

The Set of Packing and Covering Densities of Convex Disks

Włodzimierz Kuperberg

Received: 6 February 2013 / Revised: 18 August 2013 / Accepted: 27 August 2013 /
Published online: 28 September 2013
© Springer Science+Business Media New York 2013

Abstract For every convex disk K (a convex compact subset of the plane, with non-void interior), the packing density $\delta(K)$ and covering density $\vartheta(K)$ form an ordered pair of real numbers, i.e., a point in \mathbb{R}^2 . The set Ω consisting of points assigned this way to all convex disks is the subject of this article. A few known inequalities on $\delta(K)$ and $\vartheta(K)$ jointly outline a relatively small convex polygon P that contains Ω , while the exact shape of Ω remains a mystery. Here we describe explicitly a leaf-shaped convex region Λ contained in Ω and occupying a good portion of P . The sets Ω_T and Ω_L of translational packing and covering densities and lattice packing and covering densities are defined similarly, restricting the allowed arrangements of K to translated copies or lattice arrangements, respectively. Due to affine invariance of the translative and lattice density functions, the sets Ω_T and Ω_L are compact. Furthermore, the sets Ω , Ω_T and Ω_L contain the subsets Ω^* , Ω_T^* and Ω_L^* respectively, corresponding to the centrally symmetric convex disks K , and our leaf Λ is contained in each of Ω^* , Ω_T^* and Ω_L^* .

Keywords Convex disk · Packing · Covering · Density

Mathematics Subject Classification 52C15

1 Introduction: Definitions and Notation

An n -dimensional *convex body* is a compact, convex subset of \mathbb{R}^n that contains an interior point. A 2-dimensional convex body is called a *convex disk*. The convex hull of a set S is denoted by $\text{Conv}(S)$.

W. Kuperberg (✉)
Department of Mathematics & Statistics, Auburn University, Auburn, AL36849-5310, USA
e-mail: kuperwl@auburn.edu

A family $\mathcal{F} = \{K_i\}$ of subsets of \mathbb{R}^n , each congruent to a given convex body K , is a *packing* (of \mathbb{R}^n with copies of K) if their interiors are mutually disjoint, and it is a *covering* if their union is \mathbb{R}^n . If \mathcal{F} is a packing and a covering, then it is called a *tiling*, and we say that such a convex body K *admits a tiling of \mathbb{R}^n* or that K is a *tile*, for brevity.

For a (measurable) set S in \mathbb{R}^n , $|S|$ denotes the n -dimensional measure of S , and for a family of sets $\mathcal{F} = \{K_i\}$ and a region $D \subset \mathbb{R}^n$, we will denote by $|\mathcal{F} \cap D|$ the sum of the volumes of $K_i \cap D$ over all $K_i \in \mathcal{F}$. Let B_r denote the ball of radius r in \mathbb{R}^n , centered at the origin.

Given any family $\mathcal{F} = \{K_i\}$ of congruent copies of a convex body K (in particular, a packing or a covering), the *density* of \mathcal{F} is defined as

$$d(\mathcal{F}) = \lim_{r \rightarrow \infty} \frac{|\mathcal{F} \cap B_r|}{|B_r|}.$$

Here, if the limit does not exist, then in case \mathcal{F} is a packing we take the upper limit, *limsup*, and in case it is a covering we take the lower limit, *liminf*, in place of the limit.

For every convex body K , the supremum of the set of densities $d(\mathcal{F})$ taken over all packing arrangements with copies of K is called *the packing density* of K and is denoted by $\delta(K)$. Similarly, the infimum of the set of densities $d(\mathcal{F})$ taken over all covering arrangements with copies of K is called *the covering density* of K and is denoted by $\vartheta(K)$.

Often certain restrictions on the structure of the arrangements \mathcal{F} are imposed: one may consider arrangements of translated copies of K only, or just lattice arrangements of translates of K . In these cases, the corresponding densities assigned to K by analogous definitions are: the *translative* packing and covering density of K , denoted by $\delta_T(K)$ and $\vartheta_T(K)$, and the *lattice* packing and covering density of K , denoted by $\delta_L(K)$ and $\vartheta_L(K)$, respectively.

It is easily seen that these quantities satisfy the following inequalities:

$$0 \leq \delta_L(K) \leq \delta_T(K) \leq \delta(K) \leq 1 \leq \vartheta(K) \leq \vartheta_T(K) \leq \vartheta_L(K). \tag{1.1}$$

For more details on these notions and their basic properties see [3]. Here we only mention the following:

- (i) Each of the extreme densities $\delta(K)$, $\delta_T(K)$, $\delta_L(K)$, $\vartheta(K)$, $\vartheta_T(K)$, and $\vartheta_L(K)$ is attained by a corresponding arrangement \mathcal{F} . In other words, for every convex body K there is a maximum density packing and a minimum density covering in each of the three types: by arbitrary isometries, by translates only, and by lattice arrangements only.
- (ii) Each of the six densities δ , δ_T , δ_L , ϑ , ϑ_T , and ϑ_L , considered as a real-valued function defined on the hyperspace \mathcal{K}^n of all n -dimensional convex bodies furnished with the Hausdorff metric, is continuous.
- (iii) Each of the four densities δ_T , δ_L , ϑ_T , and ϑ_L is affine-invariant, hence each of them can be viewed as a (continuous) function defined on the space $[\mathcal{K}^n]$ of affine equivalence classes of all convex n -dimensional bodies—a quotient space of \mathcal{K}^n .
- (iv) $\delta(K) = 1 \iff \vartheta(K) = 1 \iff K$ is a tile.

