

Open Sets Avoiding Integral Distances

Sascha Kurz · Valery Mishkin

Received: 2 April 2012 / Revised: 13 March 2013 / Accepted: 22 April 2013 /
Published online: 23 May 2013
© Springer Science+Business Media New York 2013

Abstract We study open point sets in Euclidean spaces \mathbb{R}^d without a pair of points an integral distance apart. By a result of Furstenberg, Katznelson, and Weiss such sets must be of Lebesgue upper density 0. We are interested in how large such sets can be in d -dimensional volume. We determine the exact values for the maximum volumes of the sets in terms of the number of their connected components and dimension. Here techniques from diophantine approximation, algebra and the theory of convex bodies come into play. Our problem can be viewed as a counterpart to known problems on sets with pairwise rational or integral distances. It possibly opens a new research direction with strong links to topology and measure theory.

Keywords Excluded distances · Euclidean Ramsey theory · Integral distances · Erdős-type problems

Mathematics Subject Classification 52C10 · 52A40 · 51K99

1 Introduction

Is there a dense set S in the plane so that all pairwise Euclidean distances between the points are rational? This famous open problem was posed by Ulam in 1945, see e.g. [18, 39]. Unlike this, a construction of a countable dense set in the plane avoiding

S. Kurz (✉)
Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany
e-mail: sascha.kurz@uni-bayreuth.de

V. Mishkin
Department of Mathematics and Statistics, York University, Toronto, ON M3J1P3, Canada
e-mail: vmichkin@mathstat.yorku.ca

rational distances is not hard to find, see e.g. [27, Problems 13.4, 13.9]. If all pairwise distances between the points in S are integral and S is non-collinear, i.e. not all points are located on a line, then S is finite [2, 17]. Having heard of this result, Ulam guessed that the answer to his question would be in the negative. Of course the rational numbers form a dense subset of a coordinate line with pairwise rational distances; also, on a circle there are dense sets with pairwise rational distances, see e.g. [1, 2]. It was proved by Solymosi and De Zeeuw [37] that the line and the circle are the only two irreducible algebraic curves containing infinite subsets of points with pairwise rational distances. Point sets with rational coordinates on spheres have been considered in [34]. There is interest in a general construction of a planar point set $S(n, k)$ of size n with pairwise integral distances such that $S(n, k) = A \cup B$ where A is collinear, $|A| = n - k$, $|B| = k$, and B has no three collinear points. The current record is $k = 4$ [11]. And indeed, it is very hard to construct a planar point set, no three points on a line, no four points on a circle, with pairwise integral distances. Kreisel and Kurz [28] found such a set of size 7, but it is unknown if there exists one of size 8.

The present paper is concerned with a problem that may be considered as a counterpart to those just described, namely with *large* point sets in \mathbb{R}^d without a pair of points an integral distance apart. We write $f_d(n)$ for the supremum of the volumes $\lambda_d(\mathcal{P})$ of open point sets $\mathcal{P} \subset \mathbb{R}^d$ with n connected components without a pair of points whose distance apart is a positive integer. We determine the exact values of the function $f_d(n)$ for all d and n .

This problem is related to the famous Hadwiger–Nelson open problem of determining the (measurable) chromatic number of \mathbb{R}^d , see e.g. [12, Problem G10]. Here one can also ask for the highest density of one color class in such a coloring, that is, we may ask for the densest set without a pair of points a distance 1 apart. In [31] such a construction in \mathbb{R}^3 has been given. In the plane the best known example, due to Croft [13], consists of the intersections of hexagons with circles and attains a density of 0.2294. The upper bounds are computed in [5, 14]. Point sets avoiding a finite number k of prescribed distances are considered e.g. in [10] and [12, Problem G4], so the point sets avoiding all distances that are positive integers correspond to the case with an infinite number κ of excluded distances. It is known [21] that for each subset \mathcal{U} of the plane with positive density, there is a constant $d(\mathcal{U})$ such that all distances greater than d occur between the points of \mathcal{U} . The same result is true in higher dimensions [33]. It follows that in every dimension $d \geq 2$, the Lebesgue measurable sets avoiding integral distances, which are of interest here, must be of upper density 0, so we consider the supremum of their volumes instead.

The paper is organized as follows: in Sect. 2 we introduce the basic notation and provide characterizations of arbitrary open point sets without pairs of points an integral distance apart. After stating first relationships between the upper bounds for the maximum volumes of those sets with different numbers of connected components we continue in Sect. 3 by considering a relaxed problem. We evaluate the maximum volumes of sets avoiding integral distances in the special case where the connected components of the sets are open balls. In our crucial constructions we make use of Weyl's theorem from diophantine approximation and the fact, we derived from Mann's theorem, that the lengths of the diagonals of a regular p -gon are linearly independent over \mathbb{Q} whenever p is a prime. In Sect. 4 we approach the main problem of evaluating

the function $f_d(n)$ in the general case. For two-component opens sets ($n = 2$) we provide a complete solution in Sect. 4.1. Motivated by the necessary conditions for open point sets to avoid integral distances we consider d -dimensional open sets with n connected components of diameter at most 1 each whose intersection with every line has a total length of at most 1. At the end of Sect. 4.1 we state a conjecture on the exact values of $l_d(n)$ which bears a strong resemblance to the problems of geometric tomography, see e.g. [22]. In Sect. 4.2 we provide some upper bounds for $l_d(2)$ with $d \geq 2$. The main problem of generally evaluating $f_d(n)$ is finally settled in Sect. 4.3. In Sect. 5 we give a summary of the results obtained and draw the appropriate conclusions.

2 General Observations and Basic Notation

Denote by $\text{dist}(x, y)$ the Euclidean distance between two points $x, y \in \mathbb{R}^d$ and by $\text{dist}(V, W) := \inf\{\text{dist}(x, y) \mid x \in V, y \in W\}$ the distance between two subsets V and W of \mathbb{R}^d . The minimal width of V , i.e. the minimum distance between parallel support hyperplanes of the closed convex hull of V , will be denoted by $\text{width}(V)$, and λ_d will stand for the Lebesgue measure in \mathbb{R}^d .

At first we observe that the diameter of any connected component of an open set avoiding integral distances, i.e. having no points an integral distance apart, is at most 1.

Lemma 1 *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be an open set avoiding integral distances. Then for every connected component C of \mathcal{P} we have $\text{diam}(C) \leq 1$.*

Proof Suppose there is a connected component C with $\text{diam}(C) > 1$, then there exist $x_1, x_2 \in C$ such that $\text{dist}(x_1, x_2) > 1$. Since C is open, it is path connected, so there is a point x on the image curve of a continuous path in C joining x_1 and x_2 such that $\text{dist}(x_1, x) = 1$. □

By the isodiametric inequality the open ball $B_d \subset \mathbb{R}^d$ centered at the origin with unit diameter has the largest volume among measurable sets in \mathbb{R}^d of diameter at most 1, see e.g. [19], [6, Chap. 2]. Thus we have

$$f_d(1) = \lambda_d(B_d) = \frac{\pi^{d/2}}{2^d \cdot \Gamma(\frac{d}{2} + 1)} = \begin{cases} \frac{\pi^{\frac{d}{2}}}{2^d (\frac{d}{2})!} & \text{for } d \text{ even,} \\ \frac{(\frac{d-1}{2})! \cdot \pi^{\frac{d-1}{2}}}{d!} & \text{for } d \text{ odd.} \end{cases}$$

The first few values are given by $\lambda_1(B_1) = 1$, $\lambda_2(B_2) = \frac{\pi}{4}$, $\lambda_3(B_3) = \frac{\pi}{6}$, and $\lambda_4(B_4) = \frac{\pi^2}{32}$. We remark that the volume of the scaled ball B with diameter m in \mathbb{R}^d is $\lambda_d(B) = m^d \lambda_d(B_d)$.

Next we characterize 1-dimensional open sets containing a pair of points an integral distance apart.

Lemma 2 *A non-empty open set $\mathcal{P} \subseteq \mathbb{R}$ contains a pair of points $x, y \in \mathcal{P}$ with $\text{dist}(x, y) \in \mathbb{N}$ if and only if either $\lambda_1(\mathcal{P}) > 1$ or there is a pair of connected components (i.e. disjoint open intervals) C_1, C_2 of \mathcal{P} such that $\text{dist}(C_1, C_2) \notin \mathbb{N}$ and*

$\lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2) > \lceil \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \rceil - \text{dist}(\mathcal{C}_1, \mathcal{C}_2)$. If $\lambda_1(\mathcal{P}) \leq 1$, then there exists a shift $f : x \mapsto x + a$ of \mathbb{R} such that $f(\mathcal{P}) \cap \mathbb{Z} = \emptyset$.

Proof The restriction of the canonical epimorphism $\phi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}, x \mapsto x + \mathbb{Z} = (x - \lfloor x \rfloor) + \mathbb{Z}$, to the interval $[0, 1)$ is a continuous bijection of $[0, 1)$ onto the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the inverse map $\phi|_{[0,1)}^{-1}$ being continuous at all points except $\phi(0) = 0 + \mathbb{Z} = \mathbb{Z} \in \mathbb{T}$. We consider the retraction $\phi_1 := \phi|_{[0,1)}^{-1} \circ \phi : \mathbb{R} \rightarrow [0, 1)$, that is, $\phi_1(x) = x - \lfloor x \rfloor$ for all $x \in \mathbb{R}$ (i.e. $\phi_1(x) = x \bmod 1$ is the fractional part of x). We observe that the image under ϕ_1 of any open interval (x, y) of length $y - x < 1$ is either the open interval $(\phi_1(x), \phi_1(y)) = (x - n, y - n)$ of the same length $\phi_1(y) - \phi_1(x) = (x - n) - (y - n) = y - x$, whenever both x and y are in $(n, n + 1)$, for some $n \in \mathbb{Z}$, or the union of two disjoint connected components

$$[0, \phi_1(y)) \cup (\phi_1(x), 1) = [0, y - n) \cup (1 - (n - x), 1)$$

of the same total length $(y - n) + (n - x) = y - x$, whenever $x < n < y$, for some $n \in \mathbb{Z}$. If $y - x = 1$, then similarly either $\phi_1((n, n + 1)) = (0, 1)$ or $\phi_1((x, y)) = [0, y - n) \cup (1 - (n - x), 1) = [0, 1) \setminus \{y - n\}$ whenever $x < n < y$ for some $n \in \mathbb{N}$. Hence, in general, the total length of the connected components of $\phi_1((x, y))$ is $y - x$, whenever $y - x \leq 1$.

