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Kadets-Type Theorems for Partitions of a Convex Body

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Abstract For convex partitions of a convex body B, we try to put a homothetic copy of B into each set of the partition so that the sum of the homothety coefficients is at least 1. In the plane this can be done for an arbitrary partition, while in higher dimensions we need certain restrictions on the partition.

Keywords The Tarski plank problem · The Kadets theorem

1 Introduction

Alfred Tarski [12] proved that for any covering of the unit disk by planks (the sets $a \le \lambda(x) \le b$ for a linear function λ and two reals $a < b$), the sum of plank widths is at least 2. Thøger Bang [3] generalized this result for covering of a convex body B in $\mathbb{R}^d$ by planks showing that the sum of the widths is at least the width of B. He also posed the following question: Can the plank widths in the Euclidean metric be replaced by the widths relative to B (as in Definition 2 below)?

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Keith Ball [2] proved the conjecture of Bang for centrally symmetric bodies B or, in other words, for arbitrary normed spaces and coverings of the unit ball. For possibly nonsymmetric B, it is known (see [1]) that the Bang conjecture is equivalent to the Davenport conjecture: If a convex body B is sliced by $n-1$ hyperplane cuts, then there exists a piece that contains a translate of $\frac{1}{n}B$.

In [5, 6] András Bezdek and Károly Bezdek proved an analogue of the Davenport conjecture for binary partitions by hyperplanes. The difference is that they do not cut everything with every hyperplane; instead, they divide one part into two parts and then proceed recursively.

One of the strongest results about coverings of a unit ball for the Hilbert (and finite-dimensional Euclidean) space was proved by Vladimir Kadets [10] (see also [4] for the proof in the two-dimensional case using the idea from [12]): For any convex covering $C_1, \dots, C_k$ of the unit ball, the sum of the inscribed ball radii $\sum_{i=1}^{k} r(C_i)$ is at least 1.

The reader is referred to [7] for a detailed historical survey on the Tarski plank problem. Some other open problems can be found in [9].

In this paper we prove analogues of the Kadets theorem for inscribing homothetic copies of a (not necessarily symmetric) convex body, replacing arbitrary coverings by certain convex partitions. By a *partition* of a convex set B we mean a covering of B by a family of closed convex sets with disjoint interiors. In the two-dimensional case the analogue of the Kadets theorem for possibly nonsymmetric bodies (Theorem 2) holds for any partition, while in higher dimensions we need additional restrictions on the partition. In other words, we are solving positively certain particular cases of [7, Problem 4.4.2] about extending the Kadets theorem to Banach spaces.

We work in finite-dimensional spaces. If one needs analogues for infinite-dimensional Banach spaces, then the standard approximation argument works as in [10].

This paper is organized as follows. In Sect. 2 we define *inductive partitions* and prove a Kadets-type result for such partitions. In Sect. 3 we introduce *affine partitions* and *hierarchical affine partitions*, which are very natural examples of inductive partitions.

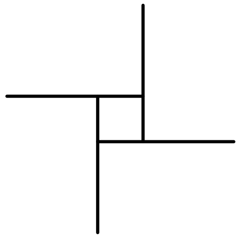
The main result of the paper, the case of two-dimensional partition, is proved in Sect. 4. In Sect. 5 we introduce *inductive coverings*. This is a special type of coverings, for which our methods prove a Kadets-type theorem. Section 6 is not connected with the previous sections directly; there we give a simplified proof of the Bezdek–Schneider analogue of the Kadets theorem on the sphere. In Sect. 7 we show that in the hyperbolic case the Kadets theorem does not hold.

2 Inductive Partitions

Let us describe the class of partitions, for which an analogue of the Kadets theorem is true:

Definition 1 A convex partition $V_1 \cup \dots \cup V_k$ of $\mathbb{R}^d$ is said to be *inductive* if for any $1 \le i \le k$, there exists an inductive partition $W_1 \cup \dots \cup W_{i-1} \cup W_{i+1} \cup \dots \cup W_k$

Fig. 1 Not inductive partition



such that $W_j \supseteq V_j$ for any $j \neq i$. A partition into one part $V_1 = \mathbb{R}^d$ is assumed to be inductive.