2 The Background

The main result, as outlined in the Abstract, concerns the packing and covering densities of convex disks. Therefore the brief survey given in this section is focused on the two-dimensional case.

The lattice packing density of the circular disk B^2 (the unit ball in \mathbb{R}^2) has been determined by Lagrange [16] in 1773: $\delta_L(B^2) = \pi/\sqrt{12}$. In 1831, Gauss [6] established that $\delta_L(B^3) = \pi/\sqrt{12}$. The first proof of the equality $\delta(B^2) = \pi/\sqrt{12}$ was given by Thue [19] in 1910. The analogous result for the covering, $\vartheta(B^2) = 2\pi/\sqrt{27}$, is due to Kershner [12].

The following significant generalization of Thue's and Kershner's theorems was given by Fejes Tóth [5]. For every convex disk K , let $h(K)$ and $H(K)$ denote the maximum area hexagon contained in K and the minimum area hexagon containing K , respectively. Then no packing of the plane with congruent copies of K can be of density greater than $|K|/|H(K)|$. Fejes Tóth's proof of the bound of $|K|/|h(K)|$ for coverings requires that the copies K_i and K_j of K *do not cross each other*, meaning that at least one of the sets $K_i \setminus K_j$ and $K_j \setminus K_i$ is connected, hence the second result is not quite analogous to the first one. However, since two translates of a convex disk cannot cross each other, the two results produce the following inequalities:

$$\delta(K) \leq \frac{|K|}{|H(K)|} \quad \text{and} \quad \vartheta_T(K) \geq \frac{|K|}{|h(K)|}. \quad (2.1)$$

Along with the theorems of Dowker [2] on inscribed and circumscribed polygons in centrally symmetric disks, the above inequalities imply that if K is centrally symmetric, then

$$\delta(K) = \delta_T(K) = \delta_L(K) = \frac{|K|}{|H(K)|} \quad \text{and} \quad \vartheta_T(K) = \vartheta_L(K) = \frac{|K|}{|h(K)|}. \quad (2.2)$$

These two equalities generalize Thue's and Kershner's theorems, respectively.

The equality $\vartheta(K) = \frac{|K|}{|h(K)|}$ for centrally symmetric disks K , conjectured by Fejes Tóth [4] (see also [5]) still remains unproven.

Chakerian and Lange [1] proved that every convex disk K is contained in a quadrilateral of area at most $\sqrt{2}|K|$. Since every quadrilateral admits a tiling of the plane, they established the inequality

$$\delta(K) \geq \sqrt{2}/2 = 0.7071 \dots \quad (2.3)$$

Henceforth, K will denote an arbitrary (not necessarily centrally symmetric) convex disk, unless otherwise explicitly assumed.

The inequality of Chakerian and Lange [13] was subsequently improved in to

$$\delta(K) \geq \sqrt{3}/2 = 0.8660 \dots \quad (2.4)$$

Concerning upper bounds on covering density, the inequality

$$\vartheta(K) \leq 8(2\sqrt{3} - 3)/3 = 1.2376043 \dots, \quad (2.5)$$

established in [15], was improved by Ismailescu [8] to

$$\vartheta(K) \leq 1.2281772 \dots. \quad (2.6)$$

The upper bound in the above inequality, though presently the best known, is not likely to be sharp. However, for the centrally symmetric convex disks K , the second equality in (2.2) and a theorem of Sas [18] on maximum area polygons inscribed in a convex disk imply that

$$\vartheta(K) \leq 2\pi/\sqrt{27} = 1.209 \dots, \quad (2.7)$$

which is sharp, since equality holds for the circular disk.

The following inequality, linking the packing density $\delta(K)$ and the covering density $\vartheta(K)$:

$$3\vartheta(K) \leq 4\delta(K) \quad (2.8)$$

was proved in [14]. The inequality is sharp, since equality holds in the case of the circular disk.

3 Packing and Covering Via Tiling

In this section we do not restrict ourselves to two dimensions. Here all convex bodies are assumed to be n -dimensional, also packing and covering means packing and covering of \mathbb{R}^n .

We begin with the following simple observation.

Assume that a convex body K is contained in a tile T . Then a space tiling with congruent replicas of T naturally yields a space packing with congruent replicas of K . Similarly, if K contains T , then the tiling yields a space covering with replicas of K . Such a packing, resp. covering, is said to be *generated* by the tile T .

