the convex polytopes that admit exceptional multiple tilings, in the sense that the multiset of translations Λ is not a finite union of 3-dimensional lattices. A class of these exceptions is easily provided by prisms (Minkowski sums of a symmetric polygon with a line segment, not in the polygons plane), for which one can lift a 2-dimensional k-tiling up into the third dimension by separately 1-tiling the tube above each projection.

By Claim 5.1 for such a tiling, the intersection property (8) cannot be true. Therefore, there exists a line (in fact a 1-dimensional subspace) $l \subseteq \mathbb{R}^3$ such that

$$l \subseteq \bigcap_{\mathbf{e},\tau_1.\tau_2} \left(\mathbf{e}^\perp \cup \boldsymbol{\tau}_1^\perp \cup \boldsymbol{\tau}_2^\perp\right). \tag{10}$$

It was already shown in [6] that a multiple tiler in $\mathbb{R}^3$ must be a zonotope, i.e. a Minkowski sum of line segments. Here we will show that given the non-discreteness of supp $\widehat{\delta_\Lambda}$, we can deduce that a zonotope is a Minkowski sum of two 2-dimensional symmetric polygons. And in Sect. 7 we provide examples of such exceptional tilings, under a mild commensurability condition for such zonotopes.

Theorem 6.1 *Suppose a polytope P tiles $\mathbb{R}^3$ with multiplicity by translations over a multiset Λ and condition* (10) *holds. Then P is a two-flat zonotope.*

Proof We let L be a plane orthogonal to l and supporting P; (10) is then equivalent to

$$\forall \mathbf{e}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \text{ either } \mathbf{e} \parallel L, \text{ or } \boldsymbol{\tau}_1 \parallel L, \text{ or } \boldsymbol{\tau}_2 \parallel L. \tag{11}$$

Let $F = L \cap P$. The dimension of the face F can be 0, 1 or 2. Consider any facet G of P that has at least one common vertex with F, and let e be an edge of G that shares exactly one vertex v with F (so $G \neq F$). Consider the 4-legged frame determined by G and e with $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ being the corresponding translation vectors. Since v is in L, by (11) one of the three vertices $v + \mathbf{e}$, $v + \boldsymbol{\tau}_1$ and $v + \boldsymbol{\tau}_2$ lies in L, and therefore lies in F as well. By our choice of $\mathbf{e}$, $v + \mathbf{e}$ is a vertex of G but not of F. Thus either $v + \boldsymbol{\tau}_1 \in F$, or $v + \boldsymbol{\tau}_2 \in F$:

(1) If $v + \boldsymbol{\tau}_1 \in F$, then $\boldsymbol{\tau}_1 \in G \cap F$, so we see that $\boldsymbol{\tau}_1$ is an edge of G. Hence G is a parallelogram.

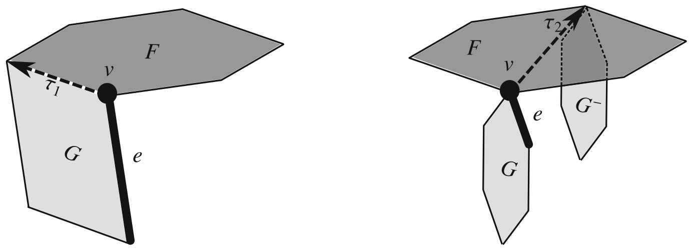
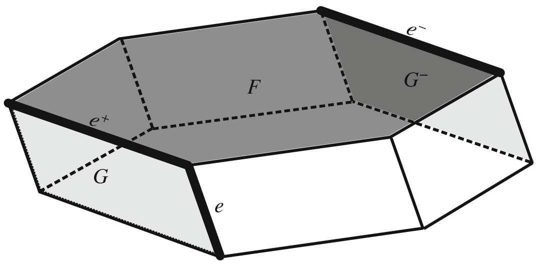


Fig. 2 The two possibilities for the facet G with respect to F. (1) $v + \boldsymbol{\tau}_1 \in F$. (2) $v + \boldsymbol{\tau}_2 \in F$

Fig. 3 This is the case that each facet adjacent to F is a parallelogram sharing an edge with F, giving us a prism



(2) If $v + \boldsymbol{\tau}_2 \in F$, then F connects G with its opposite face G^-.

See Fig. 2.

