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Tilings of Convex Polygons with Congruent Triangles

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Abstract By the spectrum of a polygon A we mean the set of triples (α, β, γ) such that A can be dissected into congruent triangles of angles α, β, γ. We propose a technique for finding the spectrum of every convex polygon. Our method is based on the following classification. A tiling is called regular if there are two angles of the triangles, α and β such that at every vertex of the tiling the number of triangles having angle α equals the number of triangles having angle β. Otherwise the tiling is irregular. We list all pairs (A, T) such that A is a convex polygon and T is a triangle that tiles A regularly. The list of triangles tiling A irregularly is always finite, and can be obtained, at least in principle, by considering the system of equations satisfied by the angles, examining the conjugate tilings, and comparing the sides and the area of the triangles to those of A. Using this method we characterize the convex polygons with infinite spectrum, and determine the spectrum of the regular triangle, the square, all rectangles, and the regular N-gons with N large enough.

Keywords Tilings with congruent triangles · Regular and irregular tilings

1 Introduction

This paper is concerned with the following problem. Suppose we are given a convex polygon A. Decide, whether or not A can be dissected into congruent triangles, and if there is such a dissection, find all triangles T such that A has a dissection into

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congruent triangles similar to T. By a dissection (or tiling) we mean a decomposition of A into pairwise nonoverlapping polygons. No other conditions are imposed on the tilings. In particular, it is allowed that two pieces have a common boundary point, but do not have a common side. By the *spectrum* of the polygon A we mean the set of triples (α, β, γ) such that A can be dissected into congruent triangles of angles α, β, γ.

A complete solution to this problem would be an algorithm producing the spectrum of every given convex polygon. Although we cannot solve the problem in this algorithmic sense of the word, we present a method which seems to work in most (and possibly, for all) cases. Our method is based on the following classification of tilings introduced in [3]. Suppose that A is dissected into triangles similar to T (we do not assume that the triangles are congruent). We say that the tiling is *regular* if there are two angles of T, say α and β such that at each vertex V of the tiling, the number of triangles having V as a vertex and having angle α at V is the same as the number of triangles having angle β at V. If this condition is not satisfied, then the tiling is called *irregular*. Thus the problem of finding the spectrum is divided into two separate questions: given A, determine those triangles T for which A has a regular (resp. irregular) tiling with congruent triangles similar to T.

We give a complete solution to the question concerning regular tilings. In Theorem 2.1 we list all pairs (A, T) such that A is a convex polygon, T is a triangle, and there is a regular tiling of A with congruent triangles similar to T. In fact, this is the main result of the paper, and its proof occupies Sects. 4–9.

As for irregular tilings, our starting point is [3, Theorem 4] stating that for every polygon A, the number of distinct non-similar triangles T such that A has an irregular tiling with triangles similar to T is finite. More precisely, the number of these triangles is at most cN^6, where N is the number of vertices of A and c is an absolute constant. The proof of [3, Theorem 4] is effective, and gives a list of triples (α, β, γ) such that the angles of every triangle which tiles A irregularly are given by one of the triples of the list.

As we shall see in Sect. 3, in many cases this list can be reduced considerably by using the system of equations satisfied by the angles, and considering conjugate tilings as in [1]. Then, assuming that the tiling consists of congruent triangles, we may compare the sides and the area of the triangles to those of A in order to obtain further number theoretical restrictions on the triples (α, β, γ). Discarding all triples violating these conditions, we arrive at the list of all triangles that tile A irregularly.

There can be two problems with the application of this argument. The first problem is that we cannot guarantee that the reduced list obtained by considering the equations, conjugate tilings and number theoretical restrictions only contains triples corresponding to a tiling. (Proving this fact would result in an algorithmic solution of the problem.) However, as we shall see in Sect. 3, in all cases we consider, each triple contained by the reduced list actually tiles A. One can hope that this happens to every convex polygon A.

The other problem is that when we arrive at a list of triples from which we cannot eliminate any item, we have to produce tilings corresponding to these triangles. In principle, this could be the most difficult step in the procedure. However, in the actual cases we consider, the existence of these tilings is either trivial (as in the case of rectangles or regular N-gons with N large enough), or relatively easy (as for the regular triangle). One can hope that this shows a general tendency.

The applications of the method outlined above will be given in Sect. 3. As an application of Theorem 2.1 we characterize the convex polygons with infinite spectrum (Theorem 3.1). Then we determine the spectrum of the regular triangle, the square, all rectangles, and the regular N-gons with N large enough (Theorems 3.3, 3.6, Corollary 3.7 and Theorem 3.4).

2 Regular Tilings of Convex Polygons

We recall the definition of regular tilings. Let A be a polygon with vertices $V_1, \dots, V_N$, and suppose that A is decomposed into nonoverlapping similar triangles $\Delta_1, \dots, \Delta_t$ of angles α, β, γ. Let $V_1, \dots, V_m$ $(m \ge N)$ be an enumeration of the vertices of the triangles $\Delta_1, \dots, \Delta_t$. For every $i = 1, \dots, m$ we shall denote by p_i (resp. q_i and r_i) the number of those triangles Δ_j whose angle at the vertex V_i is α (resp. β and γ). If $i \le N$ and the angle of A at the vertex V_i is δ_i, then

$$p_i\alpha + q_i\beta + r_i\gamma = \delta_i. \tag{1}$$

If $i > N$ then we have either

$$p_i\alpha + q_i\beta + r_i\gamma = 2\pi \tag{2}$$

or

$$p_i\alpha + q_i\beta + r_i\gamma = \pi. \tag{3}$$

Namely, (2) holds if V_i is in the interior of A and whenever V_i is on the boundary of a triangle Δ_j then necessarily V_i is a vertex of Δ_j. In the other cases (3) will hold. It is clear that the coefficients p_i, q_i, r_i must satisfy

$$\sum_{i=1}^m p_i = \sum_{i=1}^m q_i = \sum_{i=1}^m r_i = t. \tag{4}$$

The tiling will be called *regular*, if one of the following statements is true:

- $p_i = q_i$ for every $i = 1, \dots, m$;
- $p_i = r_i$ for every $i = 1, \dots, m$;
- $q_i = r_i$ for every $i = 1, \dots, m$.

Otherwise the tiling is called *irregular*. We call a polygon *rational*, if its sides are pairwise commensurable.

Theorem 2.1 *Let A be a convex N-gon, and suppose that A has a regular tiling with congruent triangles of angles α, β, γ such that at each vertex of the tiling the number of angles α is the same as that of β. Then one of the following statements is true.*

(i) *A is a parallelogram of angles γ and $\alpha + \beta$, and the ratio of the sides of A is a rational multiple of* $\sin\alpha / \sin\beta$.

(ii) *A is a rational polygon, $N-2$ of its angles equal γ, the other two angles are integer multiples of $\alpha+\beta$, and $\sin\alpha, \sin\beta, \sin\gamma$ are pairwise commensurable.*
(iii) *A is a rational polygon, $N-2$ of its angles equal $\alpha+\beta$, the other two angles are integer multiples of γ, and $\sin\alpha, \sin\beta, \sin\gamma$ are pairwise commensurable.*
(iv) *A is a regular triangle, $\gamma=\pi/3$ or $\gamma=2\pi/3$, and $\sin\alpha, \sin\beta, \sin\gamma$ are pairwise commensurable.*
(v) *A is a centrally symmetric rational N-gon, $N=2k\ge 8$, each angle of A equals $\alpha+\beta$, and the side lengths of A are $a_1,\dots,a_k,a_1,\dots,a_k$ in this order, where $a_1,\dots,a_{k-1}$ constitute a geometric progression of quotient $q=\sin\alpha/\sin\beta$, and $a_k\ge q^{k-1}\cdot a_1$.*
(vi) *A is a centrally symmetric rational hexagon, each angle of A equals $2\pi/3$, $\gamma=\pi/3$ or $\gamma=2\pi/3$, and $\sin\alpha/\sin\beta$ is rational.*
(vii) *A is a regular N-gon, $\alpha=\beta=(\pi/2)-(\pi/N)$ and $\gamma=2\pi/N$.*
(viii) *A is a rational N-gon with $N\le 6$, each angle of A equals $\pi/3$ or $2\pi/3$, $\alpha=\beta=\pi/6$ and $\gamma=2\pi/3$.*
(ix) *A is a rational hexagon, each angle of A equals $2\pi/3$, $\gamma=\pi/3$, and either $\alpha=\pi/6$, $\beta=\pi/2$ or $\alpha=\pi/2$, $\beta=\pi/6$.*

We shall prove Theorem 2.1 in Sects. 4–9. In this section first we discuss the most immediate consequences of Theorem 2.1. Then we show that the cases listed in the theorem correspond to existing tilings.