Proposition 1 *The density of a packing generated by a tile T containing K is $|K|/|T|$. Similarly, the density of a covering generated by a tile t contained in K is $|K|/|t|$.*

We omit the easy, natural proof of this proposition. Following are two simple, but quite useful statements.

Proposition 2 *Suppose a densest space packing with congruent copies of K , that is, a packing of density $\delta(K)$, is generated by a tile T containing K , and let M be a convex body “sandwiched” between K and T , that is, $K \subset M \subset T$. Then $\delta(M) = |M|/|T|$.*

Henceforth we will say that set B is *sandwiched* between sets A and C to mean that $A \subset B \subset C$ or $C \subset B \subset A$.

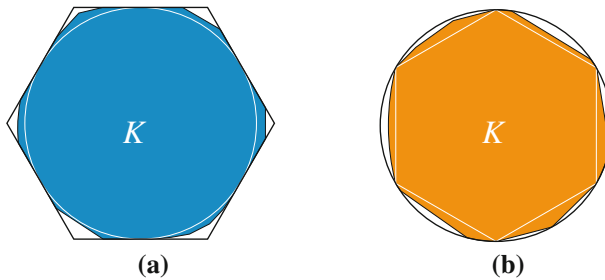


Fig. 1 Disk K sandwiched between the circle and **a** the circumscribed, **b** the inscribed, regular hexagon

Proposition 3 Suppose a thinnest space covering with congruent copies of K , that is, a covering of density $\vartheta(K)$, is generated by a tile T contained in K , and let M be a convex body sandwiched between K and T . Then $\vartheta(M) = |M|/|T|$.

Proof of Proposition 2 Indeed, T generates a packing with replicas of M that is of density $|M|/|T|$, and a packing with higher density is impossible, since such a packing would yield a packing with copies of K of density exceeding $\delta(K)$. \square

Proof of Proposition 3 is completely analogous to that of Proposition 2.

As an application of the above, we obtain the following two propositions.

Proposition 4 Let K be a convex disk sandwiched between the circular disk C and a regular hexagon H circumscribing C (see Fig. 1a). Since, by Thue's theorem, $\delta(C) = |C|/|H|$ and since H is a tile, we conclude that

$$\delta(K) = |K|/|H|.$$

Moreover, since the hexagon H is centrally symmetric, it tiles the plane in a lattice-like manner, which implies that

$$\delta(K) = \delta_T(K) = \delta_L(K).$$

Similarly:

Proposition 5 Let K be a convex disk sandwiched between the circular disk C and a regular hexagon h inscribed in C (see Fig. 1b). By the above Propositions and by Kershner's theorem, we conclude that

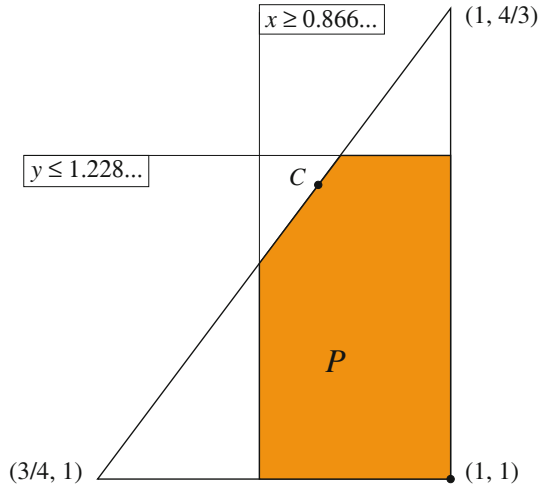
$$\vartheta(K) = |K|/|h|.$$

Moreover,

$$\vartheta(K) = \vartheta_T(K) = \vartheta_L(K).$$

Remark In view of the recent proof of the sphere packing conjecture in \mathbb{R}^3 (also known as the Kepler Conjecture) by Hales [7], the rhombic dodecahedron circumscribed about the ball B^3 generates a densest ball packing. Hence:

Fig. 2 Pentagon P that contains the set Ω



For every convex body K sandwiched between the ball and the rhombic dodecahedron circumscribed about it, the packing density $\delta(K) = \delta_T(K) = \delta_L(K)$ is the ratio between the volume of K and the volume of the rhombic dodecahedron.