Let \mathcal{P} be the disjoint union of open intervals \mathcal{C}_i , say, with total length $\lambda_1(\mathcal{P}) = \sum_i \lambda_1(\mathcal{C}_i) > 1$. Then by Lemma 1 $i \geq 2$ and $\lambda_1(\mathcal{C}_i) \leq 1$ for all i . We thus have from above that the total length of the connected components of all the images $\phi_1(\mathcal{C}_i)$ equals $\sum_i \lambda_1(\mathcal{C}_i) > 1$. Hence at least two images $\phi_1(\mathcal{C}_k)$ and $\phi_1(\mathcal{C}_j)$ must overlap, so there exists $z \in \phi_1(\mathcal{C}_k) \cap \phi_1(\mathcal{C}_j)$, that is, $x_0 - \lfloor x_0 \rfloor = y_0 - \lfloor y_0 \rfloor$ for some $x_0 \in \mathcal{C}_k$ and $y_0 \in \mathcal{C}_j$. Thus $x_0 - y_0 = \lfloor x_0 \rfloor - \lfloor y_0 \rfloor \in \mathbb{Z} \setminus \{0\}$, hence $\text{dist}(x_0, y_0) \in \mathbb{N}$.

If $\lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2) > \alpha$ for some connected components $\mathcal{C}_1 = (a, b)$ and $\mathcal{C}_2 = (c, d)$ of \mathcal{P} with $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) = c - b = m - \alpha$, where $m \in \mathbb{N}, 0 < \alpha < 1$, so that $\lceil \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \rceil - \text{dist}(\mathcal{C}_1, \mathcal{C}_2) = \alpha$, we can take a point x in the leftmost interval, say $x \in \mathcal{C}_1$ and a point $y \in \mathcal{C}_2$ so that the length of $(x, b) \cup (c, y)$ is $\alpha < \lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2) = (b - a) + (d - c)$. Then

$$\text{dist}(x, y) = (b - x) + m - \alpha + (y - c) = \alpha + m - \alpha = m \in \mathbb{N}.$$

Conversely, suppose there are $x, y \in \mathcal{P}$ with $\text{dist}(x, y) = k \in \mathbb{N}$. If x and y lie in the same connected component \mathcal{C}_i of \mathcal{P} , then $\lambda_1(\mathcal{C}_i) > k \geq 1$ because \mathcal{C}_i is open, hence $\lambda_1(\mathcal{P}) > 1$. Suppose x and y lie in distinct connected components of \mathcal{P} , say $x < y$ and $x \in \mathcal{C}_1 = (a, b), y \in \mathcal{C}_2 = (c, d)$, and let $\lambda_1(\mathcal{P}) \leq 1$. Then $(b - a) + (d - c) \leq 1$ as well whence the distance between the components is $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) = c - b \notin \mathbb{N}$, because $c - b < \text{dist}(x, y) < c - b + [(b - a) + (d - c)] \leq c - b + 1$. Let $c - b = m - \alpha$ where $m \in \mathbb{N}, 0 < \alpha < 1$. Then

$$\begin{aligned} \alpha &= m + b - c < m + 1 + b - c < (d - a) + (b - c) \\ &= (b - a) + (d - c) = \lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2), \end{aligned}$$

since $m + 1 < d - a$ because $m + 1 \leq k < d - a$. Thus either $\lambda_1(\mathcal{P}) > 1$ or there is a pair of required connected components of \mathcal{P} .

If $\lambda_1(\mathcal{P}) \leq 1$, then $\lambda_1(\mathcal{C}_i) \leq 1$ for all i , so the total length of the connected components of all the images $\phi_1(\mathcal{C}_i)$ equals $\sum_i \lambda_1(\mathcal{C}_i) = \lambda_1(\mathcal{P})$, as shown previously. If $\lambda_1(\mathcal{P}) < 1$, then clearly, $\phi_1(\mathcal{P}) \neq [0, 1)$. If $\lambda_1(\mathcal{P}) = 1$, then again $\phi_1(\mathcal{P}) \neq [0, 1)$, whenever the images $\phi_1(\mathcal{C}_i)$ are not pairwise disjoint. Suppose all the images $\phi_1(\mathcal{C}_i)$ are pairwise disjoint and $\mathcal{P} \cap \mathbb{Z} \neq \emptyset$. Then there is exactly one $\mathcal{C}_j = (a, b)$ that meets \mathbb{Z} . Hence the complement $[0, 1) \setminus \phi_1(\mathcal{C}_j) = [\phi_1(b), \phi_1(a)]$ is a non-open set in \mathbb{R} that cannot be covered by the images $\phi_1(\mathcal{C}_i)$ of the other connected components of \mathcal{P} , since they are all open intervals, so $\phi_1(\mathcal{P}) \neq [0, 1)$ as well. Thus in all the cases we have $\phi_1(\mathcal{P}) \neq [0, 1)$. Take $\phi_1(a) \in [0, 1) \setminus \phi_1(\mathcal{P})$, $a \in \mathbb{R}$, that is, $\phi_1(a) \cap \phi_1(\mathcal{P}) = \emptyset$. Then $\phi(a) \cap \phi(\mathcal{P}) = \emptyset$, i.e. $(a + \mathbb{Z}) \cap (\mathcal{P} + \mathbb{Z}) = \emptyset$, so $(\mathcal{P} - a) \cap \mathbb{Z} = \emptyset$ and the required shift is $f : x \mapsto x + (-a)$. \square

Applying Lemma 2 we establish a criterion for an open set to avoid integral distances in all dimensions.

Theorem 1 *An open point set $\mathcal{P} \subseteq \mathbb{R}^d$ does not contain a pair of points with integral distance if and only if for every line \mathcal{L}*

- (i) $\lambda_1(\mathcal{P} \cap \mathcal{L}) \leq 1$ and
- (ii) *if \mathcal{L} hits a pair of distinct connected components $\mathcal{C}_1, \mathcal{C}_2$ of \mathcal{P} in the intervals $\mathcal{C}_1 \cap \mathcal{L}, \mathcal{C}_2 \cap \mathcal{L}$ with $\text{dist}(\mathcal{C}_1 \cap \mathcal{L}, \mathcal{C}_2 \cap \mathcal{L}) = r \notin \mathbb{N}$, then $\lceil r \rceil - r \geq \lambda_1((\mathcal{C}_1 \cup \mathcal{C}_2) \cap \mathcal{L})$.*

Another criterion, which we will also be using is:

Lemma 3 *Let \mathcal{P} be a d -dimensional disconnected open set all of whose connected components are of diameter of at most 1. Then \mathcal{P} contains a pair of points with integral distance if and only if*

$$(\text{dist}(\mathcal{C}_1, \mathcal{C}_2), \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2)) \cap \mathbb{N} \neq \emptyset$$

for some of its connected components.

Proof Since all connected components of \mathcal{P} are open with diameter at most 1, any two distinct points of \mathcal{P} with integral distance must be in two different components, say \mathcal{C}_1 and \mathcal{C}_2 . Let $x \in \mathcal{C}_1, y \in \mathcal{C}_2$ with $d(x, y) = n \in \mathbb{N}$. We then select two small closed balls $\overline{B}(x, \varepsilon_1) \subsetneq \mathcal{C}_1$ and $\overline{B}(y, \varepsilon_2) \subsetneq \mathcal{C}_2$ centered at x and y respectively with radii $\varepsilon_1, \varepsilon_2 > 0$. The line \mathcal{L} through x and y meets the two balls in the intervals, say $[x_1, x_2] \subsetneq \overline{B}(x, \varepsilon_1)$ and $[y_1, y_2] \subsetneq \overline{B}(y, \varepsilon_2)$, where $x_1, x_2 \in \mathcal{C}_1$ and $y_1, y_2 \in \mathcal{C}_2$. With this we have

$$\begin{aligned} \text{dist}(\mathcal{C}_1, \mathcal{C}_2) &< \min_{1 \leq i, j \leq 2} d(x_i, y_j) < d(x, y) \\ &= n < \max_{1 \leq i, j \leq 2} d(x_i, y_j) < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2). \end{aligned}$$

Conversely, if $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < n < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2)$ for an integer n , then there exist $x_1, x_2 \in \mathcal{C}_1$ and $y_1, y_2 \in \mathcal{C}_2$ such that

$$\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < d(x_1, y_1) < n < d(x_2, y_2) < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2).$$

Joining x_1 with x_2 in \mathcal{C}_1 and y_1 with y_2 in \mathcal{C}_2 by continuous paths, we can find $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ on the image curves of these paths with $d(x, y) = n$. \square

In some situations it is helpful if we are able to assume that the components of the point sets in question are not too close to each other. Specifically, we will be using the fact that in such cases the connected components of the sets have disjoint closures.