It is our goal to find a facet G which satisfies property (2). If, to the contrary, every facet adjacent to F satisfies property (1), then each facet adjacent to F is a parallelogram sharing an edge with F. It follows that exactly three edges meet at every vertex of F, and all edges of these parallelograms that are not edges of F or parallel to F, are parallel to each other (see Fig. 3). Now since F is centrally symmetric, consider two opposite parallel edges e^+ and e^- of F, and corresponding parallelogram facets G and G^-. The facets G and G^- are parallel and therefore opposite in P, therefore G enjoys property (2).

Now that we have found a facet G such that F connects G with G^-, we also note that since P is centrally symmetric, G also connects F and F^-. We will now show that $P = F + G$, under the minor assumption that F and G do not share an edge. The case that F and G do in fact share an edge may be handled in exactly the same manner, so without loss of generality we assume throughout the rest of the proof that F and G share none of their edges. Since P is a zonotope, $P = F + G + H$ for some polytope H. To arrive at a contradiction, we assume to the contrary that H is not a single point, and let h_0 be any edge of H. Let $F = [f_1] + \cdots + [f_k]$, $G = [g_1] + \cdots + [g_\ell]$, and $H = [h_0] + [h_1] + \cdots + [h_m]$, where $k \geq 0$, $\ell \geq 2$, and $m \geq 0$. We may assume that all line segments f_i, g_i, h_i have the origin as their midpoint and thus the center of P is also at the origin. We further consider a normal vector $\mathbf{f}_\perp$ to the face F of P. When F is a 2-dimensional face of P, $\mathbf{f}_\perp$ cannot be orthogonal to any line segment $h_i \in H$ and $g_i \in G$. If F is 0 or 1 dimensional face of P (see Fig. 4), we have an infinite collection of perpendicular vectors to F and we may choose $\mathbf{f}_\perp$ to be not orthogonal to any line

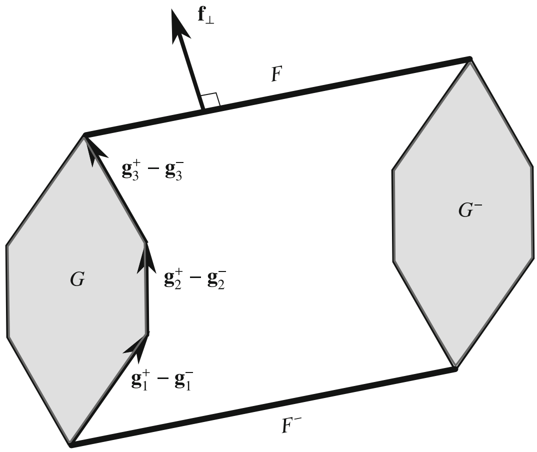


Fig. 4 Here F is a lower-dimensional face of P, namely an edge of P, and we see how we can get from the face F to the face F^- by walking along the vectors $\mathbf{g}_i^+ - \mathbf{g}_i^-$. Here $\mathbf{f}_\perp$ is chosen to be a vector orthogonal to F and not orthogonal to any of the line segments h_j

segment $h_i \in H$ and $g_i \in G$. For each edge g_i (resp. h_i) we define $\mathbf{g}_i^+$ (resp. $\mathbf{h}_i^+$) to be the vector from the origin to the endpoint of g_i (resp. h_i) such that $\langle \mathbf{g}_i^+, \mathbf{f}_\perp \rangle > 0$ (resp. $\langle \mathbf{h}_i^+, \mathbf{f}_\perp \rangle > 0$). In the same way we define $\mathbf{g}_i^-$ (resp. $\mathbf{h}_i^-$) s.t. $\langle \mathbf{g}_i^-, \mathbf{f}_\perp \rangle < 0$ (resp. $\langle \mathbf{h}_i^-, \mathbf{f}_\perp \rangle < 0$). Now one may easily see that the location of the face F in $\mathbb{R}^3$ is given by $[f_1] + \cdots + [f_k] + \mathbf{g}_1^+ + \cdots + \mathbf{g}_\ell^+ + \mathbf{h}_0^+ + \cdots + \mathbf{h}_m^+$ as a set of extremal points of the linear functional corresponding to $\mathbf{f}_\perp$. Similarly the location of the face F^- in $\mathbb{R}^3$ is given by $[f_1] + \cdots + [f_k] + \mathbf{g}_1^- + \cdots + \mathbf{g}_\ell^- + \mathbf{h}_0^- + \cdots + \mathbf{h}_m^-$. Therefore the distance between F and F^- is