4 The Subset Ω of the Plane, Consisting of Pairs $(\delta(K), \vartheta(K))$

Define $\omega : \mathcal{K}^2 \rightarrow \mathbb{R}^2$ by $\omega(K) = (\delta(K), \vartheta(K))$ for every $K \in \mathcal{K}^2$. By continuity of each of the real-valued functions δ and ϑ , the function ω is continuous. Let $\Omega = \omega(\mathcal{K}^2)$. The inequalities (2.4), (2.6), (2.7), along with the obvious ones, $\delta(K) \leq 1$ and $\vartheta(K) \geq 1$, can be interpreted as five half-planes in the (x, y) -plane \mathbb{R}^2 , each containing the set Ω , namely

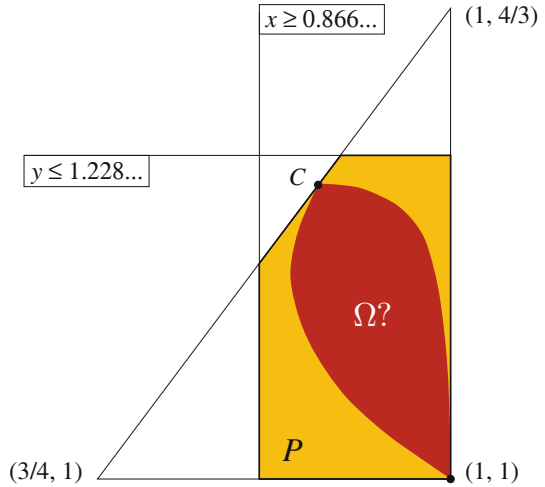
- (a) $x \geq 0.8660\dots$, (b) $x \leq 1$, (c) $y \leq 1.2281772\dots$, (d) $y \geq 1$, and
- (e) $y \leq \frac{4}{3}x$.

The intersection of these half-planes is a pentagon, we denote it P , containing Ω , see Fig. 2.

The vertex $(1, 1)$ of P corresponds to all plane-tiling polygons, and is the only point of Ω lying on the union of the two sides of P containing this vertex. The point $C = (\pi/\sqrt{12}, 2\pi/\sqrt{27})$ lies on the slant side of P and corresponds to the circular disk B^2 , and perhaps also to every ellipse—we say *perhaps*, because in general the covering density of an ellipse is unknown. It would be interesting to know if C is the only point of Ω lying on the line $y = \frac{4}{3}x$, and also if $C = \omega(K)$ for some non-elliptical convex disk K .

Remark 1 The sets Ω_T and Ω_L are defined in a similar way as Ω , just by replacing the function $\omega = (\delta, \vartheta)$ by $\omega_T = (\delta_T, \vartheta_T)$ and by $\omega_L = (\delta_L, \vartheta_L)$, respectively. Also,

Fig. 3 The conjectured teardrop shape of Ω



one can define the subset Ω^* of Ω , by restricting the function ω to the subspace \mathcal{K}^{*2} of \mathcal{K}^2 consisting of centrally symmetric disks K , and then define the sets $\Omega_T^* \subset \Omega_T$ and $\Omega_L^* \subset \Omega_L$ in the same way. Equalities (2.2) imply that $\Omega_T^* = \Omega_L^*$.

Of course, the corresponding six Ω -type sets are readily defined in the very same way for n -dimensional convex bodies with $n > 2$. However, already in dimension 3, most of the properties and inequalities analogous to those mentioned in the previous sections either do not hold or are not known to be true.

Remark 2 The Minkowski sum of convex bodies, defined by

$$K + M = \{x + y : x \in K, y \in M\},$$

is a continuous map from $\mathcal{K}^n \times \mathcal{K}^n$ to \mathcal{K}^n . Therefore the space \mathcal{K}^n is contractible, which means that there is a homotopy between the identity on \mathcal{K}^n and a constant map. This implies that the space \mathcal{K}^n is simply-connected. This in turn implies that each of the sets Ω , Ω_T , and Ω_L is pathwise-connected, being a continuous image of a pathwise-connected space. But a continuous image of a simply-connected space need not be simply-connected. However, if a simple closed curve J in Ω (in Ω_T or in Ω_L) is the image under ω (ω_T , ω_L , respectively) of a simple closed curve in \mathcal{K}^n , then J is contractible to a point in Ω (Ω_T , Ω_L , resp.). Such a contractible simple closed curve J bounds a unique topological disk in \mathcal{R}^2 , therefore the disk must be contained in Ω (Ω_T , Ω_L , resp.).

Since the Minkowski sum of two centrally symmetric sets is a centrally symmetric set, the analogous statements hold for the space \mathcal{K}^{*n} of centrally symmetric n -dimensional convex bodies, and to the corresponding sets Ω^* , Ω_T^* and Ω_L^* .

Remark 3 There are reasons to believe that the shape of Ω resembles that of a teardrop, as shown in Fig. 3. First, the lower bound $\delta(K) \geq \sqrt{3}/2 = 0.866\dots$ is very likely possible to raise, perhaps fairly close to 0.9. Also, the upper bound $\vartheta(K) \leq 1.228\dots$

is likely possible to lower, perhaps all the way down to $2\pi/\sqrt{27} = 1.209\dots$. Then, since the vertex (1.1) of P is the only point that Ω has in common with each of the two sides of P containing it, it seems reasonable to expect that if a convex disk can pack the plane very efficiently (i.e., has packing density close to 1), then it should cover the plane efficiently as well. More specifically, we conjecture that there exist a convex-to-the-left curve in P containing (1.1) and separating Ω from the ray $y = 1, x < 1$, and also there is a convex-to-the-right curve in P , containing (1.1) and separating Ω from the ray $x = 1, y > 1$. Possibly, the two curves may lie on the boundary of Ω , meeting at C , and forming the teardrop shape between them.