Lemma 4 *Let $\mathcal{C}_1, \mathcal{C}_2$ be distinct connected components of a d -dimensional open point set \mathcal{P} without a pair of points an integral distance apart. If $\lambda_d(\mathcal{C}_1 \cup \mathcal{C}_2) > \lambda_d(B_d)$, then $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1$.*

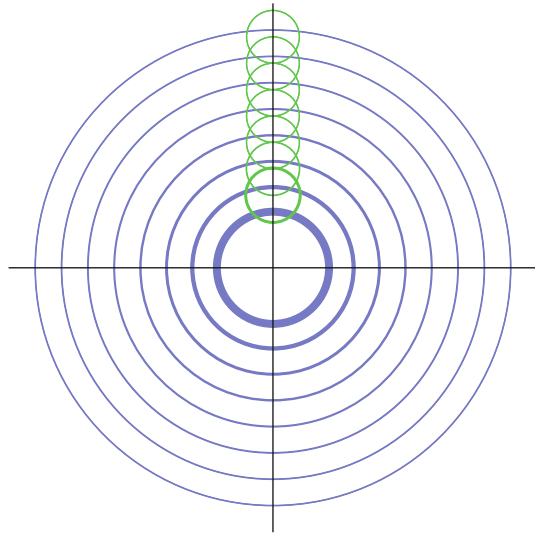
Proof Making use of the isodiametric inequality we deduce from $\lambda_d(\mathcal{C}_1 \cup \mathcal{C}_2) > \lambda_d(B_d)$ that $\text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) > 1$. By Lemma 1 we have $\text{diam}(\mathcal{C}_1) \leq 1$ and $\text{diam}(\mathcal{C}_2) \leq 1$. So we can choose $x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$ with $\text{dist}(x_1, x_2) > 1$. If $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < 1$, then there exist $\bar{x}_1 \in \mathcal{C}_1$ and $\bar{x}_2 \in \mathcal{C}_2$ such that $\text{dist}(\bar{x}_1, \bar{x}_2) < 1$. Since \mathcal{C}_1 and \mathcal{C}_2 are open, they are path connected, hence we can join x_1 and \bar{x}_1 by a continuous path in \mathcal{C}_1 and similarly x_2 and \bar{x}_2 in \mathcal{C}_2 and on the image curves of these paths we then find $x'_1 \in \mathcal{C}_1$ and $x'_2 \in \mathcal{C}_2$ such that $\text{dist}(x'_1, x'_2) = 1$, but \mathcal{P} avoids integral distances, a contradiction. Thus we have $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1$. \square

As Lemma 1 and Theorem 1(i) will be our main tools in estimating upper bounds for $f_d(n)$, we denote by $l_d(n)$ the supremum of the volumes $\lambda_d(\mathcal{P})$ of open point sets $\mathcal{P} \subseteq \mathbb{R}^d$ with n connected components of diameter at most 1 each (condition (a)), and with total length of the intersection with every line at most 1 (condition (b)). Clearly $l_d(1) = f_d(1) = \lambda_d(B_d)$ and $f_d(n) \leq l_d(n)$ for all d and n . We remark that omitting condition (b) trivializes the problem of estimating the extreme volumes, the extremal configurations obviously consist of n disjoint open d -dimensional balls of diameter 1. Dropping condition (a) makes the problem more challenging. It turns out that there are open connected d -dimensional point sets \mathcal{P} with infinite volume $\lambda_d(\mathcal{P})$ and diameter $\text{diam}(\mathcal{P})$ even though the length of the intersection of \mathcal{P} with every line \mathcal{L} is at most 1, i.e. $\lambda_1(\mathcal{P} \cap \mathcal{L}) \leq 1$ for all \mathcal{L} .

Example 1 For integers $n \geq 1$ and $d \geq 2$, denote by \mathcal{A}_n^d the d -dimensional open spherical shell, or annulus, centered at the origin with inner radius n and outer radius $n + \frac{1}{dn^d}$, i.e. \mathcal{A}_n^d are bounded by concentric $(d - 1)$ -dimensional spheres centered at the origin. These shells will guarantee that the volume of their union is unbounded as n tends to infinity. So far the constructed point set is disconnected. To obtain a connected point set, we denote by \mathcal{B}_n^d the d -dimensional open spherical shell centered on the y -axis at $n + \frac{3}{4}$ with inner radius 1 and outer radius $1 + \frac{1}{n^4}$. With this $\mathcal{P} = \bigcup_{n \geq 30} (\mathcal{A}_n^d \cup \mathcal{B}_n^d)$ is open and connected with infinite volume and diameter even though the length of its intersection with every line is at most 1. In Fig. 1 we depicted such a configuration in dimension $d = 2$ containing the first few annuli. The detailed computations demonstrating the assertions claimed are provided in the appendix, see Sect. A.1.

In order to make the problem of evaluating the functions $f_d(n)$ and $l_d(n)$ more tractable, we consider both problems in the special case, where the connected components are restricted to d -dimensional open balls. We denote the corresponding maximum volumes by $f_d^\circ(n)$ and $l_d^\circ(n)$ respectively. Clearly we have $f_d^\circ(n) \leq f_d(n)$ and

Fig. 1 Concentric annuli with infinite area but finite lengths of line intersections (color figure online)



$f_d^\circ(n) \leq l_d^\circ(n) \leq l_d(n)$. In Sect. 3 we determine the exact values of both functions $l_d^\circ(n)$ and $f_d^\circ(n)$ for all d and n .

Based on a simple averaging argument, any given upper bound on one of the four introduced maximum volumes for n components yields an upper bound for $k \geq n$ connected components in the same dimension.

Lemma 5 For each $k \geq n$ we have $l_d(k) \leq \frac{k}{n} \cdot \Lambda_1$ whenever $l_d(n) \leq \Lambda_1$ and $f_d(k) \leq \frac{k}{n} \cdot \Lambda_2$ whenever $f_d(n) \leq \Lambda_2$.

Proof Let \mathcal{P} be a d -dimensional open set with corresponding property in either case and $k \geq n$ connected components. The volume of each of the $\binom{k}{n}$ different unions of n connected components inheriting these properties is at most Λ_i . Since each component occurs exactly $\binom{k-1}{n-1}$ times in those unions, the stated inequalities hold. \square

3 Unions of Open d -Dimensional Balls

Here we consider open point sets \mathcal{P} that are unions of n disjoint d -dimensional open balls of diameter at most 1 each such that they either do not contain a pair of points with integral distance or intersect each line in the intervals with total length at most 1. As introduced in the previous section we denote the supremum of possible volumes of such \mathcal{P} by $f_d^\circ(n)$ and $l_d^\circ(n)$ respectively.

In dimension 1 we can consider one open interval of length $1 - \varepsilon$ and $n - 1$ open intervals of length $\frac{\varepsilon}{n}$, where $1 > \varepsilon > 0$, arranged in a unit interval so that they are pairwise disjoint. The set does not have a pair of points an integral distance apart and the total length of the n intervals tends to 1 as ε approaches 0. It follows from Theorem 1 that $f_1^\circ(n) = l_1^\circ(n) = 1$ for all n . For $n = 1$, by the isodiametric inequality, only the volumes of d -dimensional open balls of diameter 1 attain the maximum values $f_d^\circ(1) = l_d^\circ(1) = \lambda_d(B_d)$.

Lemma 6 For $d, n \in \mathbb{N}$, we have $l_d^\circ(n) \leq \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$.

Proof Consider n disjoint open d -dimensional balls with diameters $X_1 \leq 1, \dots, X_n \leq 1$, where we can assume w.l.o.g. that $X_1 \leq \dots \leq X_n \leq 1$. Clearly in dimension 1 we have $l_1^\circ(n) = l_1(n) = 1 = \lambda_1(B_1)$ for all $n \in \mathbb{N}$ and for all dimensions d , we have $l_d^\circ(1) = l_d(1) = \lambda_d(B_d)$, so in the cases where either d or n is 1 the stated inequality holds, hence we can assume that $d \geq 2$ and $n \geq 2$. Then by Theorem 1 we have $X_i + X_j \leq 1$ for all $1 \leq i < j \leq n$. If $X_n \leq \frac{1}{2}$, then $\sum_{i=1}^n X_i^d \leq \frac{n}{2^d}$ so the required inequality holds. Otherwise we have $X_i \leq 1 - X_n$ and it remains to maximize the function $g_d(x) := x^d + (n - 1)(1 - x)^d$ with domain $[\frac{1}{2}, 1]$. Since $g_d(x)'' = d(d - 1)x^{d-2} + d(d - 1)(n - 1)(1 - x)^{d-2} > 0$, every inner local extremum of g_d is a minimum, so the global maximum of g_d is attained at the boundary of the domain. Finally, we compute $g_d(1) = 1, g_d(\frac{1}{2}) = \frac{n}{2^d}$, hence the lemma follows. \square

Remark The special case of balls of diameter $\frac{1}{2}$ is directly related to point sets with pairwise integral distances. Let \mathcal{P} be the union of n d -dimensional open balls of diameter $\frac{1}{2}$ each without a pair of points an integral distance apart. Then the distance between the centers of the balls must be of the form $d_{i,j} + \frac{1}{2}$ for some integers $d_{i,j}$. By dilation with a factor of two we obtain the set of n centers of the balls with pairwise odd integral distances. However, it has been shown in [23] that for such sets $n \leq d + 2$, where equality is possible if and only if $d + 2 \equiv 0 \pmod{16}$. The exact maximum number of odd integral distances between points in the plane has been determined in [32].

Theorem 2 We have $l_d^\circ(n) = \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$ for all $d \geq 2$.

Proof By Lemma 6 it suffices to provide configurations (asymptotically) attaining the upper bound.

For an arbitrary $1 > \varepsilon > 0$ we consider the union of 1 d -dimensional open ball of diameter $1 - \varepsilon$ and $n - 1$ disjoint open balls of diameter $\frac{\varepsilon}{n-1}$ arranged in the interior of an open ball of diameter 1. As ε tends to 0, the volume of the union tends to $\lambda_d(B_d)$.

For the remaining part we consider the union of n open d -dimensional balls with diameter $\frac{1}{2}$ centered at the vertices of a regular n -gon with circumradius k . Clearly, for k large enough, every line hits at most two balls. \square

Another construction consists of n open d -dimensional balls with centers $(i \cdot k, i^2, 0, \dots, 0)$ and diameter $\frac{1}{2}$ for $1 \leq i \leq n$. If k is large enough then again there is no line intersecting three or more balls.

Corollary 1 We have $f_d^\circ(n) = l_d^\circ(n) = \lambda_d(B_d)$ for $d \geq 2$ and $n \leq 2^d$.

It turns out that, in fact, the equalities $f_d^\circ(n) = l_d^\circ(n) = \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$ hold in all dimensions $d \geq 2$. To explain the underlying idea, we first consider the special case where $d = 2$ and $n = 5$, i.e. the first case that is not covered by Corollary 1.