$$\left\langle \mathbf{f}_\perp, \sum_{i=1}^{\ell} \mathbf{g}_i^+ + \sum_{i=0}^{m} \mathbf{h}_i^+ - \sum_{i=1}^{\ell} \mathbf{g}_i^- - \sum_{i=0}^{m} \mathbf{h}_i^- \right\rangle > \left\langle \mathbf{f}_\perp, \sum_{i=1}^{\ell} \left(\mathbf{g}_i^+ - \mathbf{g}_i^-\right) \right\rangle.$$

On the other hand, since G connects F and F^- we have $F = F^- + \sum_{i \in I} \left(\mathbf{g}_i^+ - \mathbf{g}_i^-\right)$, for a set I of edges in G. The latter implies that the distance between F and F^- is not more than

$$\left\langle \mathbf{f}_\perp, \sum_{i=1}^{\ell} \left(\mathbf{g}_i^+ - \mathbf{g}_i^-\right) \right\rangle,$$

a contradiction. □

Remark 6.1 One of F or G can be 1-dimensional, in which case P becomes a 3-dimensional prism.

Main Theorem *Suppose a polytope P k-tiles $\mathbb{R}^3$ with a multiset Λ, and suppose that P is not a two-flat zonotope. Then Λ is a finite union of translated lattices.*

Proof If P is not a two-flat zonotope, then due to Theorem 6.1, condition (10) is violated. Therefore, the intersection property (8) in Corollary 4.1 holds. Claim 5.1 now concludes the proof. □

7 Many Two-Flat Zonotopes have Weird Tiling Sets

In this section we prove that, under a mild commensurability condition, two-flat zonotopes admit tilings which are not quasi-periodic (“weird”).

Theorem 7.1 *Suppose P is a two-flat zonotope in $\mathbb{R}^3$ which is the Minkowski sum of the segments*

$$[\mathbf{v}_1], \ldots, [\mathbf{v}_n], \quad [\mathbf{w}_1], \ldots, [\mathbf{w}_m],$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n \in H_1$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m \in H_2$ and H_1, H_2 are two different two dimensional subspaces. Suppose also that the additive group generated by $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_m$ is discrete and that the $\mathbf{v}_j$ span H_1.

Then P admits a tiling by translations at a set $\Lambda \subseteq \mathbb{R}^3$ which is not a finite union of translated lattices.

Corollary 7.1 *If $P \subseteq \mathbb{R}^3$ is a two-flat* rational *zonotope then P admits tilings by sets which are not finite unions of translated lattices.*

Proof of Theorem 7.1 We begin the analysis by noting that P can be paved by parallelepipeds, whose sides are among the vectors $\mathbf{v}_j$ and $\mathbf{w}_j$ (proof is by induction on the number of line segments whose Minkowski sum is the zonotope). Therefore we can write its indicator function as a finite sum of indicator functions of parallelepipeds.

$$\mathbf{1}_P(x) = \sum_{j=1}^{M} \mathbf{1}_{B_j}(x), \quad \text{for a.e. } x,$$

where each B_j is a parallelepiped, whose three sides are equal to some of the $\mathbf{v}_j$ and $\mathbf{w}_j$.

Suppose now that the parallelepiped B is centered at the origin and has sides parallel to the three linearly-independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. We can write the indicator function of B as a convolution

$$\mathbf{1}_B = \frac{|\det(\mathbf{a}, \mathbf{b}, \mathbf{c})|}{|\mathbf{a}| \cdot |\mathbf{b}| \cdot |\mathbf{c}|} \mu_{\mathbf{a}} * \mu_{\mathbf{b}} * \mu_{\mathbf{c}},$$

where $\mu_{\mathbf{a}}$ is the measure that equals arc-length on the line segment from $-\mathbf{a}/2$ to $\mathbf{a}/2$, and $\mu_{\mathbf{b}}$, $\mu_{\mathbf{c}}$ are similarly defined. Since (see Sect. 3)

$$\widehat{\mu_{\mathbf{a}}}(\boldsymbol{\xi}) = |\mathbf{a}| \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle}$$

and similarly for $\widehat{\mu_{\mathbf{b}}}$, $\widehat{\mu_{\mathbf{c}}}$, we obtain the formula

$$\widehat{\mathbf{1}_B}(\boldsymbol{\xi}) = |\det(\mathbf{a}, \mathbf{b}, \mathbf{c})| \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle} \cdot \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{b} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{b} \rangle} \cdot \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{c} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{c} \rangle}. \tag{12}$$