5 Certain Two Arcs in Ω and the Disk They Enclose

Let H be the regular hexagon of unit edge length let D_t ($0 \leq t \leq 1$) be a circular disk shrinking monotonically and continuously in t , beginning with being circumscribed about H and ending up inscribed in H . Let \mathcal{A} be the arc in \mathcal{K}^2 consisting of the convex disks $K_t = D_t \cap H$, see Fig. 4.

Similarly, let \mathcal{B} be the arc in \mathcal{K}^2 consisting of the convex disks $L_t = \text{Conv}(D_t \cup H)$, see Fig. 5.

Each of the arcs \mathcal{A} and \mathcal{B} connects points H and D , and $\mathcal{A} \cup \mathcal{B}$ is a simple closed curve in \mathcal{K}^2 . Each of $\alpha = \omega(\mathcal{A})$ and $\beta = \omega(\mathcal{B})$ is a path in Ω joining point $(1, 1)$ with point $C = (\pi/\sqrt{12}, 2\pi/\sqrt{27})$.

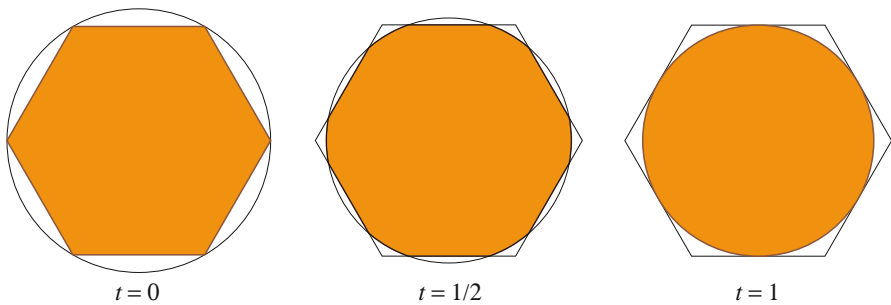


Fig. 4 The disks K_t representing points of the arc \mathcal{A} for $t = 0, t = 1/2$, and $t = 1$

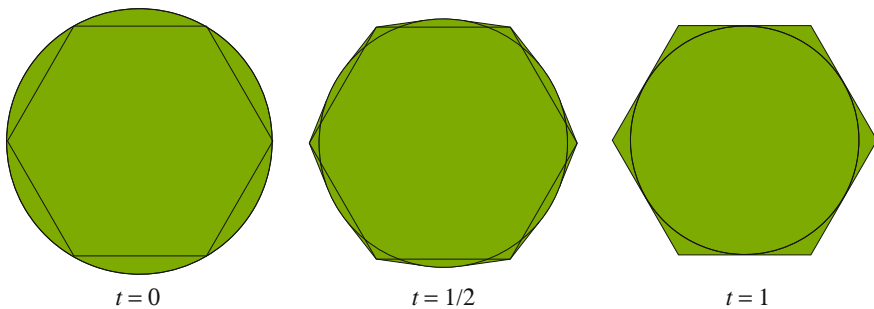


Fig. 5 The disks L_t representing points of the arc \mathcal{B} for $t = 0, t = 1/2$, and $t = 1$

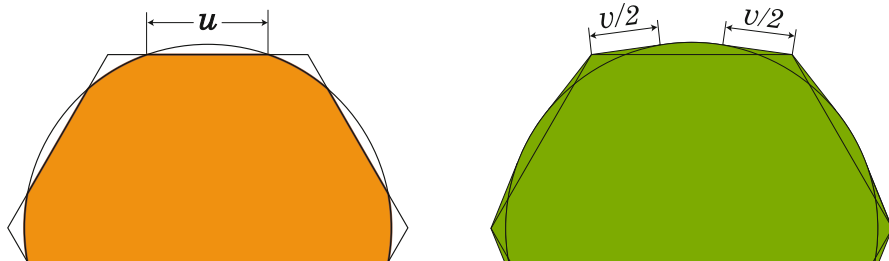


Fig. 6 The parameters u and v

For a parametric description of α and β , we observe that each disk K_t is sandwiched between the circular disk D_t and a regular hexagon inscribed in D_t , and also is sandwiched between the regular hexagon H and the circle inscribed in it. Similarly, each disk L_t is sandwiched between the circular disk D_t and a regular hexagon circumscribed about D_t , and also is sandwiched between the regular hexagon H and the circle circumscribed about it. Using Propositions 4 and 5 and by computing the areas of the disks K_t and L_t and the disks between which they are sandwiched, we obtain explicit parametric presentations of the paths α and β as follows.