Lemma 7

$$f_2^\circ(5) = \frac{5\pi}{16} \approx 0.9817477.$$

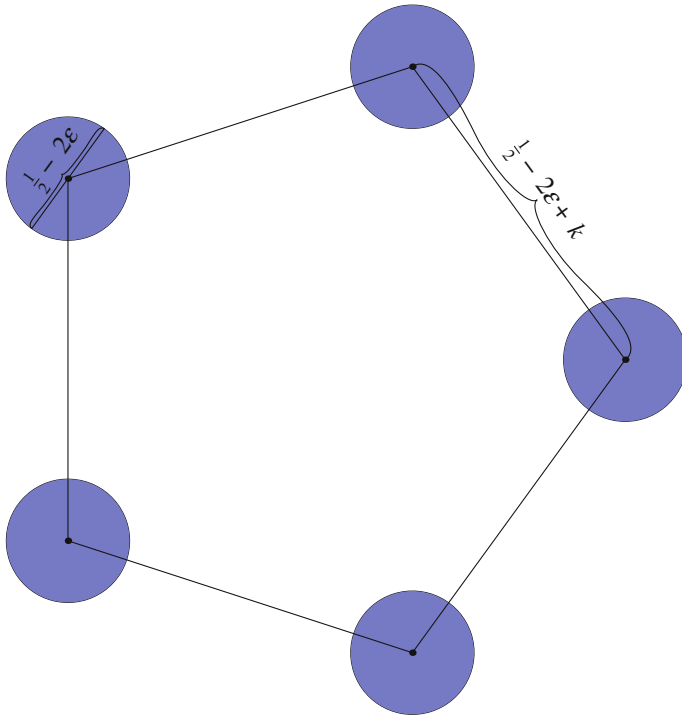


Fig. 2 Integral distance avoiding open set for $d = 2$ and $n = 5$

Proof For each integer $k \geq 2$ and $\frac{1}{7} > \varepsilon > 0$, we consider a regular pentagon P with side length $\frac{1}{2} - 2\varepsilon + k$. At each of the vertices of P we place an open circle of diameter $\frac{1}{2} - 2\varepsilon$, see Fig. 2. Since each component has diameter less than 1, there is no pair of points at integral distance inside one of the five components. For two points a and b from different components we either have

$$k < \text{dist}(a, b) < k + 1,$$

whenever the discs are adjacent with their centers located on an edge of P , or

$$\left(\frac{1 + \sqrt{5}}{2}\right) \cdot k + \frac{\sqrt{5} - 1}{4} - 2\varepsilon < \text{dist}(a, b) < \left(\frac{1 + \sqrt{5}}{2}\right) \cdot k + \frac{3 + \sqrt{5}}{4} - 5\varepsilon$$

otherwise.

Let $[\alpha]$ denote the positive fractional part of a real number α , i.e. there exists an integer l with $\alpha = l + [\alpha]$ and $0 \leq [\alpha] < 1$. If, given $\varepsilon > 0$, one can find an integer k such that

$$\left[\left(\frac{1 + \sqrt{5}}{2}\right) \cdot k + \frac{\sqrt{5} - 1}{4} - 2\varepsilon\right] < 3\varepsilon,$$

then the point set with parameters k and ε does not contain a pair of points with integral distance.

Since $\frac{1+\sqrt{5}}{2}$ is irrational, we can apply the equidistribution theorem, see e.g. [38,40], to deduce that $(\frac{1+\sqrt{5}}{2}) \cdot \mathbb{N}$ is dense (even uniformly distributed) in $[0, 1)$. The same holds true if we shift the set by the fixed real number $\frac{\sqrt{5}-1}{4} - 2\varepsilon > 0$. Thus we can find a suitable integer k for each $\varepsilon > 0$. As ε tends to 0, the total area of the five components approaches $\frac{5\pi}{16}$ which is best possible by Lemma 6. \square

We illustrate this by a short list of suitable values of k :

$$\begin{aligned} \left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{6} \right] &\approx 0.01722, \\ \left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{61} \right] &\approx 0.00909, \\ \left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{116} \right] &\approx 0.00096, \\ \left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{1103} \right] &\approx 0.00051, \end{aligned}$$

and

$$\left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{2090} \right] \approx 0.00005.$$

We shall generalize Lemma 7 to an arbitrary dimension $d \geq 2$ and arbitrary number n of connected components. The idea is to locate the centers of n small d -dimensional balls of diameter slightly less than $\frac{1}{2}$ at some points C_i in a two-dimensional sub-plane so that the set of different pairwise distances α_i between their centers are linearly independent over the rational numbers. So either the distances coincide or they are rationally independent. The appropriate candidates for the center points C_i would be the vertices of a regular p -gon, where p is an odd prime. We use a theorem of Mann, see [29], to prove the desired property of the distances. The condition that the point set in question avoids integral distances can be translated into a system of inequalities of the form $[\alpha_1 \cdot k] < \varepsilon, \dots, [\alpha_l \cdot k] < \varepsilon$, where $k \in \mathbb{N}$, and we are looking for an integer k such that the above fractional parts of the scaled pairwise distances are arbitrarily small. By a theorem of Weyl, see e.g. [40, Satz 3] or a textbook on Diophantine Approximation like e.g. [9], such systems have solutions if the α_i are irrational and linearly independent over \mathbb{Q} (Weyl actually proves equidistribution while we only need denseness, a weaker result that Weyl himself attributes to Kronecker).

We remark that a similar construction, using the vertices of a regular hexagon does not work. Here the lengths of the occurring diagonals are given by 1, $\sqrt{3}$, and 2. The necessary inequalities

$$\left[\left(k + \frac{1}{2} - 2\varepsilon\right) \cdot l_i - \left(\frac{1}{2} - 2\varepsilon\right) \right] = \left[k \cdot l_i + \left(\frac{1}{2}l_i - \frac{1}{2}\right) + (2 - 2l_i) \cdot \varepsilon \right] < 3\varepsilon,$$

where l_i denote the lengths of the diagonals, are trivially satisfied for $l_i = 1$, but cannot hold for $l_i = 2$ and small ε . We remark that very recently Mann’s theorem was used in another problem from discrete geometry see [15,35].

Theorem 3 (Mann [29]) *Suppose we have*

$$\sum_{i=1}^k a_i \zeta_i = 0,$$

with $a_i \in \mathbb{Q}$, the ζ_i roots of unity, and no sub-relations $\sum_{i \in I} a_i \zeta_i = 0$, where $\emptyset \neq I \subsetneq [k]$. Then

$$(\zeta_i / \zeta_j)^m = 1$$

for all i, j , with $m = \prod_{p \leq k} \text{prime } p$.

The vertices of a regular p -gon with a circumcircle of radius 1 centered at the origin are given by

$$\left(\cos\left(\frac{j \cdot 2\pi}{p}\right), \sin\left(\frac{j \cdot 2\pi}{p}\right) \right)$$

for $0 \leq j \leq p - 1$. Using complex notation they coincide with the p th roots of unity

$$\zeta'_j = \cos\left(\frac{j \cdot 2\pi}{p}\right) + i \cdot \sin\left(\frac{j \cdot 2\pi}{p}\right).$$

The distance between vertices 0 and j is given by $2 \sin\left(\frac{j \cdot 2\pi}{2p}\right)$. Since $\sin(\pi - \alpha) = \sin(\alpha)$, there are only $(p - 1)/2$ distinct distances in a regular p -gon, attained for $1 \leq j \leq (p - 1)/2$. We remark that this is not too far away from the minimal number of distinct distances in the plane, which is bounded below by $c \cdot \frac{p}{\log p}$ for a suitable constant c , see [24]. We can express these distances in terms of $2p$ th roots of unity

$$\zeta_j = \cos\left(\frac{j \cdot 2\pi}{2p}\right) + i \cdot \sin\left(\frac{j \cdot 2\pi}{2p}\right)$$

via

$$2 \sin\left(\frac{j \cdot 2\pi}{2p}\right) = \frac{\zeta_j - \zeta_{2p-j}}{i}$$

for all $1 \leq j \leq \frac{p-1}{2}$.

Lemma 8 *Given an odd prime p , let*

$$\alpha_j = \frac{\zeta_j - \zeta_{2p-j}}{i} \text{ for } 1 \leq j \leq \frac{p-1}{2},$$

where the ζ_j are $2p$ th roots of unity. Then the α_j are irrational and linearly independent over \mathbb{Q} .

Proof A folklore result, see e.g. [30], states that $\sin(\pi q)$, where $q \in \mathbb{Q}$, is a rational number if and only if $\sin(\pi q) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. Since p is odd this cannot occur in our context. It remains to show that the α_j are linearly independent over \mathbb{Q} . Suppose to the contrary that there are rational numbers b_j for $1 \leq j \leq l \leq \frac{p-1}{2}$ with $\sum_{j=1}^l b_j \alpha_j = 0$. We then have

$$\sum_{j=1}^l (b_j \zeta_j - b_j \zeta_{2p-j}) = 0.$$

Now let J be a subset of those indices $j, 2p - j$ such that $\sum_{j \in J} a_j \zeta_j = 0$, where $a_j \in \{\pm b_j\}$, and no sub-relation adds up to 0. We have $|J| \leq p - 1$. Hence by Mann’s Theorem $(\zeta_{j_1} / \zeta_{j_2})^2 = 1$ for all $j_1, j_2 \in J$, since

$$\gcd\left(2p, \prod_{t \leq p-1 \text{ prime}} t\right) = 2.$$

With this we obtain $j_2 = j_1 + p$ for $j_1 < j_2$. Since J is a subset of

$$\left\{1, \dots, \frac{p-1}{2}\right\} \cup \left\{2p - \frac{p-1}{2}, \dots, 2p - 1\right\},$$

this is impossible and the α_j have to be linearly independent over \mathbb{Q} . □

Theorem 4 *For $d \geq 2$ we have $f_d^\circ(n) = \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$.*

Proof Since $f_d^\circ(n) \leq l_d^\circ(n)$ we conclude from Theorem 2 the upper bound $f_d^\circ(n) \leq \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$. For the construction we fix an odd prime p with $p \geq n$. For each integer $k \geq 2$ and each $\frac{1}{4} > \varepsilon > 0$ we consider a regular p -gon P with side lengths $2k \cdot \sin\left(\frac{\pi}{p}\right)$, i.e. with radius k . At n arbitrarily chosen vertices of the p -gon P we place the centers of d -dimensional open balls with diameter $\frac{1}{2} - 2\varepsilon$. Since each of the n components has a diameter less than 1 there is no pair of points at integral distance inside one of these n components. Next we consider two points a and b from two different components. By α we denote the distance of the centers of the corresponding open balls. From the triangle inequality we conclude

$$\alpha - \left(\frac{1 - 4\varepsilon}{2}\right) < \text{dist}(a, b) < \alpha + \left(\frac{1 - 4\varepsilon}{2}\right).$$

Since the occurring distances α are given by $2k \sin\left(\frac{\pi}{p}\right)$ for $1 \leq j \leq \frac{p-1}{2}$ we look for a simultaneous solution of the system

$$\left[2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + 2\varepsilon\right] \leq 4\varepsilon$$

with $k \in \mathbb{N}$. By Lemma 8 the factors $2 \sin\left(\frac{j\pi}{p}\right)$ are irrational and linearly independent over \mathbb{Q} , so by Weyl’s Theorem [40] the systems admit a solution for all ε .