A natural example of an inductive partition is a Voronoi partition or, more generally, a projection of lower faces of a convex polytope. In the next section we generalize these examples and discuss hierarchical affine partitions, which are the only examples of inductive partitions that we know. An example of a noninductive partition is outlined on Fig. 1.

Now we define the *inradius* relative to B:

Definition 2 Let $B \subset \mathbb{R}^d$ be a convex body. For a convex set $C \subseteq \mathbb{R}^d$, define the analogue of the inscribed ball radius as follows:

$$r_B(C) = \sup\{h \geq 0 : \exists t \in \mathbb{R}^d \text{ such that } hB + t \subseteq C\},$$

and $r_B(C) = -\infty$ for empty C.

Now we are ready to state one of the main results.

Theorem 1 *Let $B \subset \mathbb{R}^d$ be a convex body, and let $C_1 \cup \dots \cup C_k = B$ be induced from an inductive partition $V_1 \cup \dots \cup V_k = \mathbb{R}^d$ (that is, $C_i = V_i \cap B$ for any i). Then*

$$\sum_{i=1}^{k} r_B(C_i) \geq 1.$$

Before proving this theorem we need a lemma about the inradius.

Lemma 1 *Let a convex polytope $C \subset \mathbb{R}^d$ be defined by linear inequalities for $i = 1, \dots, m$:*

$$\lambda_i(x) \leq 0.$$

Denote by $C(\bar{y})$ the polytope defined by the inequalities

$$\lambda_i(x) + y_i \leq 0,$$

where $\bar{y} = (y_1, \dots, y_m)$ is a vector of reals. Then $r_B(C(\bar{y}))$ is a concave function of $\bar{y}$.

Proof Denote the set of indices $[m]=\{1,\dots,m\}$. By the Helly theorem we have

$$r_B\big(C(\bar{y})\big)=\inf_{I\subseteq[m],\ |I|\le d+1} r_B\big(C_I(\bar{y})\big),$$

where $C_I(\bar{y})$ is defined by the inequalities $\lambda_i(x)+y_i\le 0$ for $i\in I$. The sets $C_I(\bar{y})$ are either Cartesian products of a linear subspace $L\subset\mathbb{R}^d$ of positive dimension with a lower-dimensional polyhedral set $C'_I(\bar{y})$, or simplicial cones, or simplices. In the first case we use induction on the dimension. In the second case we note that $r_B(C_I(\bar{y}))=+\infty$. In the third case the function $r_B(C_I(\bar{y}))$ is obviously linear. Hence, for any $C_I(\bar{y})$, the inradius $r_B(C_I(\bar{y}))$ is a concave function of $\bar{y}$. Therefore the inradius $r_B(C(\bar{y}))$ is concave as an infimum of concave functions. □

Lemma 2 *Let $C_1,\dots,C_m$ be a family of convex bodies in $\mathbb{R}^d$. Then the inradius of the intersection of translates*

$$r_B\big((C_1+y_1)\cap(C_2+y_2)\cap\cdots\cap(C_m+y_m)\big)$$

is a concave function of $\bar{y}=(y_1,\dots,y_m)\in(\mathbb{R}^d)^{\times m}$.

Proof When C_i's are polytopes, this is a particular case of Lemma 1. The general case is made by approximating C_i's by polytopes and passing to the limit. □

Remark 1 In the above lemmas we actually prove that the set of vectors $\bar{y}$ such that the considered function of $\bar{y}$ is $>-\infty$ is a convex closed set.

Proof of Theorem 1 Let us vary the vector $y\in\mathbb{R}^d$ and define $C_i(y)=B\cap(V_i+y)$. The function $r(y)=\sum_{i=1}^k r_B(C_i(y))$ is a concave function of y by Lemma 2, and the set $Y=\{y:r(y)>-\infty\}$ is a convex closed set. If y is in the boundary of Y, then at least one of $C_i(y)$ has the empty interior. In this case we can omit the corresponding V_i and consider a smaller partition $\{W_j\}_{j\ne i}$, which induces the same partition $\{(W_j+y)\cap B\}_{j\ne i}$ as $\{(V_j+y)\cap B\}$ up to sets with empty interior.