Each parallelepiped B_j in the decomposition of P is a translate of a parallelepiped of the type B, above, with some of the vectors $\mathbf{v}_j$, $\mathbf{w}_j$ in place of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Hence the Fourier Transform $\widehat{\mathbf{1}_{B_j}}$ has the same zeros as the Fourier Transform of its centered translate and these are

$$\mathsf{Z}\big(\widehat{\mathbf{1}_{B_j}}\big) = \big((\mathbb{Z}')\mathbf{a}^* + \mathbf{a}^\perp\big) \cup \big((\mathbb{Z}')\mathbf{b}^* + \mathbf{b}^\perp\big) \cup \big((\mathbb{Z}')\mathbf{c}^* + \mathbf{c}^\perp\big),$$

where $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$ and again $\mathbf{a}^* = \mathbf{a}/|\mathbf{a}|^2$ is the geometric inverse of $\mathbf{a}$, etc. Write now

$$G = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$$

for the additive subgroup (lattice) of H_1 generated by the $\mathbf{v}_j$'s and $G^* \subseteq H_1$ for its dual lattice *in* H_1

$$G^* = \{\mathbf{u} \in H_1 : \forall \mathbf{g} \in G \quad \langle \mathbf{u}, \mathbf{g} \rangle \in \mathbb{Z}\}. \tag{13}$$

We claim now that for each j

$$H_1^\perp + \big(G^* \setminus (\mathbf{v}_1^\perp \cup \cdots \cup \mathbf{v}_n^\perp)\big) \subseteq \mathsf{Z}\big(\widehat{\mathbf{1}_{B_j}}\big). \tag{14}$$

This follows since at least one side of B_j is equal to a vector $\mathbf{v}_j$ which makes the corresponding factor in (12) vanish on any element of G^* which is not orthogonal to $\mathbf{v}_j$. And since that factor in (12) is constant along $H_1^\perp$ we obtain the claim. Since (14) holds for all j we obtain

$$H_1^\perp + \big(G^* \setminus (\mathbf{v}_1^\perp \cup \cdots \cup \mathbf{v}_n^\perp)\big) \subseteq \mathsf{Z}\big(\widehat{\mathbf{1}_P}\big). \tag{15}$$

Pick now any non-zero $c_1, c_2, \ldots, c_n \in \mathbb{R}$. We claim that the measure

$$\tau := \mathbf{1}_P * \delta_G * \big[\delta_0 - \delta_{c_1\mathbf{v}_1}\big] * \cdots * \big[\delta_0 - \delta_{c_n\mathbf{v}_n}\big] = 0, \tag{16}$$

where $\delta_G = \sum_{\mathbf{g}\in G} \delta_{\mathbf{g}}$. For this it is enough to show that the Fourier Transform of the above measure

$$\widehat{\tau}(\boldsymbol{\xi}) = \widehat{\mathbf{1}_P}(\boldsymbol{\xi})(1 - e^{2\pi i c_1\langle \mathbf{v}_1, \boldsymbol{\xi}\rangle}) \cdots (1 - e^{2\pi i c_n\langle \mathbf{v}_n, \boldsymbol{\xi}\rangle})\widehat{\delta_G}$$

is identically 0. By the Poisson Summation Formula (the Fourier Transform is taken in the sense of distributions)

$$\widehat{\delta_L} = \frac{1}{\operatorname{vol} L}\delta_{L^*}, \tag{17}$$

for each lattice $L = A\mathbb{Z}^d$ in $\mathbb{R}^d$ and dual lattice $L^* = A^{-\top}\mathbb{R}^d$ (where $A \in GL(d, \mathbb{R})$), it follows that $\widehat{\delta_G}$ is a *measure* with support on the lines orthogonal to H_1 that go through the points of G^*:

$$\operatorname{supp}\widehat{\delta_G} = G^* + H_1^\perp.$$

By (15) the function $\widehat{\mathbf{1}_P}(\boldsymbol{\xi})$ kills $\widehat{\delta_G}$ except at the lines of the form $\mathbf{g}^* + H_1^\perp$ with $\mathbf{g}^* \in G^*$ is orthogonal to some $\mathbf{v}_j$. But at these lines one of the factors