In our parameterization of the curve $\alpha = \omega(\mathcal{A})$, we choose the parameter u associated with $\omega(K_t)$ to be the length of a rectilinear component of the boundary of K_t . For the curve $\beta = \omega(\mathcal{B})$, we choose the parameter v associated with $\omega(K_t)$ to be the total length of two rectilinear components of the boundary of L_t , see Fig. 6.

By elementary computations of corresponding areas we obtain the formulae for the packing and covering densities $\delta(K_t)$, $\vartheta(K_t)$ expressed as functions of u , and $\delta(L_t)$ and $\vartheta(L_t)$ expressed as functions of v , that constitute the following parametric presentations of the arcs α and β :

$$\alpha : \begin{cases} x = f_1(u) = \frac{2\pi - 12 \arcsin (u/2) + 3\sqrt{4 - u^2}}{\sqrt{3}(4 - u^2)}; \\ y = g_1(u) = \frac{2\pi - 12 \arcsin (u/2) + 3\sqrt{4 - u^2}}{3\sqrt{3}}; \end{cases} \quad (0 \leq u \leq 1)$$

and

$$\beta : \begin{cases} x = f_2(v) = \sqrt{3} \left(\frac{\pi}{6} - \arcsin (v/2) + \frac{v}{\sqrt{4 - v^2}} \right); \\ y = g_2(v) = \frac{\sqrt{3}}{3} \left((4 - v^2) \left(\frac{\pi}{6} - \arcsin (v/2) \right) + v\sqrt{4 - v^2} \right) \end{cases} \quad (0 \leq v \leq 1)$$

Path α begins at $(1, 1)$ and ends at $C = (\pi/\sqrt{12}, 2\pi\sqrt{27})$, and path β begins at $C = (\pi/\sqrt{12}, 2\pi\sqrt{27})$ and ends at $(1, 1)$, therefore $\alpha + \beta$ is a loop. Since each of the functions ψ_1 and ψ_2 is strictly monotonic (ψ_1 decreases and ψ_2 increases), each of

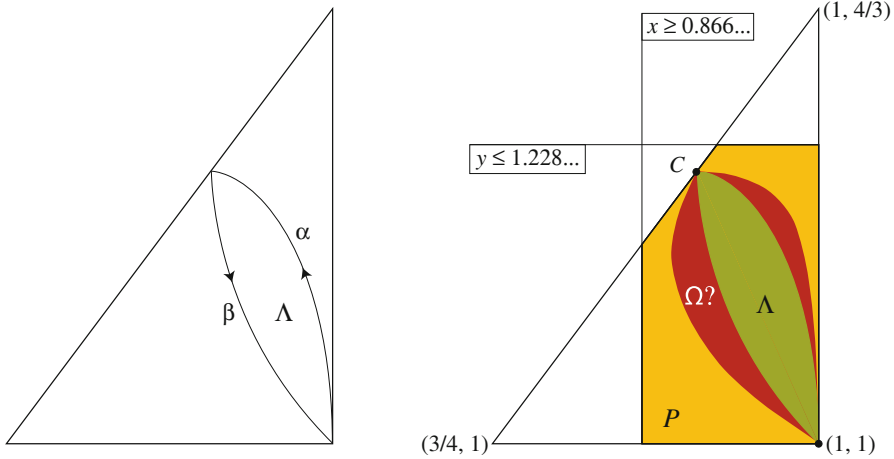


Fig. 7 The leaf Λ and its position within Ω

the paths α and β is an arc. Moreover, α is concave, and β is convex, hence the loop $\alpha + \beta$ is a simple closed curve that bounds a leaf-shaped convex disk Λ , see Fig. 7.

Propositions 4 and 5 yield immediately the equalities

$$\begin{aligned} \delta(K_t) &= \delta_T(K_t) = \delta_L(K_t), \\ \vartheta(K_t) &= \vartheta_T(K_t) = \vartheta_L(K_t), \\ \delta(L_t) &= \delta_T(L_t) = \delta_L(L_t), \end{aligned}$$

and

$$\vartheta(L_t) = \vartheta_T(L_t) = \vartheta_L(L_t),$$

which imply that the leaf Λ is contained in each of Ω , Ω_T , and Ω_L .

Moreover, since each of the disks K_t and L_t is centrally symmetric, the last observation of Remark 2, implies that Λ is contained in each of Ω^* , Ω_T^* , and Ω_L^* as well.