Thus, by induction we have $r(y)\ge 1$ on ∂Y. Along with the concavity of $r(y)$, this implies $r(y)\ge 1$ on the whole Y unless Y is a halfspace. From the obvious formula (the sum is the Minkowski sum)

$$Y=\bigcap_{i=1}^k\big(B+(-V_i)\big)$$

it follows that Y can be a halfspace if and only if every V_i contains the same halfspace. This is impossible unless $k=1$; but for $k=1$, the theorem is obviously true. □

3 Affine Partitions

In this section we describe constructively a certain class of inductive partitions.

Definition 3 For a sequence of affine (linear with possible constant term) functions $F = \{\lambda_1, \dots, \lambda_k\}$, define the *affine partition* $P(F)$ of $\mathbb{R}^d$ by

$$C_i = \{x \in \mathbb{R}^d : \forall j \neq i, \text{ we have } \lambda_i(x) \leq \lambda_j(x)\}.$$

An affine partition of a subset $X \subset \mathbb{R}^d$ is defined as a restriction of an affine partition of the whole $\mathbb{R}^d$.

Remark 2 Affine partitions are also known as *generalized Voronoi partitions* or *power diagrams*, but we use the term *affine* for brevity.

Corollary 1 *Let $B \subset \mathbb{R}^d$ be a convex body, and let $C_1 \cup \dots \cup C_k = B$ be its affine partition. Then*

$$\sum_{i=1}^{k} r_B(C_i) \geq 1.$$

Proof It suffices to show that any affine partition is inductive. Starting from $V_1 \cup \dots \cup V_k = \mathbb{R}^d$ defined by $\{\lambda_1, \dots, \lambda_k\}$, we omit λ_i from the list and obtain another affine partition $\{W_j\}_{j \neq i}$ such that $W_j \supseteq V_j$ for any $j \neq i$. So the induction step is possible. □

A straightforward generalization of an affine partition is a *hierarchical affine partition*:

Definition 4 By induction: If in a hierarchical affine partition $C_1 \cup \dots \cup C_k$ we partition some C_i by an affine partition, we obtain again a hierarchical affine partition.

Let us show that a hierarchical affine partition is a limit of affine partitions.

Lemma 3 *Suppose that B is a convex body and $C_1 \cup \dots \cup C_k = B$ is its hierarchical affine partition. Then this partition can be approximated by an affine partition with arbitrary precision in the Hausdorff metric.*

Proof From the definition we know that there exists a graded tree T with an affine function λ_v in every vertex $v \in T$ such that the sets C_i correspond to the leaves ℓ_i of T; the condition $x \in C_i$ is equivalent to $\lambda_v(x) \leq \lambda_w(x)$ for any v in the ancestors of ℓ_i and w a sibling of v.

Now we take $\varepsilon > 0$ small enough and for any C_i and its corresponding ℓ_i, consider the full chain from the root $v_0 < v_1 < \dots < v_m = \ell_i$ and the corresponding affine function

$$\lambda_{i,\varepsilon} = \lambda_{v_0} + \varepsilon\lambda_{v_1} + \dots + \varepsilon^m \lambda_{v_m}.$$

Now it is obvious that the affine partition of B corresponding to $\{\lambda_{i,\varepsilon}\}_{i=1}^k$ tends to $\{C_i\}_{i=1}^k$ as ε tends to $+0$. □

Even without this lemma it is obvious that Corollary 1 holds for hierarchical affine partitions by induction. Note that a binary partition by hyperplanes is a particular case of a hierarchical affine partition.

4 The Two-Dimensional Case

Now we are ready to prove an analogue of the Kadets theorem in the plane. The key property of an inductive partition in the proof of Theorem 1 is actually the following: we consider convex partitions $C_1 \cup \cdots \cup C_k = B$ that can be extended to a convex partition $V_1 \cup \cdots \cup V_k = \mathbb{R}^d$. Then we can translate V_i's with y so that one of the sets $C_i = V_i \cap B$ disappears, remove C_i, extend the partition $\{C_j\}_{j \neq i}$ again to a new partition of the whole space, and so on.