Therefore, for every $0 < \varepsilon < \frac{1}{4}$ we can choose a suitable value of k and construct a point set without pairs of points an integral distance apart with a volume of $n\left(\frac{1}{2} - 2\varepsilon\right)^d \cdot \lambda_d(B_d)$. As ε approaches 0 this volume tends to $\frac{n}{2^d} \cdot \lambda_d(B_d)$. For small values of n we consider, for an arbitrary $\varepsilon > 0$, one open d -dimensional ball of diameter $1 - \varepsilon$ and $d - 1$ open d -dimensional balls of diameter $\frac{\varepsilon}{d}$ arranged in an open d -dimensional ball of diameter 1, see the proof of Theorem 2. As the diameter of the whole set is less than 1 there is no pair of points with integral distance. \square

Thus, in the case of spherical components the values of $l_d^\circ(n)$ and $f_d^\circ(n)$ are completely determined. For general components the problem is more challenging for $n \geq 2$ and will be addressed in the following section.

4 Bounds for $l_d(n)$ and the Exact Value of $f_d(n)$

In dimension 1 we can consider one open interval of length $1 - \varepsilon$ and $n - 1$ open intervals of length $\frac{\varepsilon}{n}$, where $1 > \varepsilon > 0$, arranged in the unit interval so that they are pairwise non-intersecting. Obviously there is no pair of points with integral distance and the total length of the n intervals tends to 1 as ε approaches 0. Thus we can conclude $f_1(n) = l_1(n) = 1$ from Theorem 1. For $n = 1$ component the unique example achieving the maximum volume of $f_d(1) = l_d(1) = \lambda_d(B_d)$ is the d -dimensional open ball with diameter 1. For $n, d \neq 1$ the problem gets more involved. In Sect. 4.1 we treat the case of $n = 2$ components. In general we are only able to obtain some lower and upper bounds, which is executed in Sect. 4.2 for $l_d(n)$ and in Sect. 4.3 the exact value of $f_d(n)$ is determined.

4.1 Two Components

At first we will start to obtain an upper bound in the case of two components. To this end, we remark that we can reformulate the condition of Lemma 3 as $\text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) \leq \lceil \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \rceil + 1$. Next we use Lemmas 3 and 4 to deduce some structural information on the pairs of connected components $\mathcal{C}_1, \mathcal{C}_2$ in a d -dimensional integral distance avoiding set \mathcal{P} . Due to Lemma 4 there exist parallel hyperplanes \mathcal{H}_2 and \mathcal{H}_3 such that, after a possible relabeling of the components, \mathcal{C}_1 is on the left hand side of \mathcal{H}_2 , \mathcal{C}_2 is on the right hand side of \mathcal{H}_3 , and \mathcal{H}_2 is on the left hand side of \mathcal{H}_3 . W.l.o.g. we assume $\text{dist}(\mathcal{H}_2, \mathcal{H}_3) \leq \text{dist}(\mathcal{C}_1, \mathcal{C}_2)$. By Lemma 3 there exist two further hyperplanes $\mathcal{H}_1, \mathcal{H}_4$ being parallel to \mathcal{H}_2 and \mathcal{H}_3 such that \mathcal{C}_1 is on the right hand side of \mathcal{H}_1 and \mathcal{C}_2 is on the left hand side of \mathcal{H}_4 . In other words, \mathcal{C}_1 lies between \mathcal{H}_1 and \mathcal{H}_2 , and \mathcal{C}_2 lies

Table 1 Values of $\lambda_d(S_d)$ and $\lambda_d(C_d)$ for small dimensions

d	2	3	4	5
$\lambda_d(S_d)$	$\frac{\sqrt{3}}{8} + \frac{\pi}{12} \approx 0.4783$	$\frac{11\pi}{96} \approx 0.3600$	$\frac{\pi}{384} \cdot (9\sqrt{3} + 4\pi) \approx 0.2303$	$\frac{203\pi^2}{15360} \approx 0.1304$
$\lambda_d(C_d)$	$\frac{\pi}{12} - \frac{\sqrt{3}}{16} \approx 0.1535$	$\frac{5\pi}{192} \approx 0.0818$	$\frac{\pi^2}{96} - \frac{3\sqrt{3}\pi}{256} \approx 0.0390$	$\frac{53\pi^2}{30720} \approx 0.0170$

between \mathcal{H}_3 and \mathcal{H}_4 . W.l.o.g. we can assume $\text{dist}(\mathcal{H}_1, \mathcal{H}_4) \leq \text{diam}(C_1 \cup C_2)$. Thus for $d_1 := \text{dist}(\mathcal{H}_1, \mathcal{H}_2)$ and $d_2 := \text{dist}(\mathcal{H}_3, \mathcal{H}_4)$ we have $d_1 + d_2 \leq 1$ by Theorem 1(i). Clearly d_1 and d_2 are upper bounds for the width of C_1 and C_2 , respectively.

For a convex body \mathcal{K} in \mathbb{R}^d with diameter D and minimal width ω an upper bound on its d -dimensional volume V has been found in [25, Theorem 1], namely:

$$V \leq \lambda_{d-1}(B_{d-1}) \cdot D^d \int_0^{\arcsin \frac{\omega}{D}} \cos^d \theta \, d\theta. \tag{1}$$

Equality holds if and only if \mathcal{K} is the d -dimensional spherical symmetric slice with diameter D and minimal width ω . In the planar case some more inequalities relating several descriptive parameters of a convex set can be found in [36]. Since we will extensively use d -dimensional spherical symmetric slices with diameter 1 and width $\frac{1}{2}$, we denote them by S_d . Viewing S_d as a truncated d -dimensional ball of unit diameter we denote the two isomorphic cut-off bodies by C_d and call them caps, i.e. we have $\lambda_d(B_d) = \lambda_d(S_d) + 2 \cdot \lambda_d(C_d)$.

$$\lambda_d(S_d) = \lambda_{d-1}(B_{d-1}) \int_0^{\frac{\pi}{6}} \cos^d \theta \, d\theta, \tag{2}$$

$$\lambda_d(C_d) = \frac{1}{2} \cdot (\lambda_d(B_d) - \lambda_d(S_d)). \tag{3}$$

In Table 1 we give the first exact volumes and refer to the Appendix, i.e. Sect. A.2, for further equivalent expressions.

Lemma 9 For $d \geq 2$, we have $f_d(2) \leq 2\lambda_d(S_d)$.

Proof With the notation as introduced above we estimate the total volume of the closed convex hulls of the two connected components $\text{conv}(\bar{C}_1)$, $\text{conv}(\bar{C}_2)$,

$$\lambda_d(\text{conv}(\mathcal{C}_1)) + \lambda_d(\text{conv}(\mathcal{C}_2)),$$

where both components have a diameter of at most 1, C_1 has a width of at most d_1 , and C_2 has a width of at most d_2 , using Inequality (1). We thus have

$$\lambda_d(\text{conv}(\mathcal{C}_1)) \leq \lambda_d(B_{d-1}) \int_0^{\arcsin d_1} \cos^d \theta \, d\theta$$

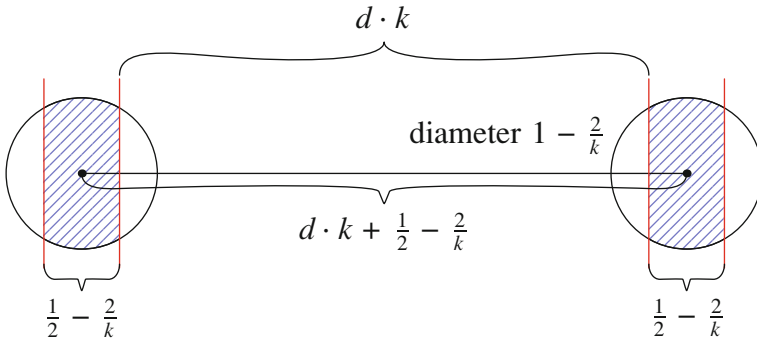


Fig. 3 Truncated circles—a construction of two components without integral distances

and

$$\lambda_d(\text{conv}(\mathcal{C}_2)) \leq \lambda_d(B_{d-1}) \int_0^{\arcsin d_2} \cos^d \theta \, d\theta.$$

Since both right hand sides are strictly monotone in d_1, d_2 , respectively, we can assume w.l.o.g. that $d_1 + d_2 = 1$. So it suffices to maximize the function

$$\int_0^{\arcsin x} \cos^d \theta \, d\theta + \int_0^{\arcsin(1-x)} \cos^d \theta \, d\theta$$

for $x \in [0, 1]$. After a straightforward calculation we conclude that the unique maximum is attained at $x = \frac{1}{2}$. □

Lemma 10

$$f_d(2) \geq 2 \cdot \lambda_d(S_d).$$

Proof For an arbitrary integer $k \geq 5$ we place a d -dimensional ball with diameter $1 - \frac{2}{k}$ at the origin and cut off the spherical cap at the hyperplanes with value $\pm(\frac{1}{4} - \frac{1}{k})$ of the first coordinate. By \mathcal{S}_1 we denote the resulting truncated ball. Another such truncated ball \mathcal{S}_2 is located with a shift of $dk + \frac{1}{2} - \frac{2}{k}$ in the direction of the first coordinate (see Fig. 3 for a drawing of the two-dimensional case). Both \mathcal{S}_1 and \mathcal{S}_2 have a diameter less than 1 for all $k \in \mathbb{N}$ so that they contain no pair of points with integral distance. For two points $a \in \mathcal{S}_1$ and $b \in \mathcal{S}_2$ we have

$$dk < \text{dist}(a, b) < \sqrt{(d-1)\left(1 - \frac{2}{k}\right)^2 + \left(dk + 1 - \frac{4}{k}\right)^2} \leq dk + 1,$$

so that $\mathcal{S}_1 \cup \mathcal{S}_2$ contains no pair of points with integral distance.