Theorem 2.2 *The set of those non-similar convex polygons which are not parallelograms and have a regular tiling with congruent triangles is countable.*

Proof It is enough to check that the set of those non-similar convex polygons which satisfy the conditions formulated in the cases (ii)–(ix) of Theorem 2.1 is countable.

First note that the set of non-similar rational triangles is countable, and thus the set of those triples (α,β,γ) for which $\sin\alpha, \sin\beta, \sin\gamma$ are pairwise commensurable is countable. From this observation it is clear that the set of those non-similar convex polygons which satisfy the conditions of cases (ii)–(iv) is countable.

Now consider the case (v). Then the angles of A are equal $\pi-(2\pi/N)$. Since the sides of A are commensurable, it is clear that the number of these polygons is countable.

The statement concerning the cases (vi)–(ix) is trivial. □

Remark 2.3 The analogous statement about irregular tilings is false. In fact, for every N, there are continuum many non-similar convex N-gons that have irregular tilings with congruent triangles. To see this, let XYZ be a triangle such that $\overline{XY}=\overline{XZ}$ and the angle $\gamma=YXZ\measuredangle$ satisfies $\gamma<\pi/(N-2)$. Let T_i denote the triangle obtained from XYZ by rotating it about the vertex X by the angle $i\cdot\gamma$ $(i=0,\dots,N-3)$. Then $A_\gamma=\bigcup_{i=0}^{N-3}T_i$ is a convex N-gon tiled with the triangles $T_0,\dots,T_{N-3}$. It is clear that different values of γ lead to non-similar polygons A_γ.

One can show that the set of non-similar convex N-gons having irregular tilings with congruent triangles can be decomposed into countable many families depending on some continuous parameters. Unfortunately, the description or enumeration of these families seems to be difficult.

Theorem 2.4 *Let A be a convex N-gon, and suppose that A has a regular tiling with congruent triangles of sides a, b, c. Then at least one of the following statements is true.*

(i) $N \le 6$.
(ii) *A is a regular N-gon.*
(iii) *A is rational and centrally symmetric.*
(iv) *The lengths a, b, c are pairwise commensurable.*

Proof This is clear from Theorem 2.1. □

We close this section by showing that each case listed in Theorem 2.1 occurs. More precisely, we show that in each case (with the possible exception of (ix)), whenever the polygon A and the angles α, β, γ satisfy the conditions, then A has a regular tiling with congruent triangles of angles α, β, γ.

(i) Let A be a parallelogram of sides d_1, d_2 and angles γ and $\alpha + \beta$ such that $d_1/d_2 = (p/q) \cdot (\sin\alpha/\sin\beta)$, where p and q are positive integers. Then we put $\tau = d_1/(p \sin\alpha)$. If $a = \tau \sin\alpha$ and $b = \tau \sin\beta$, then $d_1 = p \cdot a$ and $d_2 = q \cdot b$. Thus A can be decomposed into pq congruent parallelograms of sides a and b. Each of these parallelograms can be decomposed into two congruent triangles of angles α, β, γ and of sides a, b and $c = \tau \sin\gamma$. It is clear that the tiling obtained is regular.

Next suppose that A and α, β, γ satisfy the conditions formulated in (ii) or (iii). Then $\cos\gamma$ is rational, since

$$\cos\gamma = \frac{\sin^2\alpha + \sin^2\beta - \sin^2\gamma}{2\sin\alpha\sin\beta}.$$

Therefore, putting $\delta = \gamma$ if A satisfies (ii) and $\delta = \alpha + \beta$ if A satisfies (iii), we find that $\cos\delta$ is rational, $N - 2$ angles of A equal δ, and the other two angles are integer multiples of $\pi - \delta$.

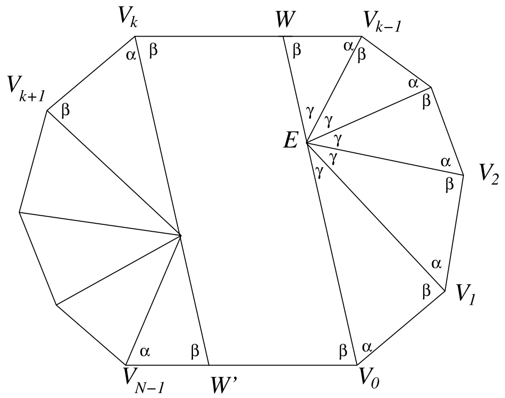
Now it can be shown by induction on N that under these conditions A can be decomposed into finitely many nonoverlapping rational symmetric trapezoids of angles δ and $\pi - \delta$. (See also Lemma 8 of [3], where a more general statement is proved, except that the rationality of the polygon A is not supposed, and thus the trapezoids obtained are not necessarily rational either. However, one can check that if A is rational and $\cos\delta$ is rational, then the construction of [3, Lemma 8] yields rational trapezoids.) It follows from Lemma 2.2 of [2] that each of these symmetric trapezoids can be tiled with congruent copies of a triangle of angles α, β, γ. Then, by Lemma 2.1 of [2], A itself can be tiled with congruent copies of a triangle of angles α, β, γ. It is easy to check that the tiling obtained this way is regular.

(iv) See Theorem 3.1 of [2].

(v) By assumption, $q = a_2/a_1$ is a positive rational number. The conditions imply that $\alpha + \beta = \pi - (2\pi/N)$, and $\gamma = 2\pi/N = \pi/k$.

Let $V_1, \dots, V_N = V_0$ be the vertices of A listed counterclockwise such that $\overline{V_{i-1}V_i} = a_i$ for every $i = 1, \dots, N$ (see Fig. 1). For every $i = 1, \dots, N$, let h_i be the halfline starting from V_i such that the angle between the side $V_{i-1}V_i$ and h_i equals β, and the angle between the side V_iV_{i+1} and h_i equals α. Let E be the inter-

Fig. 1



section of h_N and h_1. Then $\overline{EV_1}/\overline{EV_0} = \sin\alpha/\sin\beta = q$, and thus $\overline{EV_1} = q\cdot\overline{EV_0}$. Since $\overline{V_1V_2} = q\cdot\overline{V_0V_1}$ and $EV_1V_2\measuredangle = \alpha$, it follows that the triangle EV_1V_2 is similar to the triangle EV_0V_1, and thus E is the intersection of h_1 and h_2. Repeating this argument we find that E is on the halflines h_i for every $i = 1,\dots,k-1$.

Since $a_k \ge q^k\cdot a_1$, there is a point W on the side $V_{k-1}V_k$ such that $\overline{V_{k-1}W} = q^{k-1}\cdot a_1$. (The point W coincides with V_k if $a_k = q^k\cdot a_1$.) Since $\overline{WV_{k-1}} = q\cdot\overline{V_{k-2}V_{k-1}}$ and $EV_{k-1}W\measuredangle = \alpha$, it follows that the triangle $EV_{k-1}W$ is also similar to the triangle EV_0V_1. By $k\gamma = \pi$ we find that the points V_0, E and W are collinear. Thus the convex polygon $V_0,\dots,V_{k-1}WV_0$ is tiled by k triangles of angles α,β,γ such that the similarity ratio of any two of these triangles is rational.

The central symmetry of A implies that there is a point W' on the side $V_{N-1}V_0$ such that $\overline{V_{N-1}W'} = q^{k-1}\cdot a_1$. Also, the convex polygon $V_k,\dots,V_{N-1}W'V_k$ is tiled by k triangles of angles α,β,γ such that the similarity ratio of any two of these triangles is rational.

The angles of the parallelogram $W'V_0WV_k$ are $\alpha+\gamma$ and β. Since $\overline{WV_k}$ is a rational multiple of $\overline{V_0V_1}$ and $\overline{V_0W}$ is rational multiple of $\overline{EV_0}$, it follows that the ratio $\overline{WV_k}/\overline{V_0W}$ is a rational multiple of

$$\overline{V_0V_1}/\overline{EV_0} = \frac{\sin\gamma}{\sin\beta} = q\cdot\frac{\sin\gamma}{\sin\alpha}.$$

Therefore, by (i), the parallelogram $W'V_0WV_k$ can be tiled by congruent triangles of angles α,β,γ. In this way we decomposed A into nonoverlapping triangles of angles α,β,γ. It is easy to check that the similarity ratio of any two of these triangles is rational.

Let $\Delta_1,\dots,\Delta_n$ be the triangles of this decomposition, where each Δ_i is similar to Δ_1. Then there are positive rational numbers r_i such that Δ_i is obtained from Δ_1 by a similarity transformation with ratio r_i. Let p_i and q be positive integers such that $r_i = p_i/q$ $(i = 1,\dots,n)$. For every i, we can dissect Δ_i into p_i^2 congruent triangles similar to Δ_1. In this way we obtain a dissection of A into $\sum_{i=1}^n p_i^2$ congruent triangles similar to Δ_1. It is easy to see that the tiling obtained is regular.