In the plane the extension is always possible by the following:

Lemma 4 *Any convex partition $C_1 \cup \cdots \cup C_k = B \subset \mathbb{R}^2$ can be extended to a partition $V_1 \cup \cdots \cup V_k = \mathbb{R}^2$.*

Proof The boundary ∂B consists of parts of the boundaries ∂C_i. Denote the vertices of this partition by $a_1, a_2, \ldots, a_n$. Denote the polygon $a_1 a_2 \cdots a_n$ by A. Note that for some C_is, we may have more than one corresponding part of ∂B.

Obviously, from each point a_i it is possible to draw a ray ℓ_i outside B with the following property: For any set C_j and its corresponding boundary segment $[a_i a_{i+1}]$ (the indices are understood cyclically $a_{n+1} = a_1$), the union of C_j and the area that is bounded by $[a_i a_{i+1}]$, ℓ_i, and ℓ_{i+1} is convex. For the rays ℓ_i, one can take the extension of the interior with respect to B side of C_j after a_i.

Our goal is to erase parts of the rays ℓ_i and obtain a partition of $\mathbb{R}^2 \setminus A$ into n convex parts. At the start the rays may partition $\mathbb{R}^2 \setminus A$ into a larger number of parts.

We perform erasing as follows. Suppose that b_j is a point of transversal intersection of two rays ℓ_s and ℓ_t that is closer to A than other points of transversal intersection of the remaining rays. Note that the segments $b_j a_s$ and $b_j a_t$ do not intersect with other rays ℓ_i transversally (in this case the point of intersection would be closer to A than b_j). Erase part of one of the rays ℓ_s or ℓ_t after b_i and start a new iteration of this process again. Some rays actually become segments, but it does not matter.

After each step we have a convex partition of $\mathbb{R}^2 \setminus A$, and finally we obtain a partition into exactly n parts.

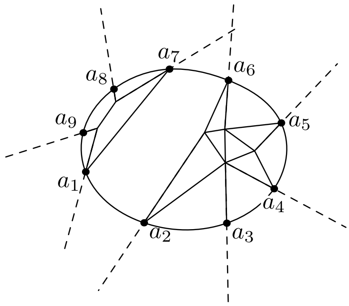
After taking union of these parts with their corresponding sets C_i, we obtain the required extension of the partition to the whole plane, see Fig. 2. □

Now we are ready to state the result.

Theorem 2 *Let $B \subset \mathbb{R}^2$ be a convex body, and let $C_1 \cup \cdots \cup C_k = B$ be its convex partition. Then*

$$\sum_{i=1}^{k} r_B(C_i) \geq 1.$$

Fig. 2 Extending the partition



Proof We extend the partition $C_1 \cup \cdots \cup C_k = B$ to $V_1 \cup \cdots \cup V_k = \mathbb{R}^2$ by Lemma 4. Then the function

$$r(y) = \sum_{i=1}^{k} r_B\big(B \cap (V_i + y)\big)$$

is again concave, so by varying y we can make one of $B \cap (V_i + y)$ have the empty interior without increasing $r(y)$. Then we omit V_i, obtain a partition of B into fewer parts, and use the inductive assumption. □

5 Possible Extension to Coverings

Theorem 2 is quite close to the plane case of the Bang conjecture, which we restate here: If $B \subset \mathbb{R}^2$ is covered by a set of planks $W_1 \cup \cdots \cup W_k \supseteq B$, then $\sum_{i=1}^{k} r_B(W_i) \geq 1$. The key difference is that in the Bang conjecture we have a covering, not a partition. Intuitively, a partition is something smaller than covering and therefore has smaller sum of "inradii." But already in the case of $\mathbb{R}^2$ there exist coverings that do not contain partitions. A simple example is a set of planks C_i passing through the center of a disk B, forming a "sunflower" so that each of the sets $C_i \cap \partial B$ consists of two disjoint arcs and these arcs partition ∂B.

It is easily verified that Theorem 1 holds (with the same proof literally) for coverings instead of partitions if we define an inductive covering by the following:

Definition 5 Call a convex covering (by closed sets) $V_1 \cup \cdots \cup V_k$ of $\mathbb{R}^d$ *inductive* if for any $1 \leq i \leq k$, there exists an inductive covering $W_1 \cup \cdots \cup W_{i-1} \cup W_{i+1} \cup \cdots \cup W_k$ such that $W_j \subseteq V_j \cup V_i$ for any $j \neq i$. A covering by one set $V_1 = \mathbb{R}^d$ is assumed to be inductive.