6 Questions

- (Q1) *Are the sets Ω and Ω^* compact?* **Comment.** Since each of the functions δ_T , δ_L , ϑ_T and ϑ_L is affine invariant (see Sect. 1, (iii)), and since the space $[\mathcal{K}^n]$ of affine equivalence classes of all convex n -dimensional bodies is compact (see Macbeath [17]), it follows that each of the sets Ω_T , Ω_L , and $\Omega_T^* = \Omega_L^*$ is compact.
- (Q2–Q4) *Is each of the five Ω -type sets simply-connected? Is each of them a topological disk? Is each of them convex?*
- (Q5) *What are the vertical lines of support from the left for each of the five Ω -type sets?*

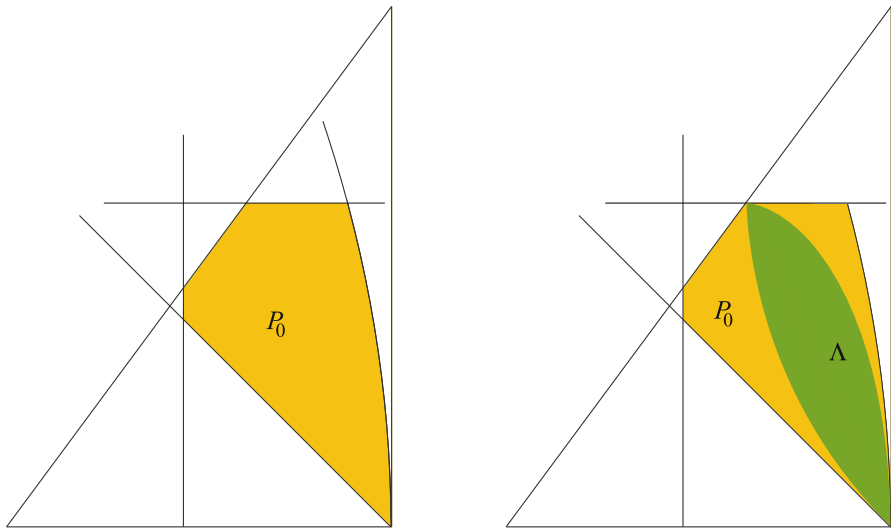


Fig. 8 The inequalities of Ismailescu bound the region P_0 containing Ω_L^* ; the leaf Λ lies in it

- (Q6) *What are the horizontal lines of support from above for each of the five Ω -type sets?*
- (Q7) The inequalities of Ismailescu [9, 10]

$$\delta_L(K) + \vartheta_L(K) \geq 2$$

and

$$\vartheta_L(K) \leq 1 + \frac{5}{4}\sqrt{1 - \delta_L(K)}$$

for every centrally symmetric convex disk K can be expressed as: the set $\Omega_L^* = \Omega_T^*$ lies between the line $x + y = 2$ and the curve $y = 1 + \sqrt{1 - x}$. Does the same hold for the set Ω ?

In addition, the upper bound (2.7) on $\vartheta_T(K)$ for centrally symmetric convex disks further restricts the region P_0 containing Ω_L^* , as shown in Fig. 8.

The inequalities of Ismailescu were inspired by the numerical studies in his own doctoral thesis [9], where he devised an algorithm to compute the maximum area of a hexagon contained in any given centrally symmetric octagon and the minimum area of a hexagon containing it. The algorithm enabled him to compute $\delta(K)$ and $\vartheta(K)$ for every centrally symmetric octagon K and he used it to plot points in Ω_L^* corresponding to a large number of randomly generated centrally symmetric octagons (see Fig. 9a). Moreover, based on the algorithm, he gave a complete, analytical description

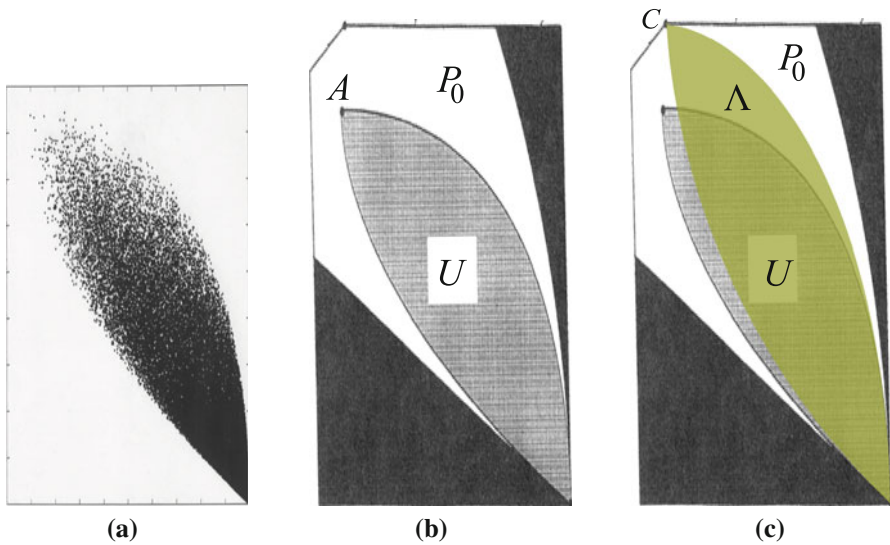


Fig. 9 The Ismailescu range U and the leaf Λ . (Diagrams used and modified by permission of Dan Ismailescu.)

of the subset U of Ω_L^* consisting of points that correspond to all centrally symmetric octagons, namely:

$$U = \left\{ (x, y) \in \mathbb{R}^2 : 1 \leq y \leq 4 - 2\sqrt{2}, \frac{5y^2 - 12y + 8}{2y^2 - 5y + 4} \leq x \leq \frac{y(y + 4 + \sqrt{y^2 - 8y + 8})}{4y + 2} \right\},$$

see Fig. 9b.