(vi) Let a_1,a_2,a_3,a_1,a_2,a_3 denote the lengths of the sides of A. It is easy to see that A can be decomposed into three parallelograms of sides a_1,a_2; a_3,a_1; and

a_2, a_3, respectively. Therefore, these parallelograms are rational, and their angles equal $\pi/3$ and $2\pi/3$. By (i), each of them can be tiled with congruent triangles of angles α, β, γ. In this way we decomposed A into nonoverlapping triangles of angles α, β, γ such that the similarity ratio of any two of these triangles is rational. From this we obtain a tiling of A into congruent triangles of angles α, β, γ as in the previous case. Clearly, the tiling obtained is regular.

(vii) Connecting the center of A with the vertices we obtain a tiling of A into congruent triangles of angles α, β, γ. Then we can label the angles $(\pi/2) - (\pi/N)$ of the triangles with α and β in such a way that we obtain a regular tiling.

(viii) Since A is rational and each angle of A equals $\pi/3$ or $2\pi/3$, it follows that A can be decomposed into congruent regular triangles. We can decompose each of these triangles into three triangles of angles α, β, γ. It is clear that we can label the acute angles of these triangles with α and β in such a way that we obtain a regular tiling.

(ix) It is clear that a regular triangle can be tiled with two congruent triangles having angles $\alpha = \pi/2$, $\beta = \pi/6$, $\gamma = \pi/3$. Since every rational polygon with angles $\pi/3$ and $2\pi/3$ can be tiled with congruent regular triangles, it follows that these polygons can be tiled with congruent triangles having angles α, β, γ. However, we are looking for regular tilings, and this extra condition excludes the cases when A is a triangle or a trapezoid or a pentagon. (We shall prove this in Sect. 8.) It is not clear whether or not every rational hexagon with angles $2\pi/3$ has a regular tiling with triangles having angles $\alpha = \pi/2$, $\beta = \pi/6$, $\gamma = \pi/3$. For some hexagons there is such a tiling. For example, if A is centrally symmetric, then the existence of such a tiling follows from (vi), since $\sin\alpha/\sin\beta = 2$.

3 Applications of Theorem 2.1

For every polygon A we shall denote by $c(A)$ the cardinality of the spectrum of A.

Theorem 3.1 *Let A be a convex N-gon. Then $c(A) = \infty$ if and only if A satisfies one of the following conditions.*

(i) *A is a regular triangle.*
(ii) *A is a parallelogram.*
(iii) *A is rational, and there is a δ such that $\cos\delta$ is rational, $N - 2$ angles of A equal δ, and the other two angles are integer multiples of $\pi - \delta$.*

Proof Let A be a given convex polygon. By [3, Theorem 4], the number of triples (α, β, γ) such that A has an irregular tiling with congruent triangles of angles (α, β, γ) is finite. Therefore, if $c(A) = \infty$, then there are infinitely many triples (α, β, γ) such that A has a regular tiling with congruent triangles of angles (α, β, γ). In particular, there are such triples different from those listed in cases (v), (vii), (viii) and (ix) of Theorem 2.1. (Note that in (v) of Theorem 2.1 the triple (α, β, γ) is uniquely determined. Indeed, γ must be equal to $2\pi/N$, and then the condition $\sin\alpha/\sin\beta = q$ determines α and β as well.) Therefore, A must satisfy one of the conditions of (i)–(iv) and (vi) of Theorem 2.1. It is clear that in each case, one of (i)–(iii) of Theorem 3.1 holds. This proves the "only if" part of the theorem.

Now we prove the "if" part. If Δ is a rational triangle and one of its angles equals $\pi/3$ or $2\pi/3$ then, by Theorem 3.1 of [2], the regular triangle can be tiled with congruent triangles similar to Δ. In Lemma 3.2 of [2] it is shown that there are infinitely many such triangles Δ, and thus $c(A) = \infty$ holds for the regular triangle.

If A is a parallelogram of sides a, b and of angles γ and $\pi - \gamma$, then A can be tiled with congruent triangles of angles α, β, γ whenever $\sin\alpha/\sin\beta$ is a rational multiple of a/b. It is easy to see that there are infinitely many such triples (α, β, γ), and thus $c(A) = \infty$.

Next suppose that A satisfies (iii). We claim that there are infinitely many triples (α, β, γ) such that α, β, γ are the angles of a triangle, $\gamma = \delta$, and $\sin\alpha, \sin\beta, \sin\gamma$ are pairwise commensurable. For each of these triples, A and (α, β, γ) satisfy the conditions of (iii) of Theorem 2.1 and thus, as we saw in the previous section, there is a tiling of A with congruent triangles of angles α, β, γ. This will prove $c(A) = \infty$.

Let $0 < \gamma < \pi$ be given such that $\cos\gamma$ is rational. Then $d = \sin^2\gamma = 1 - \cos^2\gamma$ is a positive rational number. It is well-known that there are infinitely many points on the ellipse $x^2 + d \cdot y^2 = 1$ having rational coordinates; moreover, the set of these points is everywhere dense in the ellipse. Indeed, for every $s \in \mathbb{Q}$, the point with coordinates $x = \pm(s^2 - d)/(s^2 + d)$ and $y = 2s/(s^2 + d)$ satisfies the equation $x^2 + d \cdot y^2 = 1$, and the set of these points is everywhere dense in the ellipse.

Let (x, y) be such a point with a small positive y. We define $\alpha \in (0, \pi/2)$ by $\sin\alpha = y \cdot \sin\gamma$. Then $\sin\alpha/\sin\gamma = y$ is rational, and

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - y^2 \cdot \sin^2\gamma} = \sqrt{1 - y^2 \cdot d} = |x|.$$

If y is small enough, then $\alpha + \gamma < \pi$, and we can define β by $\beta = \pi - \alpha - \gamma$. Then we have

$$\frac{\sin\beta}{\sin\gamma} = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\sin\gamma} = y \cdot \cos\gamma + \cos\alpha = y \cdot \cos\gamma + |x|,$$

and thus $\sin\alpha/\sin\gamma$ and $\sin\beta/\sin\gamma$ are both rational. It is clear that this construction gives infinitely many triples (α, β, γ) with the required properties. □

In the following applications we consider the cases when A is a regular triangle, a square, a rectangle or a regular N-gon with N large enough, and determine the set of triples (α, β, γ) such that A can be tiled with congruent triangles of angles (α, β, γ). We shall need the following lemma.

Lemma 3.2 *Suppose that the convex polygon A has an irregular tiling with congruent triangles of angles α, β, γ. Then α, β, γ are linear combinations of the angles of A with rational coefficients.*

Proof Consider (1), (2), and (3) at the vertices of the tiling. Since the tiling is irregular, it follows from [3, Lemma 10] that the determinant

$$D_{ij} = \begin{vmatrix} 1 & 1 & 1 \\ p_i & q_i & r_i \\ p_j & q_j & r_j \end{vmatrix}$$

is nonzero for at least one pair of indices (i, j). Then the corresponding system of equations

$$\begin{aligned} \alpha + \beta + \gamma &= \pi, \\ p_i\alpha + q_i\beta + r_i\gamma &= \delta_i, \\ p_j\alpha + q_j\beta + r_j\gamma &= \delta_j \end{aligned}$$

determines α, β, γ (here δ_i and δ_j are either angles of A or equal π or 2π). Applying Cramer's rule, we find that α, β, γ are linear combinations of the angles of A and of π with rational coefficients. Since π equals the sum of the angles of A divided by $N - 2$, we obtain the statement of the lemma. □

Theorem 3.3 *The regular triangle can be tiled with congruent triangles of angles* α, β, γ *if and only if a permutation of* (α, β, γ) *satisfies one of the following conditions*:

(i) $\alpha = \beta = \pi/6$ *and* $\gamma = 2\pi/3$;
(ii) $\alpha = \pi/6$, $\beta = \pi/2$, $\gamma = \pi/3$;
(iii) $\gamma \in \{\pi/3, 2\pi/3\}$ *and* $\sin\alpha, \sin\beta, \sin\gamma$ *are pairwise commensurable.*

Proof We saw already that if (α, β, γ) is one of the triples listed in the theorem, then there exists a tiling with the required properties.

Next suppose that the regular triangle is tiled with congruent triangles of angles α, β, γ. If the tiling is regular then one of (ii)–(iv), (vii) and (viii) of Theorem 2.1 must hold. It is clear that in each of these cases the statement of Theorem 3.3 is true.