$$(1 - e^{2\pi i c_1 \langle \mathbf{v}_1, \boldsymbol{\xi}\rangle}) \cdots (1 - e^{2\pi i c_n \langle \mathbf{v}_n, \boldsymbol{\xi}\rangle})$$

vanishes, so indeed $\widehat{\tau}$ is zero. Now rewrite the measure $(\delta_0 - \delta_{c_1\mathbf{v}_1}) \cdots (\delta_0 - \delta_{c_n\mathbf{v}_n})$ in the form

$$\sum_{k=1}^{N} \delta_{\mathbf{u}_k^+} - \sum_{k=1}^{N} \delta_{\mathbf{u}_k^-} \quad (\text{where } N = 2^{n-1}).$$

Equivalently, we can rewrite (16) as the equality

$$\mathbf{1}_P * \delta_G * \sum_{k=1}^{N} \delta_{\mathbf{u}_k^+} = \mathbf{1}_P * \delta_G * \sum_{k=1}^{N} \delta_{\mathbf{u}_k^-}. \tag{18}$$

Define the multisets

$$S = G + \{\mathbf{u}_1^+, \ldots, \mathbf{u}_N^+\} \quad \text{and } T = G + \{\mathbf{u}_1^-, \ldots, \mathbf{u}_N^-\} \tag{19}$$

whose ground sets are the supports of the discrete measures

$$\delta_G * \sum_{k=1}^{N} \delta_{\mathbf{u}_k^+} \text{ and } \delta_G * \sum_{k=1}^{N} \delta_{\mathbf{u}_k^-},$$

and their multiplicities at each point are those described by these measures.

In what follows we exploit (18) to give an example of a multiple tiling by P with a *discrete set* Λ, which by no means can be expressed as a finite union of translated lattices.

We notice first that since P is a zonotope decomposing into parallelepipeds of sides among the vectors $\mathbf{v}_j$, $\mathbf{w}_j$, it k-tiles $\mathbb{R}^3$, for some k, with the lattice

$$\Gamma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_m \rangle$$

generated by the $\mathbf{v}_j$, $\mathbf{w}_j$. The reason is that each of the parallelepipeds B_j tiles with a subgroup of Γ (the group generated by its side vectors) and therefore it tiles multiply with Γ itself. Clearly, P also (Nk)-tiles $\mathbb{R}^3$ by the union of N translations of the lattice Γ by the vectors $\mathbf{u}_1^+, \ldots, \mathbf{u}_N^+$.

Let $\{\gamma_j : j \in \mathbb{Z}\}$ be a complete set of coset representatives of G in Γ. Define the set of translates

$$\Lambda = \bigcup_{j \in \mathbb{Z}} (E_j + \gamma_j),$$

where for each $j \in \mathbb{Z}$ we choose $E_j = S$ or $E_j = T$ arbitrarily. We claim that for any such choice of the E_j the Λ-translates of P form a (Nk)-tiling of $\mathbb{R}^3$. Indeed the claim is true if all $E_j = S$ as it is a restatement of the fact that P (Nk)-tiles with $\Gamma + \{\mathbf{u}_1^+, \dots, \mathbf{u}_N^+\}$, which itself follows from the fact that P k-tiles with Γ. Observe now that if we change any single E_j from S to T we are adding the quantity

$$\mathbf{1}_P * \delta_G * \sum_{i=1}^{N} \delta_{\mathbf{u}_i^-} * \delta_{\gamma_j} - \mathbf{1}_P * \delta_G * \sum_{i=1}^{N} \delta_{\mathbf{u}_i^+} * \delta_{\gamma_j} \tag{20}$$

to the constant function

$$\mathbf{1}_P * \delta_\Lambda,$$

which therefore remains the same since (20) is identically 0 by (18). We conclude that we have a (Nk)-tiling no matter how the E_j are chosen (one has to make the remark here that in any given bounded region of space the fact that $P + \Lambda$ is a (Nk)-tiling or not is affected by finitely many choices for the E_j).