Returning to the Bang conjecture, we see the unpleasant thing: If we cover some part of $\mathbb{R}^2$ by planks and the remaining part is covered by the corresponding (possibly infinite) polygons, then none of the polygons can be deleted, so this is an example of a noninductive covering of the plane.

6 Notes on the Spherical Kadets Theorem

In [8] Károly Bezdek and Rolf Schneider proved the following version of the Kadets theorem in the spherical geometry:

Theorem 3 (K. Bezdek, R. Schneider, 2010) *If the sphere $\mathbb{S}^n$ is covered by spherical convex sets K_i, then we have the following inequality for the inradii*:

$$\sum_i r(K_i) \geq \pi.$$

Here we use the standard intrinsic metric of the unit sphere. By a convex set we mean the intersection of this sphere and a convex cone whose apex is at the center of the sphere.

This theorem gives rise to the following:

Problem 1 (K. Bezdek, R. Schneider, 2010) Suppose that $B_\rho \subset \mathbb{S}^n$ is a ball of radius ρ in the spherical geometry. Suppose that B_ρ is covered by spherical convex sets K_i. Prove that

$$\sum_i r(K_i) \geq \rho.$$

As it is noted in [8], Theorem 3 solves this problem for $\rho \geq \pi/2$, the solution being essentially volumetric. But if $\rho \to 0$, then this problem approaches the original Kadets theorem, which has no volumetric solution for $n > 2$. So it seems that solving this problem for ρ in the range $(0, \pi/2)$ must require a new approach.

Let us outline the proof of Theorem 3. This proof is essentially the same as that given in [8]; but we simplify it and split into several lemmas, some of which may be of interest on their own.

Lemma 5 *Let μ be a spherically symmetric absolute continuous measure on $\mathbb{R}^n$, B be a ball centered at the origin, and T be a* 0*-starshaped body. Then*

$$\mu(B \cap T)\mu\big(\mathbb{R}^n\big) \geq \mu(B)\mu(T).$$

Proof The proof will use a very simple case of the needle decomposition (see [11] for example). Let us split $\mathbb{R}^n$ into convex cones V_i of equal measures $\mu(V_i)$. Note that the sets $V_i \cap B$ will also have equal measures because of the spherical symmetry of μ. The lemma will follow from the inequality

$$\mu(B \cap T \cap V_i)\mu(V_i) \geq \mu(B \cap V_i)\mu(T \cap V_i) \tag{1}$$

by summation. The partition can be made so that every V_i gets arbitrarily close to a one-dimensional ray, and the limit case of (1) becomes an inequality for nonnegative functions:

$$\int_0^{\min\{x,y\}} f(t)\,dt \cdot \int_0^{+\infty} f(t)\,dt \geq \int_0^x f(t)\,dt \cdot \int_0^y f(t)\,dt,$$

which simply follows from the observation that $\min\{X, Y\}Z \geq XY$ for any $X, Y \in [0, Z]$. □

Lemma 6 *Let us work in the spherical geometry. Suppose that $H \subset \mathbb{S}^n$ is a hemisphere with center o, B is a ball of radius $\leq \pi/2$ centered at o, and T is an o-starshaped body in H. Then*

$$\sigma(B \cap T)\sigma(H) \geq \sigma(B)\sigma(T)$$

for the standard measure σ on the sphere.

Proof Follows from Lemma 5 by central projection of H onto $\mathbb{R}^n$ such that o goes to 0. □

Lemma 7 *Let X be s subset of the sphere $\mathbb{S}^n$ not contained in an open hemisphere, and X_0 be a set consisting of two antipodal points on the sphere. Then for their ε-neighborhoods* (*in the spherical geometry*), *we have*

$$\sigma(X + \varepsilon) \geq \sigma(X_0 + \varepsilon).$$

Proof Without loss of generality, let $X = \{o_1, \dots, o_m\}$ be finite. Consider the hemispheres H_i with respective centers o_i and the Voronoi regions V_i of o_i. Note that $V_i \subseteq H_i$ for every i. Denote the measure of the whole sphere $\mathbb{S}^n$ by σ_n.