Therefore, we may assume that the tiling is irregular. Then, by Lemma 3.2, α, β, γ are rational multiples of π. Then we can apply Theorem 5.1 of [2], and find that either a permutation of (α, β, γ) satisfies (i), (ii) or (iii) of Theorem 3.3, or a permutation of (α, β, γ) equals one of the triples

$$\left(\frac{\pi}{3}, \frac{\pi}{12}, \frac{7\pi}{12}\right), \left(\frac{\pi}{3}, \frac{\pi}{30}, \frac{19\pi}{30}\right), \left(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30}\right). \tag{5}$$

We prove that none of the triples listed in (5) gives an irregular tiling of the regular triangle. Suppose first $(\alpha, \beta, \gamma) = (\pi/3, \pi/12, 7\pi/12)$. Since the tiling is irregular, there is an equation $p\alpha + q\beta + r\gamma = \delta$ such that $q < r$ and $\delta \in \{\pi/3, \pi, 2\pi\}$. Multiplying the equation by $12/\pi$ we find that $4p + q + 7r$ is divisible by 4. Then $r - q$ is also divisible by 4. However, we have $0 < r - q \le r \le 3$, which is impossible.

Next suppose $(\alpha, \beta, \gamma) = (\pi/3, \pi/30, 19\pi/30)$. Since the tiling is irregular, there is an equation $p\alpha + q\beta + r\gamma = \delta$ such that $q < r$. Multiplying the equations by $30/\pi$

we find that $10p + q + 19r$ is divisible by 10. Then $r - q$ is also divisible by 10. However, we have $0 < r - q \le r \le 3$, which is impossible.

A similar argument works in the case when $(\alpha, \beta, \gamma) = (\pi/3, 7\pi/30, 13\pi/30)$. $\square$

Let R_N denote the regular N-gon. Connecting the center of R_N with the vertices, we obtain a tiling of R_N with congruent triangles of angles $\alpha = \beta = (\pi/2) - (\pi/N)$ and $\gamma = 2\pi/N$. Another tiling is obtained by decomposing each of these isosceles triangles into two right triangles. The angles of the triangles of the new tiling are $(\pi/2) - (\pi/N), \pi/2, \pi/N$. This shows that for every N the regular N-gon can be tiled with congruent triangles with angles given by any of the triples

$$\left(\frac{\pi}{2} - \frac{\pi}{N}, \frac{\pi}{2} - \frac{\pi}{N}, \frac{2\pi}{N}\right), \left(\frac{\pi}{2} - \frac{\pi}{N}, \frac{\pi}{2}, \frac{\pi}{N}\right). \tag{6}$$

For $N = 3, 4, 6$ there are other tilings of R_N. Moreover, in these cases there are infinitely many other triangles tiling R_N since, by Theorem 3.1, we have $c(R_3) = c(R_4) = c(R_6) = \infty$. Next we show that this behavior is exceptional among the regular polygons.

Theorem 3.4

(i) *If* $N \ne 3, 4, 6$, *then* $c(R_N) < \infty$.
(ii) *If* $N \ne 3, 4, 6$ *and* R_N *has a regular tiling with congruent triangles*, *then the angles of the triangles are* $\alpha = \beta = (\pi/2) - (\pi/N),\ \gamma = 2\pi/N$.
(iii) *If* $N > 420$, *then* $c(R_N) = 2$.

Lemma 3.5 *If* $N \ne 3, 4, 6$, *then* R_N *cannot be tiled with congruent triangles of angles* $\alpha = \beta = \pi/N$, $\gamma = \pi - (2\pi/N)$.

Proof Suppose there is such a tiling. We may assume that the sides of the triangles are a, a, c, where $a = \sin\alpha$ and $c = \sin(\pi - (2\pi/N)) = \sin 2\alpha = 2\sin\alpha\cos\alpha$. Then the side of R_N equals $xa + yc$, where x, y are nonnegative integers. If the number of tiles is t then, comparing the areas we obtain

$$N \cdot \frac{(xa + yc)^2}{4} \cdot \cot\frac{\pi}{N} = \frac{t}{2} \cdot \sin^2\frac{\pi}{N} \cdot \sin\frac{2\pi}{N}.$$

Using $a = \sin\alpha$, $c = 2\sin\alpha\cos\alpha$ and $\cot(\pi/N) = \cos\alpha/\sin\alpha$, we obtain

$$\frac{N}{4} \cdot (x + 2y\cos\alpha)^2 = t \cdot \sin^2\alpha = t \cdot \left(1 - \cos^2\alpha\right)$$

and

$$\left(Ny^2 + t\right)\cos^2\alpha + Nxy\cos\alpha + \left(\left(Nx^2/4\right) - t\right) = 0. \tag{7}$$

Thus the degree of $\cos\alpha$ is at most 2. However, by Theorem 3.9 of [5], the degree of $\cos\alpha = \cos(\pi/N)$ is $\phi(2N)/2$, and thus $\phi(2N) = 2$ or $\phi(2N) = 4$. Since $N \ne 3, 4, 6$, the only possibility is $N = 5$. Then $\cos\alpha = \cos\pi/5 = (\sqrt{5} + 1)/4$, and thus the minimal polynomial of $\cos\alpha$ is $4X^2 - 2X - 1$. Then (7) gives $Ny^2 + t = -4 \cdot ((Nx^2/4) - t)$ and $Nxy = 2((Nx^2/4) - t)$. From the first equation we obtain $t = N(x^2 + y^2)/3 > Nx^2/4$. Thus the second equation gives $Nxy < 0$, which is impossible. This contradiction completes the proof. □

Proof of Theorem 3.4 (i) Suppose $c(R_N) = \infty$. The angles of R_N equal $\delta = \pi - (2\pi/N)$. Then, by Theorem 3.1, $\cos\delta$ is rational, and then so is $\cos 2\pi/N = -\cos(\pi - \delta)$. Thus $\cos 2\pi/N = 0, \pm 1/2, \pm 1$ by [5, Corollary 3.12], and hence $N \in \{3, 4, 6\}$.

(ii) Let α, β, γ be the angles of the triangles of a regular tiling of R_N. If $N \neq 3, 4, 6$, then R_N has to satisfy one of (ii), (iii), (v) and (vii) of Theorem 2.1. If R_N satisfies (vii), then there is nothing to prove.

Suppose R_N satisfies (ii). Then $\gamma = \pi - (2\pi/N)$ and $\sin\alpha, \sin\beta, \sin\gamma$ are commensurable. Since $2\cos\gamma = (\sin^2\alpha + \sin^2\beta - \sin^2\gamma)/(\sin\alpha \sin\beta)$, it follows that $\cos\gamma$ is rational. Then, as we saw above, we have $N \in \{3, 4, 6\}$ which is impossible.

If R_N satisfies (iii), then $\gamma = 2\pi/N$ and the same argument works.

Finally, suppose that R_N satisfies (v). Then $\alpha + \beta = \pi - (2\pi/N)$ and $\gamma = 2\pi/N$. Since the sides of R_N are equal, we have $\sin\alpha/\sin\beta = 1$, $\alpha = \beta$, and $\alpha = \beta = (\pi/2) - (\pi/N)$.

(iii) We have to prove that if $N > 420$ and R_N has a tiling with congruent triangles, then the angles of the triangles are one of given by (6). By (ii), this is true if the tiling is regular, so we may assume that it is irregular. By Lemma 3.2, this implies that α, β, γ are rational multiples of π. Now we shall apply the following result proved in [4]: *Let* $N > 420$*, and suppose that* R_N *has a tiling with similar triangles of angles* α, β, γ*. If* α, β, γ *are rational multiples of* π*, then* (α, β, γ) *is one of the triples of* (6)*, or equals* $(\pi - (2\pi/N), \pi/N, \pi/N)$. Since, by Lemma 3.5, the latter case is impossible, this completes the proof. □

It is very likely that $c(R_N) = 2$ for every $N \neq 3, 4, 6$. In order to prove this, we have to show that if $N \leq 420$ and $N \neq 3, 4, 6$, then there is no irregular tiling of R_N. This amounts to checking a given finite set of triples. Unfortunately, the number of cases to consider is enormous, and it seems to be hopeless to exclude these triples without the use of computer. We plan to return to this computation in the forthcoming paper [4].

Our next aim is to determine those triangles that tile a rectangle.

Theorem 3.6 *If the congruent copies of a triangle* T *tile a rectangle, then* T *is a right triangle, and the ratio of the sides of the rectangle is a rational multiple of the ratio of the perpendicular sides of* T*. Therefore, a rectangle* A *can be tiled with congruent triangles of angles* α, β, γ *if and only if a permutation of* (α, β, γ) *satisfies the following condition:* $\gamma = \pi/2$ *and* $\sin\alpha/\sin\beta$ *is a rational multiple of the ratio of the sides of* A.