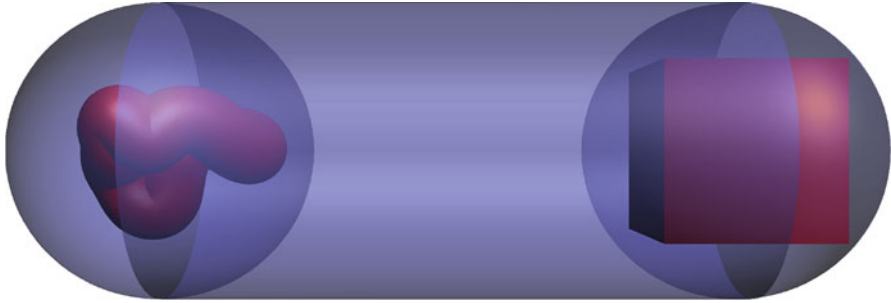


Fig. 4 Two 3-dimensional components with the enclosing balls and enclosing cylinder

Finally we remark that the volume of $\mathcal{S}_1 \cup \mathcal{S}_2$ approaches $2 \cdot \lambda_d(S_d)$ as k tends to ∞ . \square

Combining Lemma 9 with Lemma 10 yields:

Corollary 2 For $d \geq 2$ we have $f_d(2) = 2\lambda_d(S_d)$.

One might conjecture that the upper bound from Lemma 9 is also valid for $l_d(2)$, see Conjecture 1. Technically, we have used Lemmas 3 and 4, but it might be possible to come up with another approach which does not rely on these properties. In this context we would like to remark that related problems can be quite complicated, e.g. it is quite hard to determine the equilateral n -gon with diameter 1 and maximum area [3,4].

Conjecture 1 For $n \geq 2$ and $d \geq 2$ we have $l_d(n) = n \cdot \lambda_d(S_d)$.

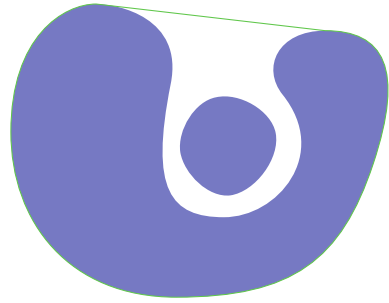
4.2 Bounds for $l_d(n)$

Using exhaustion over lines, we can conclude two first upper bounds for $l_d(n)$.

Lemma 11 For $d \geq 2$, we have $l_d(2) \leq \lambda_{d-1}(B_{d-1}) \cdot \left(\sqrt{\frac{2d}{d+1}}\right)^{d-1}$.

Proof By Lemma 1 both connected components, denoted by \mathcal{C}_1 and \mathcal{C}_2 , are of diameter at most 1, so Jung's theorem [16,26] yields the enclosing balls $\mathcal{B}_1, \mathcal{B}_2$ for these components of diameter $\sqrt{\frac{2d}{d+1}}$. So there is an enclosing cylinder, having a $(d-1)$ -dimensional ball of diameter $\sqrt{\frac{2d}{d+1}}$ as its base, containing the closed convex hull $\text{conv}(\overline{\mathcal{B}_1 \cup \mathcal{B}_2})$. The diagram is depicted in Fig. 4, where we remark that the two enclosing balls $\mathcal{B}_1, \mathcal{B}_2$ might intersect in general. By exhausting the cylinder with the lines parallel to the line trough the centers of \mathcal{B}_1 and \mathcal{B}_2 , i.e. using a suitable Riemann integral or Fubini's theorem, and applying Theorem 1(i) we conclude that the volume of $\mathcal{C}_1 \cup \mathcal{C}_2$ is at most $\lambda_{d-1}(B_{d-1}) \cdot \left(\sqrt{\frac{2d}{d+1}}\right)^{d-1}$. \square

Fig. 5 A component contained in the convex hull of another



The estimates for the first few upper bounds of $l_d(2)$ in Lemma 11 are:

$$l_2(2) \leq \frac{2}{\sqrt{3}} \approx 1.1547, \quad l_3(2) \leq \frac{3\pi}{8} \approx 1.1781,$$

$$l_4(2) \leq \frac{8\sqrt{2}\pi}{15\sqrt{5}} \approx 1.0597, \quad l_5(2) \leq \frac{25\pi^2}{288} \approx 0.8567$$

and $l_d(2)$ tends to 0 as the dimension d approaches infinity.

The enclosing balls are a bit wasteful. The universal cover problem, first stated in a personal communication of Lebesgue in 1914, asks for the minimum area A of a convex set U containing a congruent copy of any planar set of diameter 1, see [8]. For the currently best known bounds $0.832 \leq A \leq 0.844$ and generalizations to higher dimensions we refer the interested reader to [7, Sect. 11.4]. In this paper we will not use this concept. The restriction to d -dimensional open balls as components has already been treated in Sect. 3.

In dimension $d = 2$ the upper bound from Lemma 11 can be improved easily.

Lemma 12

$$l_2(2) \leq 1.$$

Proof Let \mathcal{P} be a planar open point set consisting of two components \mathcal{C}_1 and \mathcal{C}_2 each of diameter at most 1. If one of them is contained in the closed convex hull of the other, see Fig. 5 for an example, then we have $\lambda_2(\mathcal{P}) \leq \lambda_2(B_2) = \frac{\pi}{4} < 1$. Otherwise, we select any support line \mathcal{L} through the boundary points of \mathcal{C}_1 and \mathcal{C}_2 so that both regions are in the same half-plane determined by \mathcal{L} . We then consider the strip parallel to this line with smallest possible width w containing both regions, see Fig. 6. Since both \mathcal{C}_1 and \mathcal{C}_2 have diameter at most 1 we have $w \leq 1$. By exhausting the strip with the lines parallel to \mathcal{L} and applying Theorem 1 we conclude that the area of $\mathcal{C}_1 \cup \mathcal{C}_2$ is at most 1. \square

4.3 The Exact Value of $f_d(n)$

By combining Lemmas 9 and 5 we obtain the upper bound $f_d(n) \leq n\lambda_d(S_d)$. In the remaining part of this subsection we will describe a construction which asymptotically achieves this upper bound.

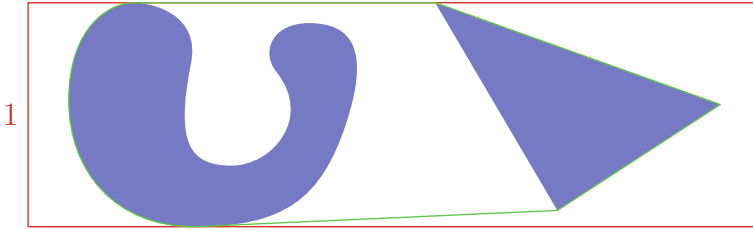


Fig. 6 Two components between two parallel lines

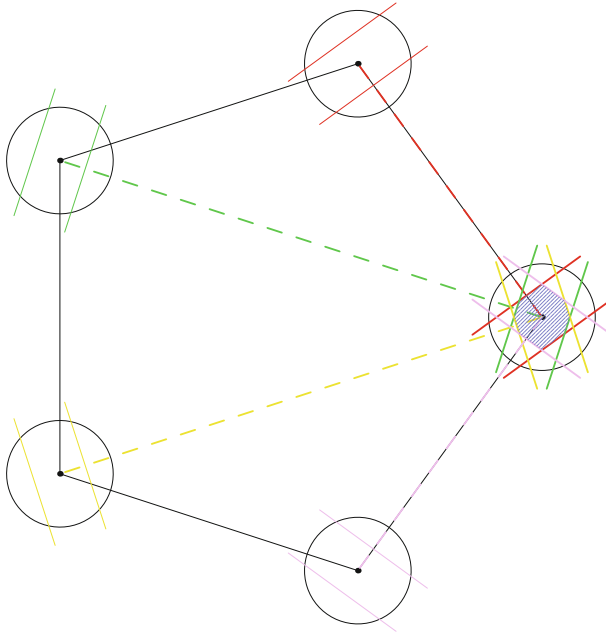


Fig. 7 p -gon Construction: integral distance avoiding point set for $d = 2$ and $p = n = 5$

As a first step we remark that we can improve slightly the construction from Theorem 4. For $d \geq 2$ we choose a prime $p \geq n$ and locate n open balls of diameter $1 - 2\varepsilon$, where ε is suitably chosen, at n out of the p vertices of the regular p -gon. For each two balls we cut off spherical caps in the directions of the lines connecting two centers resulting in a width of $\frac{1}{2} - 2\varepsilon$. If the radius of the regular p -gon approaches infinity we can assume that ε can tend to 0. So to compute the asymptotic volume of this construction it suffices to consider a regular p -gon P of fixed radius > 2 , where we locate n open balls at the vertices of P and cut off spherical caps such that the components have a width of $\frac{1}{2}$ in the direction of each line connecting two used vertices, i.e. the centers of the n balls. For future reference we call this construction the p -gon construction. An example with $p = n = 5$ in dimension $d = 2$ is depicted in Fig. 7.

Theorem 5 For $d \geq 2$ and $n \geq 2$ we have $f_d(n) = n\lambda_d(S_d)$.

Proof Lemmas 9 and 5 yield the upper bound $f_d(n) \leq n\lambda_d(S_d)$. By Lemma 10 we can assume that $n \geq 3$. For arbitrary ε let $S_{d,\varepsilon}$ be a d -dimensional spherical symmetrical

slice with diameter $1 - 2\varepsilon$ and minimal width $\frac{1}{2} - 2\varepsilon$. As ε tends to 0, the volume of $S_{d,\varepsilon}$ approaches $\lambda_d(S_d)$. In the following we provide a construction of an open n -component point set \mathcal{P}' avoiding integral distances each of whose connected n components contains a congruent copy of $S_{d,\varepsilon}$.

Consider a regular p -gon P with circumradius k , the parameters p and k to be specified. We enumerate clockwise the vertices of P from 1 to p and assume w.l.o.g. that the line through vertices 1 and 2 is the x -axis. At each vertex $1 \leq i \leq n \leq p$ we place the center of an open d -dimensional ball of diameter $1 - \varepsilon$. For each pair of the n balls we cut off spherical caps in the direction of the lines through their centers resulting in a width of $\frac{1}{2} - \varepsilon$. We denote the union of the resulting n open sets by \mathcal{P} .