Then by Lemma 6

$$\sigma\big(V_i \cap B_{o_i}(\varepsilon)\big)\frac{\sigma_n}{2} \geq \sigma\big(B_{o_i}(\varepsilon)\big)\sigma(V_i),$$

hence,

$$\sigma\big(V_i \cap (X + \varepsilon)\big)\frac{\sigma_n}{2} \geq \sigma\big(B_{o_i}(\varepsilon)\big)\sigma(V_i),$$

and then by summing over i and multiplying by 2,

$$\sigma(X + \varepsilon)\sigma_n \geq 2\sigma\big(B_*(\varepsilon)\big)\sigma_n,$$

where $B_*(\varepsilon)$ is any ball (on the sphere) of radius ε. So we obtain

$$\sigma(X + \varepsilon) \geq 2\sigma\big(B_*(\varepsilon)\big) = \sigma(X_0 + \varepsilon).$$ □

Lemma 8 *Let μ be an absolutely continuous spherically symmetric measure on $\mathbb{R}^n$. Suppose that K is a convex body in $\mathbb{R}^n$ with inscribed ball B, centered at the origin. Then*

$$\mu(K) \leq \mu(T),$$

where T is a plank with inscribed ball B.

Proof Representing the measure μ as an integral, it suffices to prove the inequality

$$\sigma(K \cap S) \leq \sigma(T \cap S) \tag{2}$$

for any sphere S centered at the origin. Let the radii of B and S be r and R, respectively. For $R \leq r$, inequality (2) is obvious, so we consider $R > r$.

The set $T \cap S$ is the complement of the ε-neighborhood of two opposite points X_0 in S, where $\varepsilon = \arccos r/R$. Since K has B as the inscribed ball, the set $X' = \partial K \cap B$ contains the origin in its convex hull. It is easy to see that the set $X = R/rX'$ is not contained in an open hemisphere and that its ε-neighborhood is disjoint with $K \cap S$. By Lemma 7,

$$\sigma(X + \varepsilon) \geq \sigma(X_0 + \varepsilon),$$

hence

$$\sigma(K \cap S) \leq \sigma\big(S \setminus (X + \varepsilon)\big) \leq \sigma\big(S \setminus (X_0 + \varepsilon)\big) = \sigma(T \cap S),$$

which is exactly (2). □

Now we deduce the following (the same as [8, Theorem 2]):

Lemma 9 *Suppose that K is a convex body in $\mathbb{S}^n$ with inscribed ball B. Then*

$$\sigma(K) \leq \sigma(K_0),$$

where $K_0 = H_0 \cap H_1$ is an intersection of two hemispheres with inscribed ball B. Note that $\sigma(K_0)$ equals $\frac{\alpha\sigma_n}{2\pi}$, where α is the angle between H_0 and H_1.

Proof Obtained from Lemma 8 by central projection that takes the center of B to the origin in $\mathbb{R}^n$. □

Now Theorem 3 follows from Lemma 9 by bounding from above the volume of every K_i in terms of $r(K_i)$.

7 The Hyperbolic Kadets Theorem

It is interesting that the Kadets theorem does not hold for hyperbolic space unlike the spherical case mentioned above. We skip the calculation here because of the negativity of this result, but the figures should be sufficiently convincing.

Consider a sufficiently large disk Ω and a regular hexagon inscribed in it. Let us cover this disk by two convex shapes, which are drawn in Fig. 3 (this is the Poincaré model). The maximal inscribed disk of a shape is drawn by a dashed line. Since it does not contain the center of Ω, its radius is less than half of the radius of Ω.

Note that this counterexample uses essentially that C_1 and C_2 do intersect. The authors do not know whether the Kadets theorem holds for partitions in the hyperbolic space.

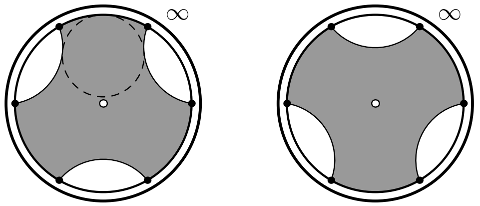


Fig. 3 Disk covering

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