We note the following consequence of Theorem 3.6.

Corollary 3.7 *The square can be tiled with congruent triangles of angles* α, β, γ *if and only if a permutation of* (α, β, γ) *satisfies the following condition:* $\gamma = \pi/2$ *and*

$\sin\alpha/\sin\beta$ is rational. Consequently, in every tiling of the square with congruent triangles, the pieces must be right triangles with commensurable perpendicular sides.

Lemma 3.8 *Suppose that a rectangle A is tiled with congruent copies of a right triangle. Then the ratio of the perpendicular sides of the triangle is a rational multiple of the ratio of the sides of A.*

Proof Let α, β, γ be the angles of the triangle, where $\gamma = \pi/2$ and $\alpha \ge \beta$. If $\alpha = \beta$, then the triangles are isosceles right triangles, and thus the sides of A are commensurable by [1, Theorem 2]. In this case the statement of the lemma is true. Therefore, we may assume that $\alpha > \beta$, and thus $\pi/4 < \alpha < \pi/2$. If the tiling is regular, then one of (i), (ii), (iii) and (vii) of Theorem 2.1 must hold. It is clear that in each of these cases the statement of the lemma is true.

Therefore, we may assume that the tiling is irregular. Then there is an equation $p\alpha + q\beta + r\gamma = \delta$ with $p > q$ and $\delta \in \{\pi/2, \pi, 2\pi\}$. This implies $(p-q)\alpha + q(\alpha + \beta) + r\gamma = \delta$, and thus $\alpha + \beta = \gamma = \pi/2$ gives $m\alpha = s\pi/2$, where $m = p - q$ and s are both positive integers. Note that $s \le 2\delta/\pi \le 4$. Since $\pi/4 < \alpha < \pi/2$, we have $1/2 < s/m < 1$, and thus s/m is one of the fractions

$$\frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \frac{4}{5}, \frac{4}{7}. \tag{8}$$

Suppose $s/m = 2/3$. Then $\alpha = \pi/3$ and $\beta = \pi/6$. We may assume that the sides of the triangles are $1, 2, \sqrt{3}$. Then the sides of A are $x\sqrt{3} + y$ and $z\sqrt{3} + u$, where x, y, z, u are nonnegative integers. If the tiling contains t triangles, then comparing the areas we obtain

$$(x\sqrt{3} + y) \cdot (z\sqrt{3} + u) = \frac{t}{2} \cdot \sqrt{3}.$$

This implies $x = u = 0$ or $y = z = 0$. In both cases, one side of A is an integer multiple of $\sin\alpha = \sqrt{3}/2$, and the other side is an integer multiple of $\cos\alpha = 1/2$; that is, the statement of the lemma is true.

Next suppose $s/m = 3/4$. Then $\alpha = 3\pi/8$ and $\beta = \pi/8$. We may assume that the sides of the triangles are $\sin\alpha, \cos\alpha$ and 1. Then the sides of A are $x\cos\alpha + y\sin\alpha + z$ and $u\cos\alpha + v\sin\alpha + w$, where x, y, z, u, v, w are nonnegative integers. If the tiling contains t triangles, then, comparing the areas we obtain

$$(x\cos\alpha + y\sin\alpha + z) \cdot (u\cos\alpha + v\sin\alpha + w) = \frac{t}{2} \cdot \cos\alpha \cdot \sin\alpha. \tag{9}$$

By Theorem 3.9 of [5], the degree of the algebraic numbers $\cos\alpha$ and $\sin\alpha$ equals four. On the other hand, $\tan\alpha = \sqrt{2} + 1$, and thus each of the numbers $\cos^2\alpha, \sin^2\alpha, \sin\alpha\cos\alpha, \tan\alpha$ belongs to the field $\mathbb{Q}(\sqrt{2})$. Thus the left hand side of (9) equals $I + J$, where $I = (xw + uz)\cos\alpha + (yw + zv)\sin\alpha$ and $J = xu\cos^2\alpha + (xv + yu)\cos\alpha\sin\alpha + yv\sin^2\alpha + zw \in \mathbb{Q}(\sqrt{2})$. Since the right hand side of (9) belongs to $\mathbb{Q}(\sqrt{2})$, we find that $I \in \mathbb{Q}(\sqrt{2})$, and thus

$$\cos\alpha \cdot \left[(xw + uz) + (yw + zv)\tan\alpha\right] \in \mathbb{Q}(\sqrt{2}).$$

By $\cos\alpha \notin \mathbb{Q}(\sqrt{2})$ we obtain $I = 0$. Now each term of I is nonnegative, and thus $I = 0$ implies $xw = uz = yw = zv = 0$. If $z \neq 0$, then we get $u = v = 0$. Since $u\cos\alpha + v\sin\alpha + w = w$ equals a side of A, we have $w \neq 0$, and thus $x = y = 0$. Then the left-hand side of (9) is zw, while the right-hand side of (9) is irrational. This is a contradiction, and thus $z = 0$. The same argument shows $w = 0$. Then, dividing both sides of (9) by $\cos^2\alpha$ we obtain

$$(x + y\tan\alpha)\cdot(u + v\tan\alpha) = \frac{t}{2}\cdot\tan\alpha$$

or $yv\tan^2\alpha + (yu + xv - (t/2))\tan\alpha + xu = 0$. Since the minimal polynomial of $\tan\alpha = \sqrt{2} + 1$ is $X^2 - 2X - 1$, it follows that $xu = -yv$, and thus $xu = yv = 0$. Then we have either $x = v = 0$ or $u = y = 0$. In both cases, one side of A is an integer multiple of $\sin\alpha$, and the other side is an integer multiple of $\cos\alpha$; that is, the statement of the lemma is true.

Finally, suppose that $s/m \neq 2/3, 3/4$. Then s/m is one of the fractions $3/5$, $4/5$, $4/7$. We can see, applying Theorem 3.9 of [5], that in each case the degree of $\sin\alpha$ is greater than the degree of $\cos\alpha$. Consequently, $\sin\alpha$ is not an element of the field $\mathbb{Q}(\cos\alpha)$. Note also that $\cos\alpha$ is irrational.

We may assume that the sides of the triangles are $\sin\alpha, \cos\alpha$ and 1. Then the sides of A are $x\cos\alpha + y\sin\alpha + z$ and $u\cos\alpha + v\sin\alpha + w$, where x, y, z, u, v, w are nonnegative integers. If the tiling contains t triangles, then, comparing the areas we obtain (9). Since $\sin^2\alpha \in \mathbb{Q}(\cos\alpha)$ but $\sin\alpha \notin \mathbb{Q}(\cos\alpha)$, (9) implies

$$xu\cos^2\alpha + xw\cos\alpha + yv\sin^2\alpha + uz\cos\alpha + zw = 0 \tag{10}$$

and

$$\big[(xv + yu)\cos\alpha + (yw + zv)\big]\cdot\sin\alpha = (t/2)\cdot\cos\alpha\cdot\sin\alpha. \tag{11}$$

By (10) we have $xu = xw = yv = uz = zw = 0$. Also, (11) gives $(xv + yu)\cos\alpha + (yw + zv) = (t/2)\cdot\cos\alpha$. Since $\cos\alpha$ is irrational, this implies $yw = zv = 0$. Since $\max(x, y, z) > 0$ and $\max(u, v, w) > 0$, these equations imply either $x = z = v = w = 0$ or $y = z = u = w = 0$. In both cases, one side of A is an integer multiple of $\sin\alpha$, and the other side is an integer multiple of $\cos\alpha$, which completes the proof. □

Proof of Theorem 3.6 Suppose that the rectangle A is tiled with congruent copies of a triangle T. By [1, Theorem 23], one of the following must hold: (i) T is a right triangle; (ii) the angles of T are $(\pi/6, \pi/6, 2\pi/3)$; or (iii) the angles of T are given by one of the following triples:

$$\left(\frac{\pi}{8}, \frac{\pi}{4}, \frac{5\pi}{8}\right),\ \left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}\right),\ \left(\frac{\pi}{12}, \frac{\pi}{4}, \frac{2\pi}{3}\right). \tag{12}$$

If T is a right triangle, then the statement of the theorem is true by Lemma 3.8.