Consider further all $2 \cdot \binom{n}{2}$ cutting hyperplanes that cut off the spherical caps from the initial open balls. As the number p of vertices of the p -gon P increases, with n fixed, all those hyperplanes tend to be orthogonal to the x -axis. Now choose a prime p large enough so that each connected component of \mathcal{P} contains a d -dimensional spherical symmetrical slice with diameter $1 - 2\varepsilon$ and minimal width $\frac{1}{2} - 2\varepsilon$ whose cutting hyperplanes are orthogonal to the x -axis. By \mathcal{P}' we denote the subset of \mathcal{P} which is the union of those $S_{d,\varepsilon}$'s.

There exists a number k_1 such that for $k \geq k_1$ each line hits at most two connected components of \mathcal{P}' . Since the diameter of each of its connected components is at most $1 - 2\varepsilon$, the pairwise distances between the points within the same component are non-integral. Let a and b be two points in different connected components. By the construction the distance between the corresponding centers is given by $2k \cdot \sin\left(\frac{j\pi}{p}\right)$ for a suitable integer j . Thus

$$\text{dist}(a, b) \geq 2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + \varepsilon.$$

There exists a number k_2 such that for $k \geq k_2$, we have

$$\text{dist}(a, b) \leq 2k \cdot \sin\left(\frac{j\pi}{p}\right) + \frac{1}{2} - \varepsilon,$$

since all the lines joining the centers of the connected components of \mathcal{P}' then to be horizontal, as k increases (compare the proof of Lemma 10).

Thus, provided that $k \geq \max(K_1, k_2)$, the system of inequalities

$$\left[2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + \varepsilon\right] \leq 2\varepsilon$$

has a solution, the distance $\text{dist}(a, b)$ cannot be integral, so \mathcal{P}' does not contain a pair of points with integral distance. By Lemma 8 and the Weyl theorem the above system admits indeed a solution. □

5 Conclusion

Problems related to point sets with pairwise rational or integral distances were one of Erdős' favorite subjects in combinatorial geometry. In the present paper we study

some kind of a counterpart to this type of problems by asking for the largest open d -dimensional set \mathcal{P} of points without a pair of points an integral distance apart, i.e. that with the largest possible volume $f_d(n)$, where n denotes the number of connected components of \mathcal{P} . As a relaxation we have also considered d -dimensional open point sets consisting of n connected components, each having a diameter at most 1, such that the intersection with every line has a total length of at most 1. The corresponding maximum volume was denoted by $l_d(n)$. While the assumption on diameters of the connected components seems to be a bit technical, geometrical objects with specified intersections with lines or higher-dimensional subspaces are interesting in their own right. So far we were not able to prove that the maximum volumes $l_d(n)$ are bounded if we drop the condition on the diameters. In this context we just mention the famous Kakeya problem of whether a Kakeya set in \mathbb{R}^d , i.e. a compact set containing a unit line segment in every direction, has Hausdorff dimension d , see e.g. the review [41] or [12, Problem G6].

By restricting the shapes of the connected components to d -dimensional balls, we were able to determine the exact values of the corresponding maximum volumes $f_d^\circ(n)$ and $l_d^\circ(n)$, respectively. Also the values of $f_d(n)$ could be determined exactly, while for $l_d(n)$ we only have the lower bound $l_d(n) \geq f_d(n)$, which we conjecture to be tight.

Acknowledgments The authors thank the anonymous referees for their extensive comments, which helped very much to improve the presentation of this paper, Juris Steprans, Peter Biryukov, Robert Connelly, and Andrey Verevkin for valuable comments and discussions, Tobias Kreisel for producing several graphics, Thomas Kriecherbauer and Benoit Cloitre for some remarks on integrals and integer sequences.

Appendix

In order to keep the main part of the paper more accessible we have moved some side remarks and technical calculations to this appendix.

A. 1 Details of the Annuli Construction

We have to ensure that Construction 1 satisfies the proposed properties. First we remark that both \mathcal{A}_n^d and \mathcal{A}_{n+1}^d meet \mathcal{B}_n^d for $n \geq 1$. Thus \mathcal{P} is a connected open set in \mathbb{R}^d . The volume $\lambda_d(\mathcal{A}_n^d)$ is given by

$$\begin{aligned} & \lambda_d(B_d) \cdot \left(\left(2n + \frac{2}{dn^d} \right)^d - (2n)^d \right) \\ & = \lambda_d(B_d) \cdot 2^d \cdot \left(\left(n + \frac{1}{dn^d} \right)^d - n^d \right) \geq \lambda_d(B_d) \cdot 2^d \cdot \frac{1}{n}. \end{aligned}$$

Since the harmonic series diverges to infinity, the d -dimensional volume of \mathcal{P} is unbounded.

Now we consider the intersection of a line \mathcal{L} with a d -dimensional annulus $\mathcal{C}_d(r_1, r_2)$ with inner radius r_1 and outer radius r_2 centered at the origin. Due to symmetry we can assume that \mathcal{L} is parallel with the x -axis, i.e.

$$\mathcal{L} = \{ (1 \ 0 \ \dots \ 0)^T \cdot \lambda + (0 \ a_2 \ \dots \ a_d)^T \mid \lambda \in \mathbb{R} \}.$$

By symmetry we can further assume $a_i \geq 0$ for all $2 \leq i \leq d$. To ease notation we set $l := \sqrt{\sum_{i=2}^d a_i^2}$. We remark $\mathcal{C}_d(r_1, r_2) \cap \mathcal{L} = \emptyset$ for $l^2 > r_2^2$. The x -coordinates of the intersections of \mathcal{L} with the d -dimensional sphere of radius r_1 are given by $\pm\sqrt{r_1^2 - l^2}$, as long as $l^2 \leq r_1^2$. Similarly the x -coordinates of the intersections of \mathcal{L} and the d -dimensional sphere of radius r_2 are given by $\pm\sqrt{r_2^2 - l^2}$, as long as $l^2 \leq r_2^2$. For $l^2 \leq r_1^2$ we have

$$\lambda_1(\mathcal{C}_d(r_1, r_2) \cap \mathcal{L}) = 2 \cdot \underbrace{\left(\sqrt{r_2^2 - \sum_{i=2}^d a_i^2} - \sqrt{r_1^2 - \sum_{i=2}^d a_i^2} \right)}_{=:h_1(a_2, \dots, a_d)}.$$

Since

$$\frac{\partial h_1}{\partial a_i}(a_2, \dots, a_d) = a_i \cdot \left(\frac{1}{\sqrt{r_1^2 - \sum_{i=2}^d a_i^2}} - \frac{1}{\sqrt{r_2^2 - \sum_{i=2}^d a_i^2}} \right) \geq 0,$$

we can assume $l^2 \geq r_1^2$ for the maximum length of the intersection. If the a_i are restricted by an inequality $l^2 \leq k^2 \leq r_1^2$ the maximum length of the intersection is bounded from above by $2\sqrt{r_2^2 - k^2} - 2\sqrt{r_1^2 - k^2}$.

For $r_1^2 \leq \sum_{i=2}^d a_i^2 \leq r_2^2$ we have

$$\lambda_1(\mathcal{C}_d(r_1, r_2) \cap \mathcal{L}) = 2 \cdot \underbrace{\sqrt{r_2^2 - \sum_{i=2}^d a_i^2}}_{=:h_2(a_2, \dots, a_d)}$$

and

$$\frac{\partial h_1}{\partial a_2}(a_2, \dots, a_d) = -a_i \cdot \frac{1}{\sqrt{r_2^2 - \sum_{i=2}^d a_i^2}} \leq 0,$$

so that the extremal values are taken at $\sum_{i=2}^d a_i^2 = r_1^2$ where we have $\lambda_1(\mathcal{C}_d(r_1, r_2) \cap \mathcal{L}) \leq 2\sqrt{r_2^2 - r_1^2}$.

Thus for an arbitrary line \mathcal{L} we have

$$\lambda_1(\cup_{n \geq 30} \mathcal{B}_n^d \cap \mathcal{L}) \leq \sum_{n=30}^{\infty} 2\sqrt{\left(1 + \frac{1}{n^4}\right)^2 - 1^2} \leq \sum_{n=30}^{\infty} \frac{2\sqrt{3}}{n^2} < 0.12.$$

For the remaining part we restrict ourselves on lines being parallel to the x -axis. If $l < 30$ then

$$\begin{aligned} \lambda_1(\cup_{n \geq 30} \mathcal{A}_n^d \cap \mathcal{L}) &\leq 2\sqrt{(30 + \frac{1}{d \cdot 30^d})^2 - 30^2} \\ &\quad + \sum_{n=31}^{\infty} 2\sqrt{(n + \frac{1}{dn^d})^2 - l^2} - 2\sqrt{n^2 - l^2} \\ &\leq 0.366 + 2 \sum_{n=31}^{\infty} \frac{\frac{2}{n}}{2\sqrt{n^2 - 30^2}} < 0.47. \end{aligned}$$

For $l \geq 30$ we have

$$\begin{aligned} \lambda_1(\cup_{n \geq 30} \mathcal{A}_n^d \cap \mathcal{L}) &\leq 4\sqrt{(\lfloor l \rfloor + \frac{1}{d \cdot \lfloor l \rfloor^d})^2 - \lfloor l \rfloor^2} \\ &\quad + \sum_{n=\lfloor l+2 \rfloor}^{\infty} 2\sqrt{(n + \frac{1}{dn^d})^2 - l^2} - 2\sqrt{n^2 - l^2} \\ &\leq 0.732 + 2 \int_{\lfloor l+1 \rfloor}^{\infty} \frac{1}{x\sqrt{x^2 - l^2}} dx \\ &= 0.732 + \frac{2}{l} \cdot \arcsin\left(\frac{l}{\lfloor l+1 \rfloor}\right) \\ &\leq 0.732 + \frac{2}{l} \cdot \frac{\pi}{2} < 0.84. \end{aligned}$$

Since $0.12 + \max(0.47, 0.84) < 1$ we have $\lambda_1(\mathcal{P} \cap \mathcal{L}) < 1$ for each line \mathcal{L} .