Choose now all $E_j = S$ with the exception of $E_0 = T$. We claim that the corresponding set Λ is not a finite union of translated fully-dimensional lattices. Indeed, by the Poisson Summation Formula (17) we have that, if

$$\Lambda' = \bigcup_{j \in \mathbb{Z}} \left(S + \gamma_j\right) = \Gamma + \{\mathbf{u}_1^+, \dots, \mathbf{u}_N^+\}$$

then $\widehat{\delta_{\Lambda'}}$ is a discrete measure in $\mathbb{R}^3$ and this should also be true for $\widehat{\delta_\Lambda}$ if Λ too were a finite union of translated lattices. Thus the difference

$$\widehat{\delta_{\Lambda'}} - \widehat{\delta_\Lambda}$$

would also be a discrete measure. But

$$\begin{aligned} \delta_{\Lambda'} - \delta_\Lambda &= \delta_{S+\gamma_0} - \delta_{T+\gamma_0} \\ &= \delta_{\gamma_0} * \delta_G * \sum_{i=1}^{N} \left(\delta_{\mathbf{u}_i^+} - \delta_{\mathbf{u}_i^-}\right) \\ &= \delta_{\gamma_0} * \delta_G * (\delta_0 - \delta_{c_1 \mathbf{v}_1}) * \cdots * (\delta_0 - \delta_{c_n \mathbf{v}_n}). \end{aligned}$$

So

$$\widehat{\delta_{\Lambda'}} - \widehat{\delta_\Lambda} = e^{2\pi i \langle \gamma_0, \boldsymbol{\xi} \rangle} \prod_{j=1}^{n} (1 - e^{2\pi i c_j \langle \mathbf{v}_j, \boldsymbol{\xi} \rangle}) \widehat{\delta_G}. \tag{21}$$

Recall now that the support of the measure $\widehat{\delta_G}$ are all straight lines orthogonal to H_1 passing through a point of G^*, the dual lattice to G in H_1. The factors in the right hand side of (21) vanish at the set

$$\bigcup_{j=1}^{n}\left(\mathbb{Z}\frac{\mathbf{v}_j^*}{c_j}+\mathbf{v}_j^{\perp}\right). \tag{22}$$

Each set in this union consists of a series of planes normal to $\mathbf{v}_j$ and spaced by a length of $\left(c_j|\mathbf{v}_j|\right)^{-1}$. Each of the straight lines that make up the support of $\widehat{\delta_G}$ is parallel to each such plane and, therefore, each such line is either entirely contained in (22) or is disjoint from it. It follows that, since the right-hand side of (21) is not identically zero, its support contains at least one straight line of the direction $H_1^{\perp}$ and is not a discrete set, as we had to show. □

Remark 7.1 One may easily extend the previous construction of Λ to the examples that cannot be expressed as a linear combination of finitely many possibly lower-dimensional lattices.

Proof In the previous construction of Λ we could let E_j to be either S, or T for each j and still get a legitimate (Nk)-tiling of $\mathbb{R}^3$. In general we could have a big family (of cardinality $2^{\mathbb{Z}}$) of possible Nk-tilings of $\mathbb{R}^3$. We call a tiling *weird* if it is not quasi-periodic. In what follows we show that our big family has a weird member $\Lambda^{\dagger}$. In our construction we will need the following claim. □

Claim 7.1 *The set of integers $\mathbb{Z}$ can be colored with two colors in such a way that every arithmetic progression has infinitely many numbers of each color.*

Proof There are countably many arithmetic progressions in $\mathbb{Z}$. We enumerate them all denoting A_i the ith progression in the enumeration, such that any progression appears infinitely many times. We begin to color $\mathbb{Z}$ step by step in such a way, that on the nth step all progressions A_i for i from 1 to n have numbers of both colors. At the nth step we find two numbers of A_n that are not yet colored, and color them differently. It is always possible to do so, because at step n only finitely many numbers of $\mathbb{Z}$ are already colored, and A_n has infinitely many numbers. With such a coloring every arithmetic progression would contain infinitely many integers of each of the colors. □

We can pick our complete set of the coset representatives $\{\gamma_j\ :\ j\in\mathbb{Z}\}$ so that it contains $\gamma_1\cdot\mathbb{Z}$ as a subset.

In order to construct $\Lambda^{\dagger}$, we consider a coloring of $\gamma_1\cdot\mathbb{Z}$ with two colors (red and black) so that every arithmetic progression there has infinitely many red and infinitely many black members. We let $E_j=S$ if corresponding coset representative $\gamma_j\notin\gamma_1\cdot\mathbb{Z}$. For coset representatives in $\gamma_1\cdot\mathbb{Z}$, if the point $\gamma_1\cdot j$ is red we choose $E_j=S$, if the point $\gamma_1\cdot j$ is black we choose $E_j=T$ in $\Lambda^{\dagger}$.