Suppose that the angles of T are $(\pi/6, \pi/6, 2\pi/3)$. We may assume that the sides of the triangles are $1, 1, \sqrt{3}$. Then the sides of A are $x\sqrt{3} + y$ and $z\sqrt{3} + u$, where

x, y, z, u are nonnegative integers. If the tiling contains t triangles, then, comparing the areas we obtain

$$(x\sqrt{3}+y)\cdot(z\sqrt{3}+u)=\frac{t}{2}\cdot\sqrt{3}.$$

This implies $x=u=0$ or $y=z=0$. In both cases, there is a side XY of A such that $\overline{XY}$ is an integer. Let $X=U_0, U_1, \dots, U_k=Y$ be a division of XY such that each subinterval $U_{i-1}U_i$ $(i=1,\dots,k)$ is a side of a triangle T_i of the tiling. Since $\overline{U_{i-1}U_i}=1$ for every i, it follows that the angle of T_i at one of the vertices U_{i-1} and U_i equals $2\pi/3$. However, the angle of T_1 at $U_0=X$ and the angle of T_k at $U_k=Y$ must be $\pi/6$, and thus there is a $0<i<k$ such that the angle of both T_{i-1} and T_i at U_i equals $2\pi/3$. Then the triangles T_{i-1} and T_i overlap, which is impossible.

In order to complete the proof of Theorem 3.6, we have to prove the following.

Lemma 3.9 *Suppose that the angles of T are given by one of the triples of (12). Then no rectangle can be tiled with congruent copies of T.*

Proof First we suppose that a rectangle A of vertices V_1, V_2, V_3, V_4 is tiled with congruent triangles with angles $\alpha=\pi/8$, $\beta=5\pi/8$, $\gamma=\pi/4$. We may assume that V_1 is the origin and V_2 is the point $(1,0)$. Since each of the numbers $\cot\pi/8=\sqrt{2}+1$, $\cot 5\pi/8=\sqrt{2}-1$ and $\cot\pi/4=1$ belongs to $\mathbb{Q}(\sqrt{2})$, it follows from [1, Theorem 2] that the coordinates of the vertices of the triangles belong to $\mathbb{Q}(\sqrt{2})$.

Let a, b, c denote the sides of the triangles. Then $a/c=(\sin\pi/8)/(\sin\pi/4)$ and $b/c=(\sin 5\pi/8)/(\sin\pi/4)$. By Theorem 3.9 of [5], the degree of $\sin\pi/8$ and of $\sin 5\pi/8$ equals 4, and thus the ratios a/c and b/c do not belong to $\mathbb{Q}(\sqrt{2})$.

There is a division $V_1=U_0,\dots,U_k=V_2$ of the side V_1V_2 such that $U_{i-1}U_i$ is the side of a triangle T_i of the tiling for every $i=1,\dots,k$. Let x_i denote the first coordinate of U_i. Since $x_i-x_{i-1}\in\mathbb{Q}(\sqrt{2})$ for every $i=1,\dots,k$ and $a/c, b/c\notin\mathbb{Q}(\sqrt{2})$, it follows that either $x_i-x_{i-1}=\overline{U_{i-1}U_i}\in\{a,b\}$ for every i, or $x_i-x_{i-1}=\overline{U_{i-1}U_i}=c$ for every i.

Suppose $\overline{U_{i-1}U_i}=c$ for every i. Then the angles of T_i at the points U_{i-1} and U_i are $\pi/8$ and $5\pi/8$ in some order. Since $5\pi/8>\pi/2$, it follows that the angle of T_1 at $U_0=V_1$ and the angle of T_k at $U_k=V_2$ must be $\pi/8$, and thus there is a $0<i<k$ such that the angle of both T_{i-1} and T_i at U_i equals $5\pi/8$. Then the triangles T_{i-1} and T_i overlap, which is impossible. This proves that $\overline{U_{i-1}U_i}\in\{a,b\}$ for every i.

Next we prove that either $\overline{U_{i-1}U_i}=a$ for every i, or $\overline{U_{i-1}U_i}=b$ for every i. In order to prove this we shall need to consider a conjugate tiling as described in [1].

Let ϕ denote the automorphism of the field $\mathbb{Q}(\sqrt{2})$ defined by $\phi(x+y\sqrt{2})=x-y\sqrt{2}$ $(x,y\in\mathbb{Q})$. Then $\Phi(x_1,x_2)=(\phi(x_1),\phi(x_2))$ defines a collineation on the set of vertices of the tiling. Let X' denote the image of X under Φ. Then $V_1'=V_1$ and $V_2'=V_2$. The points V_1', V_2', V_3', V_4' are the vertices of a rectangle A' and, according to [1], the images of the triangles are nonoverlapping and constitute a tiling of A'.

Since the images of the triangles T_i are nonoverlapping, it follows that the points $U_0',\dots,U_k'$ constitute a division of the segment V_1', V_2' in this order. Since $U_i'=(\phi(x_i),0)$ for every i, it follows that the sequence $(\phi(x_i))_{i=0}^k$ is strictly increasing.

Suppose that there are indices $1 \le i, j \le k$ such that $x_i - x_{i-1} = a$ and $x_j - x_{j-1} = b$. Then $\phi(x_i) = \phi(x_{i-1}) + \phi(a)$ and $\phi(x_j) = \phi(x_{j-1}) + \phi(b)$. Therefore, the numbers $\phi(a), \phi(b)$ are positive. On the other hand, $a/b = (\sin \pi/8)/(\sin 5\pi/8) = \sqrt{2} - 1$, and thus

$$\phi(a)/\phi(b) = \phi(a/b) = -\sqrt{2} - 1 < 0,$$

which is a contradiction. This proves that either $\overline{U_{i-1}U_i} = a$ for every i, or $\overline{U_{i-1}U_i} = b$ for every i.

Thus $\overline{V_1V_2}$ is an integer multiple of either a or b. The same is true for the side V_2V_3, and thus the area of A is an integer multiple of one of the numbers a^2, b^2, ab. On the other hand, the area of any of the triangles is $ab(\cos\gamma)/2 = ab\sqrt{2}/4$, and thus the area of A is an integer multiple of $ab\sqrt{2}/4$. Therefore, one of the numbers $(ab\sqrt{2})/a^2$, $(ab\sqrt{2})/b^2$, $(ab\sqrt{2})/(ab)$ is rational. However, $a/b = \sqrt{2} - 1$, and thus each of these numbers is irrational, a contradiction. This proves that no rectangle can be tiled with congruent triangles with angles $\alpha = \pi/8,\ \beta = 5\pi/8,\ \gamma = \pi/4$.

Next suppose that a rectangle A of vertices $V_1 = (0,0)$, $V_2 = (1,0)$ and V_3, V_4 is tiled with congruent triangles with angles $\alpha = \pi/4,\ \beta = 5\pi/12,\ \gamma = \pi/3$. Since each of the numbers $\cot \pi/4 = 1$, $\cot 5\pi/12 = 2 - \sqrt{3}$, $\cot \pi/3 = \sqrt{3}/3$ belongs to $\mathbb{Q}(\sqrt{3})$, it follows from [1, Theorem 2] that the coordinates of each vertex of A and of each triangle belong to $\mathbb{Q}(\sqrt{3})$.

Let a, b, c denote the sides of the triangles. Then

$$a/c = (\sin \pi/4)/(\sin \pi/3) = \sqrt{2}/\sqrt{3} \notin \mathbb{Q}(\sqrt{3}).$$

Since $\sin 5\pi/12 = (\sqrt{3} + 1)/(2\sqrt{2})$, we have

$$b/c = (\sin 5\pi/12)/(\sin \pi/3) = (\sqrt{3} + 1)/(\sqrt{6}),$$

and thus b/c does not belong to $\mathbb{Q}(\sqrt{3})$ either.

There is a division $V_1 = U_0, \dots, U_k = V_2$ of the side V_1V_2 such that $U_{i-1}U_i$ is the side of a triangle T_i of the tiling for every $i = 1, \dots, k$. Since $a/c, b/c \notin \mathbb{Q}(\sqrt{3})$, it follows that either $\overline{U_{i-1}U_i} \in \{a, b\}$ for every i, or $\overline{U_{i-1}U_i} = c$ for every i.

Suppose $\overline{U_{i-1}U_i} = c$ for every i. Then the angles of T_i at the points U_{i-1} and U_i are $\pi/4$ and $5\pi/12$. It is clear that the angle of T_1 at $U_0 = V_1$ must be $\pi/4$ and, similarly, the angle of T_k at $U_k = V_2$ must be $\pi/4$. Therefore, there exists an index $0 < i < k$ such that the angle of T_{i-1} at U_i is $5\pi/12$ and the angle of T_i at U_i is also $5\pi/12$. Since $\pi - 2 \cdot (5\pi/12) = \pi/6 < \min(\alpha, \beta, \gamma)$, this is clearly impossible. Thus $\overline{U_{i-1}U_i} \in \{a, b\}$ for every i.