A. 2 Volumes of Truncated Balls and Caps

In Table 1 we have stated the volumes of truncated d -dimensional balls of unit diameter S_d and the cut-off bodies, i.e. caps, C_d for small dimensions d . Due to Eqs. (2) and (3) it suffices to compute the values $v(d) := \int_0^{\frac{\pi}{6}} \cos^d(x) dx$. The first few values are given by $v(1) = \frac{1}{2}$, $v(2) = \frac{1}{8} \cdot \sqrt{3} + \frac{1}{12} \cdot \pi$, $v(3) = \frac{11}{24}$, and $v(4) = \frac{9}{64} \cdot \sqrt{3} + \frac{1}{16} \cdot \pi$. Using integration by parts we can immediately compute

$$v(d) = \begin{cases} \frac{(2m-1)!!}{(2m)!!} \cdot \left(\frac{1}{2} \cdot \sum_{k=0}^{m-1} \frac{(2k)!!}{(2k+1)!!} \cdot \frac{\sqrt{3}}{2} \cdot \left(\frac{3}{4}\right)^k + \frac{\pi}{6}\right) & \text{for } d = 2m, \\ \frac{(2m)!!}{(2m+1)!!} \cdot \frac{1}{2} \cdot \sum_{k=0}^m \frac{(2k-1)!!}{(2k)!!} \cdot \left(\frac{3}{4}\right)^k & \text{for } d = 2m + 1. \end{cases}$$

Given the integer sequence A091814 from the “On-line encyclopedia of integer sequences” $v(d)$ can be written as

$$\frac{A091814(d) \cdot \left(\frac{d-1}{2}\right)!}{d! \cdot 2^{\frac{d+1}{2}}}$$

for all odd d . Benoit Cloitre contributed the following second order recursion formula: $v(1) = \frac{1}{2}, v(3) = \frac{11}{24}$, and

$$v(2n - 1) = \frac{1}{8n - 4} \cdot ((14n - 17) \cdot v(2n - 3) - 6(n - 2) \cdot v(2n - 5))$$

for $n \geq 3$. A similar recursion formula can be obtained for even d , where $v(d)$ can be written as

$$q(d) \cdot \sqrt{3} + \frac{\binom{d-1}{\frac{d}{2}}}{2^d \cdot 3} \cdot \pi$$

for rational numbers $q(d)$.

Additionally one can compute the corresponding ordinary generating function:

$$\begin{aligned} F(z) &:= \sum_{k=0}^{\infty} v(k)z^k = \sum_{k=0}^{\infty} \int_0^{\frac{\pi}{6}} (z \cos t)^k dt = \int_0^{\frac{\pi}{6}} \frac{dt}{1 - z \cos t} \\ &= \frac{2}{\sqrt{1 - z^2}} \arctan\left(\sqrt{\frac{1+z}{1-z}} \cdot \tan \frac{\pi}{12}\right). \end{aligned}$$

We will apply singularity analysis in order to determine the asymptotic behavior of $a_n := F_\alpha(z)[z^n]$, where slightly more generally

$$F_\alpha(z) := \frac{2}{\sqrt{1 - z^2}} \arctan\left(\sqrt{\frac{1+z}{1-z}} \cdot \alpha\right),$$

see e.g. [20, chapter VI]. The main singularity is at $z = 1$ since there is a compensation for $z = -1$. From

$$\begin{aligned} \arctan\left(\sqrt{\frac{1+z}{1-z}} \cdot \alpha\right) &= \frac{\pi}{2} + O((1-z)^{\frac{1}{2}}), \\ \frac{2}{\sqrt{1+z}} &= \sqrt{2} + O(1-z), \quad \text{and} \\ [z^n] \frac{1}{\sqrt{1-z}} &= \frac{1}{\sqrt{\pi n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \end{aligned}$$

we conclude

$$a_n = \sqrt{\frac{\pi}{2n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

So we have $v(d) \sim \sqrt{\frac{\pi}{2d}}$.

References

1. Anderson, W., Simons, W., Mauldon, J.G., Smith, J.C.: Elementary problems and solutions: A dense subset of the unit circle. *Am. Math. Mon.* **86**(3), 225–234 (1979)
2. Anning, N.H., Erdős, P.: Integral distances. *Bull. Am. Math. Soc.* **51**, 598–600 (1945)
3. Audet, C., Hansen, P., Messine, F.: Extremal problems for convex polygons. *J. Glob. Optim.* **38**(2), 163–179 (2007)
4. Audet, C., Hansen, P., Messine, F.: Extremal problems for convex polygons—an update. In: Pardalos, P.M. et al. (eds.) *Lectures on Global Optimization. Papers Based on the Presentations at the Workshop on Global Optimization: Methods and Applications*, Toronto, Canada, May 11–12, 2007. American Mathematical Society (AMS), Providence, RI; The Fields Institute for Research in Mathematical Sciences, Toronto. *Fields Institute. Communications* **55**, 1–16 (2009)
5. Bachoc, C., Nebe, G., de Oliveira Filho, F., Vallentin, F.: Lower bounds for measurable chromatic numbers. *Geom. Funct. Anal.* **19**, 645–661 (2009)
6. Bollobás, B. (ed.): *Littlewood’s Miscellany*, revised edn., vol. VII. Cambridge University Press, Cambridge (1986)
7. Brass, P., Moser, W., Pach, J.: *Research Problems in Discrete Geometry*. Springer, New York (2005)
8. Brass, P., Sharifi, M.: A lower bound for Lebesgue’s universal cover problem. *Int. J. Comput. Geom. Appl.* **15**(5), 537–544 (2005)
9. Bugeaud, Y.: *Distribution Modulo One and Diophantine Approximation*, Cambridge Tracts in Mathematics, vol. 193. Cambridge University Press, Cambridge (2012)
10. Bukh, B.: Measurable sets with excluded distances. *Geom. Funct. Anal.* **18**(3), 668–697 (2008)
11. Campbell, G., Goins, E.H.: Heron triangles, diophantine problems and elliptic curves. Preprint. www.swarthmore.edu/NatSci/gcampbel/papers/heron-Campbell-Goins.pdf (2004)
12. Croft, H.T., Falconer, K.J., Guy, R.K.: Unsolved problems in geometry. In: *Unsolved Problems in Intuitive Mathematics*, vol. II. Springer, New York (1994)
13. Croft, H.: Incidence incidents. *Eureka* **30**, 22–26 (1967)
14. de Oliveira Filho, F.M., Vallentin, F.: Fourier analysis, linear programming, and densities of distance avoiding sets in \mathbb{R}^n . *J. Eur. Math. Soc.* **12**(6), 1417–1428 (2010)
15. de Zeeuw, F.: *An algebraic view of discrete geometry*, Ph.D. Thesis, The University of British Columbia (2011). <https://circle.ubc.ca/handle/2429/38158>
16. Dekster, B.V.: The Jung theorem in metric spaces of curvature bounded above. *Proc. Am. Math. Soc.* **125**(8), 2425–2433 (1997)
17. Erdős, P.: Integral distances. *Bull. Am. Math. Soc.* **51**, 996 (1945)
18. Erdős, P.: Ulam, the man and the mathematician. *J. Graph Theory* **9**(4), 445–449 (1985)
19. Evans, L.C., Gariépy, R.F.: *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)
20. Flajolet, P., Sedgewick, R.: *Analytic Combinatorics*. Cambridge University Press, Cambridge (2009)
21. Furstenberg, H., Katznelson, Y., Weiss, B.: *Ergodic theory and configurations in sets of positive density*. Mathematics of Ramsey Theory, Coll. Pap. Symp. Graph Theory, Prague/Czech. *Algorithms Comb.* **5**, 184–198 (1990)
22. Gardner, R.J.: *Geometric Tomography*, Encyclopedia of Mathematics and Its Applications, vol. 58, 2nd edn. Cambridge University Press, Cambridge (2006)
23. Graham, R.L., Rothschild, B.L., Straus, E.G.: Are there $n + 2$ points in E^n with odd integral distances? *Am. Math. Mon.* **81**, 21–25 (1974)
24. Guth, L., Katz, N.H.: On the Erdős distinct distance problem in the plane, 37pp. <http://arXiv:1011.4105v3> (2011, submitted)

25. Hernández Cifre, M.A., Salinas, G.: Two optimization problems for convex bodies in the n -dimensional space (contributions to algebra and geometry). *Beiträge Algebra Geom.* **45**(2), 549–555 (2004)
26. Jung, H.: Über die kleinste Kugel, die eine räumliche Figur einschließt. *J. Reine Angew. Math.* **123**, 241–257 (1901) (in German)
27. Komjáth, P., Totik, V.: *Problems and Theorems in Classical Set Theory*. Springer, New York (2005)
28. Kreisel, T., Kurz, S.: There are integral heptagons, no three points on a line, no four on a circle. *Discrete Comput. Geom.* **39**(4), 786–790 (2008)
29. Mann, H.B.: On linear relations between roots of unity. *Math. Lond.* **12**, 107–117 (1965)
30. Olmsted, J.M.H.: Discussions and notes: rational values of trigonometric functions. *Am. Math. Mon.* **52**(9), 507–508 (1945)
31. Payne, M.S., Coulson, D.: A dense distance 1 excluding set in \mathbb{R}^3 . *Aust. Math. Soc. Gaz.* **34**(2), 97–102 (2007)
32. Piepmeyer, L.: The maximum number of odd integral distances between points in the plane. *Discrete Comput. Geom.* **16**(1), 113–115 (1996)
33. Quas, A.: Distances in positive density sets. *J. Comb. Theory A* **116**, 979–987 (2009)
34. Schmutz, E.: Rational points on the unit sphere. *Cent. Eur. J. Math.* **6**, 482–487 (2008)
35. Schwartz, R., Solymosi, J., de Zeeuw, F.: Rational distances with rational angles. *Mathematika* **58**(2), 409–418 (2012)
36. Scott, P.R., Awyong, P.W.: Inequalities for convex sets. *J. Inequal. Pure Appl. Math.* **1**(1), 6 (2000)
37. Solymosi, J., de Zeeuw, F.: On a question of Erdős and Ulam. *Discrete Comput. Geom.* **43**(2), 393–401 (2010)
38. Stein, E.M., Shakarchi, R.: *Fourier Analysis. An Introduction*, Princeton Lectures in Analysis, I. Princeton University Press, Princeton (2003)
39. Ulam, S.M.: *A Collection of Mathematical Problems (Interscience Tracts in Pure and Applied Mathematics. No. 8.)*, vol. XIII. Interscience, New York (1960)
40. Weyl, H.: Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.* **77**, 313–352 (1916) (in German)
41. Wolff, T.: Recent work connected with the Kakeya problem. In: *Prospects in Mathematics*, pp. 129–162. American Mathematical Society, Princeton, NJ (1999)