Figure 9 shows side-by-side: (a)—the plot of 50,000 points in Ω_L^* corresponding to random computer-generated centrally symmetric octagons; (b)—the analytically described range U , whose apex A corresponds to the regular octagon; and (c)—the leaf-like disk Λ , described in Sect. 5, superimposed on the Ismailescu diagram; the apex C of Λ corresponds to the circular disk.

Note the unexpected resemblance between the shapes of U and Λ . Also note that, somewhat surprisingly, a randomly generated centrally symmetric octagon seems quite likely to be close to a planar tile. *Is the same true for a randomly chosen n -gon with $n \geq 3$?*

A very recent result of Ismailescu and Kim [11] includes the inequality $\delta_L(K)\vartheta_L(K) \geq 1$ for every centrally symmetric K , which is stronger than Ismailescu’s inequality $\delta_L(K) + \vartheta_L(K) \geq 2$ mentioned above, further restricting the convex region P_0 containing Ω_L^* shown in Fig. 8. Still, observe that at the left vertical edge of the region P_0 and at the upper-right corner of it (see Fig. 9c) there seem to be further areas in P_0 free from points of Ω_L^* , as if inviting discovery of additional restricting inequalities.

The questions of describing explicitly the sets Ω , Ω_T , Ω_L , Ω^* , and Ω_L^* remain open and appear to be extremely difficult. However, asking for some better approximations from outside and from inside seems reasonable.

References

1. Chakerian, G.D., Lange, L.H.: Geometric extremum problems. *Math. Mag.* **44**, 57–69 (1971)
2. Dowker, C.H.: On minimum circumscribed polygons. *Bull. AMS* **50**, 120–122 (1944)
3. Fejes Tóth, G., Kuperberg, W.: Packing and covering with convex sets, chapter 3.3. In: Gruber, P.M., Wills, J.M. (eds.) *Handbook of Convex Geometry*, pp. 799–860. Elsevier, Amsterdam (1993)
4. Fejes Tóth, L.: Some packing and covering theorems. *Acta Sci. Math. Szeged* **12/A**, 62–67 (1950)
5. Fejes Tóth, L.: *Lagerungen in der Ebene, auf der Kugel und im Raum*, 2nd edn, 1972. Springer, Berlin (1953)
6. Gauss, C.F.: Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber. *Göttingische gelehrte Anzeigen*, July 9, 1831. Reprinted in: *Werke*, Vol. 2, Königliche Gesellschaft der Wissenschaften, Göttingen, 1863, 188–196; *J. Reine Angew. Math.* **20**(1840), 312320
7. Hales, T.C.: A proof of the Kepler conjecture. *Ann. Math.* **162**(3), 10651185 (2005). Second Series
8. Ismailescu, D.: Covering the plane with copies of a convex disk. *Discrete Comput. Geom.* **20**(2), 251–263 (1998)
9. Ismailescu, D.: *Efficient Packing and Covering and Other Extremal Problems in Geometry*. Thesis (Ph.D.). New York University, p. 97 (2001)
10. Ismailescu, D.: Inequalities between lattice packing and covering densities of centrally symmetric plane convex bodies. *Discrete Comput. Geom.* **25**(3), 365–388 (2001)
11. Ismailescu, D., Kim, B.: *Packing and Covering with Centrally Symmetric Convex Disks*. Preprint, 2013
12. Kershner, R.: The number of circles covering a set. *Am. J. Math.* **61**, 665–671 (1939)
13. Kuperberg, G., Kuperberg, W.: Double-lattice packings of convex bodies in the plane. *Discrete Comput. Geom.* **5**, 389–397 (1990)
14. Kuperberg, W.: An inequality linking packing and covering densities of plane convex bodies. *Geom. Dedic.* **23**(1), 59–66 (1987)
15. Kuperberg, W.: Covering the plane with congruent copies of a convex body. *Bull. Lond. Math. Soc.* **21**(1), 82–86 (1989)
16. Lagrange, J.L.: *Recherches d'Arithmétique*. *Nouv. Mem. de l'Acad. Roy. Sci. et Belles-Lettres Berlin* 265–312 (1773) [Oeuvres, Vol. III, (Gauthier-Villars, Paris 1869) pp. 693795]
17. Macbeath, A.M.: A compactness theorem for affine equivalence-classes of convex regions. *Can. J. Math.* **3**, 54–61 (1951)
18. Sas, E.: Über eine Extremumeigenschaft der Ellipsen. *Compos. Math.* **6**, 468–470 (1939)
19. Thue, A.: Über die dichteste Zusammenstellung von kongruenten Kreisen in der Ebene. *Christiania Vid. Selsk. Skr.* **1**, 3–9 (1910)