Next we prove that either $\overline{U_{i-1}U_i} = a$ for every i, or $\overline{U_{i-1}U_i} = b$ for every i. Let ψ denote the automorphism of the field $\mathbb{Q}(\sqrt{3})$ defined by $\psi(x + y\sqrt{3}) = x - y\sqrt{3}$ $(x, y \in \mathbb{Q})$. Then $\Psi(x_1, x_2) = (\psi(x_1), \psi(x_2))$ defines a collineation on the set of vertices of the tiling. Let X' denote the image of X under Ψ. Then $V_1' = V_1$ and $V_2' = V_2$. The points V_1', V_2', V_3', V_4' are the vertices of a rectangle A' and the images of the triangles are nonoverlapping.

Therefore, the points $U_0', \dots, U_k'$ constitute a division of the segment V_1', V_2' in this order. Let x_i denote the first coordinate of U_i. Then $U_i' = (\psi(x_i), 0)$ for every i, and hence the sequence $(\psi(x_i))_{i=0}^k$ is strictly increasing.

Suppose that there are indices $1 \le i, j \le k$ such that $x_i - x_{i-1} = a$ and $x_j - x_{j-1} = b$. Then $\psi(x_i) = \psi(x_{i-1}) + \psi(a)$ and $\psi(x_j) = \psi(x_{j-1}) + \psi(b)$. Therefore, the numbers $\psi(a), \psi(b)$ are positive. On the other hand, $a/b = (\sin\pi/4)/(\sin 5\pi/12) = \sqrt{3} - 1$, and thus

$$\psi(a)/\psi(b) = \psi(a/b) = -\sqrt{3} - 1 < 0,$$

which is a contradiction. This proves that either $\overline{U_{i-1}U_i} = a$ for every i, or $\overline{U_{i-1}U_i} = b$ for every i.

Thus $\overline{V_1V_2}$ is an integer multiple of either a or b. The same is true for the side V_2V_3, and thus the area of A is an integer multiple of one of the numbers a^2, b^2, ab. On the other hand, the area of any of the triangles is $ab(\cos\gamma)/2 = ab\sqrt{3}/4$, and thus the area of A is an integer multiple of $ab\sqrt{3}/4$. Therefore, one of the numbers $(ab\sqrt{3})/a^2$, $(ab\sqrt{3})/b^2$, $(ab\sqrt{3})/(ab)$ is rational. However, $a/b = \sqrt{3} - 1$, and thus each of these numbers is irrational, a contradiction. This proves that no rectangle can be tiled with congruent triangles with angles $\alpha = \pi/4$, $\beta = 5\pi/12$, $\gamma = \pi/3$.

Finally, suppose that a rectangle is tiled with congruent triangles with angles $\alpha = \pi/4$, $\beta = \pi/12$, $\gamma = 2\pi/3$. It is easy to check that the conjugate tiling corresponding to the automorphism of the field $\mathbb{Q}(\sqrt{3})$ defined by $\psi(x + y\sqrt{3}) = x - y\sqrt{3}$ $(x, y \in \mathbb{Q})$ is a tiling of a rectangle with congruent triangles with angles $\alpha = \pi/4$, $\beta = 5\pi/12$, $\gamma = \pi/3$ (see [1], p. 291). As we proved above, this is impossible. □

4 Proof of Theorem 2.1: Some Preliminary Results

We start with the following simple observation.

Lemma 4.1 *Let A be a parallelogram of angles γ and $\pi - \gamma$. Suppose that A is tiled with congruent copies of a triangle of angles α, β, γ and of sides a, b, c. If one of the sides of A is an integer multiple of a, then the ratio of the sides of A is a rational multiple of* $\sin\alpha/\sin\beta$, *and thus* (i) *of Theorem* 2.1 *holds.*

Proof If the other side of A is of length d, then the area of A equals $kad \cdot \sin\gamma$ with an integer k. Since the area of each triangle of the tiling equals $(ab\sin\gamma)/2$, we obtain $2kad\sin\gamma = t \cdot ab\sin\gamma$ and $d = (t/2k) \cdot b$, where t is the number of triangles of the tiling. Thus the ratio of the sides of A equals $ka/d = (2k^2/t) \cdot (a/b) = (2k^2/t) \cdot (\sin\alpha/\sin\beta)$. □

In the sequel we fix a regular tiling of the convex polygon A with the congruent triangles $\Delta_1, \dots, \Delta_t$ of angles α, β, γ such that equations (1)–(3) are of the form $p_i(\alpha + \beta) + r_i\gamma = \delta_i$, $p_i(\alpha + \beta) + r_i\gamma = 2\pi$ and $p_i(\alpha + \beta) + r_i\gamma = \pi$, respectively. The vertex V_i of the tiling is called *normal*, if $p_i = r_i$. If $p_i \ne r_i$ and V_i is not a vertex of A, then we say that the vertex V_i is *exceptional*. Whenever

$$p(\alpha + \beta) + r\gamma = v\pi \tag{13}$$

is an equation satisfied by α, β, γ, then we shall call $(p - s)(\alpha + \beta) + (r - s)\gamma = (v - s)\pi$ the *reduced form of* (13), where $s = \min(p, r)$. Thus the reduced form of the equation at a normal vertex is $0 \cdot (\alpha + \beta) + 0 \cdot \gamma = 0 \cdot \pi$.

Let a, b, c denote the sides of the triangles Δ_i opposite to the angles α, β, γ. Then $a/b = \sin\alpha / \sin\beta$ and $b/c = \sin\beta / \sin\gamma$.

We shall say that a triangle Δ is *supported by a segment* UV, if one of the sides of Δ is a subset of UV.

First we assume that a is not a linear combination of b and c with nonnegative rational coefficients. We define a directed graph Γ_a on the set of vertices of the tiling as follows. Let XY be a maximal segment belonging to the union of the boundaries of the triangles Δ_i ($i = 1, \dots, t$), and suppose that the segment XY lies in the interior of A, except perhaps the endpoints X and Y. Then there are divisions $X = U_0, U_1, \dots, U_k = Y$ and $X = V_0, V_1, \dots, V_\ell = Y$ of the segment XY such that each subinterval $U_{i-1}U_i$ ($i = 1, \dots, k$) is a side of a triangle T_i of the tiling supported by XY and lying on the same side of the segment XY, and each subinterval $V_{j-1}V_j$ ($j = 1, \dots, \ell$) is a side of a triangle T'_j of the tiling supported by XY and lying on the other side of XY. Suppose that exactly one of the lengths $\overline{U_0U_1}$ and $\overline{V_0V_1}$ equals a. By symmetry, we may assume that $\overline{U_0U_1} = a \neq \overline{V_0V_1}$. Since a is not a linear combination of b and c with nonnegative rational coefficients, it follows that there is a unique index $1 \le i_0 < k$ such that $\overline{U_{i-1}U_i} = a$ for every $i \le i_0$ and $\overline{U_{i_0}U_{i_0+1}} \neq a$. If all these conditions are satisfied, then we connect the vertices X and U_{i_0} by a directed edge $\overrightarrow{XU_{i_0}}$. Note that U_{i_0} belongs to the interior of A and is different from each of the points V_j ($j = 0, \dots, \ell$). Thus U_{i_0} is in the interior of one of the sides of the triangle T'_j for a suitable j.

Let Γ_a denote the set of all directed edges defined as above. It is clear that the in-degree of any vertex is zero or one. As we saw above, a vertex V can have an incoming edge only if V belongs to the interior of A and if V is in the interior of one of the sides of a triangle of the tiling.

Lemma 4.2 *Suppose that a is not a linear combination of b and c with nonnegative rational coefficients. If V is a normal vertex and V is the endpoint of an edge of Γ_a, then at least one edge of Γ_a starts from V.*

Proof Let $\overrightarrow{XV}$ be an edge, and let $V = U_{i_0}$, where XY is a maximal segment belonging to the union of the boundaries of the triangles Δ_i, and $X = U_0, U_1, \dots, U_k = Y$ and $T_1, \dots, T_k$ are as in the definition of the graph. Put $T = T_{i_0}$ and $T' = T_{i_0+1}$. Then the side of T lying on the segment XY equals a, and the side of T' lying on the segment XY is different from a, and thus the angle of T off the line XY is α, and the angle of T' off the line XY is different from α. Since V is normal, the equation at V must be $\alpha + \beta + \gamma = \pi$. Therefore, V is the common vertex of three triangles.