We further notice that due to the freedom to choose c_k's in the definition of $\mathbf{u}_k^-$'s and $\mathbf{u}_k^+$'s, one can pick c_k's so that multisets S and T have different multiplicities at 0. Indeed, we may pick c_k's so that for any set of indexes $I\subset[n]$ the corresponding linear combination $\sum_{k\in I}c_k\cdot\mathbf{v}_k\notin G$. Then $G+\sum_{k=1}^{N}\mathbf{u}_k^+$ has 0 at multiplicity 1, while $G+\sum_{k=1}^{N}\mathbf{u}_k^-$ has 0 at multiplicity 0. Furthermore, at each point $\gamma_1\cdot\ell$ of $\gamma_1\cdot\mathbb{Z}$ the multisets $S+\ell\cdot\gamma_1$ and $T+\ell\cdot\gamma_1$ have different multiplicities as well. Thus we get irregular behavior of $\Lambda^\dagger$ on the line $\mathbb{Z} \cdot \gamma_1$. In particular, $\Lambda^\dagger$ simultaneously contains and misses infinitely many members of each infinite coset of $\mathbb{Z} \cdot \gamma_1$.

In other words, we construct such $\Lambda^\dagger$, that $\Lambda^\dagger$ simultaneously hits and misses infinitely many points of any 1-dimensional sublattice of $\mathbb{Z} \cdot \gamma_1$ (note that (i) we can pick sufficiently small c_i's in the definition of $\mathbf{u}_i^+$'s and $\mathbf{u}_i^-$'s so that the sets S and T are disjoint; (ii) that $\mathbb{Z} \cdot \gamma_1$ intersects each $S + \gamma_1 \cdot j$ exactly at one point).

Now if we assume that $\Lambda^\dagger$ may be expressed as a finite linear combination of translated lattices $\delta_{\Lambda^\dagger} = q_1 \cdot \delta_{\Lambda_1} + \dots + q_m \cdot \delta_{\Lambda_m}$ (to simplify notations, we will write $q_1 \cdot \Lambda_1 + \dots + q_m \cdot \Lambda_m$ instead of $\delta_{\Lambda^\dagger}$), then

$$\Lambda^\dagger \cap \mathbb{Z} \cdot \gamma_1 = \sum_{i=1}^{m} q_i \cdot \big(\Lambda_i \cap \mathbb{Z} \cdot \gamma_1\big).$$

Each $\Lambda_i \cap \mathbb{Z} \cdot \gamma_1$ is a coset of $\mathbb{Z} \cdot \gamma_1$. Therefore, $\Lambda_i \cap \mathbb{Z} \cdot \gamma_1$ is either empty, or is a single point, or is an arithmetic progression in $\mathbb{Z} \cdot \gamma_1$ with the common difference d_i. We denote the set of all the indices of the latter Λ_i's by $M \subset \{1, \dots, m\}$. We further consider an arithmetic progression A of $\mathbb{Z} \cdot \gamma_1$ with the common difference $\prod_{i \in M} d_i$. We notice that for any $i \in M$ either $A \cap \Lambda_i = A$, or $A \cap \Lambda_i = \emptyset$. Since $A \subset \mathbb{Z} \cdot \gamma_1$, we have

$$\Lambda^\dagger \cap A = \sum_{i=1}^{m} q_i \cdot \big(\Lambda_i \cap A\big) = \sum_{i \notin M} q_i \cdot \big(\Lambda_i \cap A\big) + A \cdot \sum_{\substack{i \in M: \\ \Lambda_i \cap A \neq \emptyset}} q_i .$$

According to the definition of M the set $\sum_{i \notin M} q_i \cdot \delta_{\Lambda_i \cap A}$ has finite support. Since A is an arithmetic progression in $\mathbb{Z} \cdot \gamma_1$ and due to our construction of $\Lambda^\dagger$, the support of

$$\delta_{\Lambda^\dagger \cap A} - \delta_A \cdot \sum_{\substack{i \in M: \\ \Lambda_i \cap A \neq \emptyset}} q_i$$

cannot be finite, a contradiction. □

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