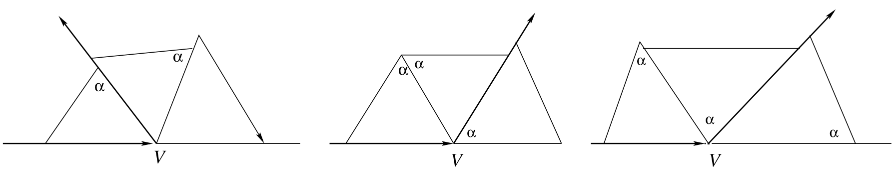
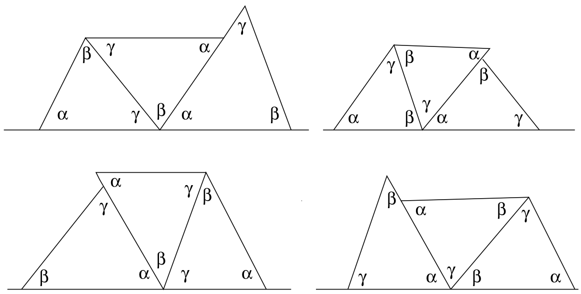


Fig. 2

Fig. 3



Considering all possible positions of the angle α in these triangles, we can see that in each case at least one edge starts from V (see Fig. 2). Note that $a \neq b$ and $a \neq c$. $\square$

Lemma 4.3 *Suppose that a is not a linear combination of b and c with nonnegative rational coefficients. Let XY be a side of A, and suppose that every vertex V lying in the interior of the segment XY is normal and is such that no edge of the graph Γ_a starts from V. Then each of the following statements is true.*

(i) *At least one of the angles of A at X and Y is such that the corresponding equation is of the form $p(\alpha+\beta)=\delta$ with a positive integer p.*
(ii) *If there is a triangle Δ supported by XY such that its angles on the side XY are β and γ, then every triangle supported by XY is a translated copy of Δ. In particular, the length of XY equals $k \cdot a$, where k is a positive integer.*
(iii) *If no edge of the graph Γ_a starts from X or from Y then at least one of the angles of A at X and Y is such that the corresponding equation is different from $\alpha+\beta=\delta$.*

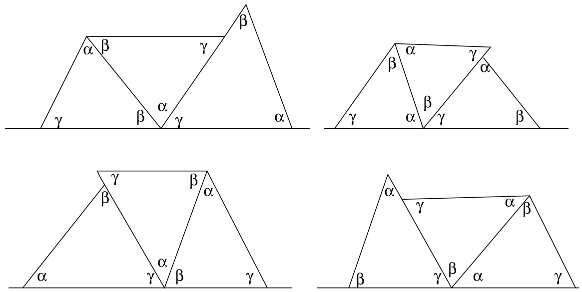
Proof There is a division $X=U_0, U_1, \dots, U_k=Y$ of the segment XY such that each subinterval $U_{i-1}U_i$ $(i=1,\dots,k)$ is a side of a triangle T_i of the tiling supported by XY. By assumption, U_i is normal, and no edge starts from U_i for every $1 \le i \le k-1$. It is easy to check, by inspecting the possible cases, that either T_{i+1} is a translated copy of T_i, or one of the four cases presented in Fig. 3 holds.

It follows from the convexity of A that the equation at X is of the form $p(\alpha+\beta)=\delta$ or $r\gamma=\delta$. Suppose it is $r\gamma=\delta$. Then the angle of T_1 at the vertex $U_0=X$ is γ. Then, considering the possible cases according to Fig. 3, we can see that each triangle T_i is either a translated copy of T_1, or its angle at the vertex U_i is α. This implies that the angle of T_k at Y is α or β, and thus the equation at Y is $p(\alpha+\beta)=\delta$. This proves (i).

If T_i has angles β and γ on XY then, considering again the possible cases according to Fig. 3, we can see that each triangle T_j is a translated copy of T_i. This proves (ii).

Suppose that the equation at the vertex X is of the form $\alpha+\beta=\delta$. If no edge starts from X, then either the angle of T_1 at X is β and at U_1 is γ, or its angle at the vertex X is α. In the first case each triangle T_i must be a translated copy of T_1. Then the angle of T_k at Y equals γ, and thus the equation at Y cannot be of the form $\alpha+\beta=\delta$. Suppose that the angle of T_1 at X is α. Then, as Fig. 3 shows, the angle

Fig. 4



of T_i at U_{i-1} is α for every i. In particular, the angle of T_k at U_{k-1} is α. If the angle of T_k at Y equals γ, then the equation at Y cannot be of the form $\alpha+\beta=\delta$. On the other hand, if the angle of T_k at Y equals β and the equation at Y is $\alpha+\beta=\delta$, then there is an edge starting from the vertex Y. This proves (iii). □

Next we consider the case when c is not a linear combination of a and b with nonnegative rational coefficients. We define the directed graph Γ_c the same way as we defined Γ_a, except that we replace a by c in the definition.

Lemma 4.4 *Suppose that c is not a linear combination of a and b with nonnegative rational coefficients. Let XY be a side of A, and suppose that every vertex V lying in the interior of the segment XY is normal and is such that no edge of the graph Γ_c starts from V. Then*

(i) *at least one of the angles of A at X and Y is such that the corresponding equation is of the from $p(\alpha+\beta)=\delta$ with a positive integer p; and*
(ii) *if there is a triangle Δ supported by XY such its angles on the side XY are α and β, then every triangle supported by XY is a translated copy of Δ. In particular, the length of XY equals $k\cdot c$, where k is a positive integer.*

Proof There is a division $X=U_0,U_1,\dots,U_k=Y$ of the segment XY such that each subinterval $U_{i-1}U_i$ $(i=1,\dots,k)$ is a side of a triangle T_i of the tiling supported by XY. It is easy to check, by inspecting the possible cases, that either T_{i+1} is a translated copy of T_i, or one of the presented in Fig. 4 four cases holds. Then we can repeat the argument of the proof of Lemma 4.3. □

Lemma 4.5 *Suppose that c is not a linear combination of a and b with nonnegative rational coefficients. Let the segment XY belong to the union of the boundaries of the triangles, and let T and T' be two triangles supported by XY and lying on the same side of XY. Suppose that T and T' have a common vertex V on the segment XY and that V is either on the boundary of A, or is an inner point of a side of a triangle. If the side of T lying on XY equals c and the side of T' lying on XY is different from c, then there is an edge of Γ_c starting from V.*

Proof If V is normal, then we can easily check that an edge starts from V (see Fig. 2 with α replaced by γ).

Suppose the vertex V is exceptional. Then the equation at V is of the form $r\gamma = \pi$ or $p(\alpha + \beta) = \pi$. If the equation is $r\gamma = \pi$, then neither of the triangles T and T' can have a side of length c lying on XY, which is impossible. Thus the equation is $p(\alpha + \beta) = \pi$. Let $T_1, \dots, T_{2p}$ denote the triangles having V as a vertex listed clockwise. We may assume, by symmetry, that $T_1 = T$ and $T_{2p} = T'$. There are halflines h_i $(i = 0, \dots, 2p)$ starting from V such that T_i lies in the angular domain bounded by h_{i-1} and h_i for every $i = 1, \dots, 2p$. Let α_i and β_i denote the angles of T_i at its vertices lying on the halflines h_{i-1} and h_i, respectively. Then $\alpha_1 \neq \gamma$ and $\beta_{2p} = \gamma$ by the conditions on T and T'. Since the angle of T_i at V is different from γ for every i, it follows that there exists an index $i < 2p$ such that $\alpha_i \neq \gamma$ and $\beta_{i+1} = \gamma$ or $\alpha_i = \gamma$ and $\beta_{i+1} \neq \gamma$. Then an edge of Γ_c starts from V along the halfline h_i, which completes the proof of the lemma. □

Lemma 4.6 *If a and b are commensurable and c is not a rational multiple of a, then the graph Γ_c is empty*: *it has no edge.*

Proof By Lemma 4.5, the in-degree is not greater than the out-degree at each vertex. Therefore, the in-degree equals the out-degree at each vertex. Since no edge arrives at the boundary of A, it follows that no edge of Γ_c starts from the boundary of A.

Suppose that Γ_c is not empty, and let G denote the set of vertices of nonzero out-degree. Then each vertex $V \in G$ is the starting point of an edge and also the endpoint of another edge. Let V be a vertex of the convex hull of G. Then $V \in G$, and thus V is the endpoint of an edge $\overrightarrow{XV}$. Let $V = U_{i_0}$, where XY is a maximal segment belonging to the union of the boundaries of the triangles, and let $X = U_0, U_1, \dots, U_k = Y$ and $T_1, \dots, T_k$ be as in the definition of the graph. Since $X \in G$, it follows that the segment VY is outside the convex hull of G, and thus no edge starts or arrives at any point of VY except at V.

Suppose that an inner point W of the segment XY is the endpoint of an edge $\overrightarrow{ZW}$. Then W is an inner point of a side of a triangle which must be supported by XY. Thus the starting point Z of the edge must be either X or Y. It cannot be Y, as Y is outside the convex hull of G, and thus $Y \notin G$. Thus $Z = X$, and then $W = V$. Therefore, among the vertices that are inner points of XY only V belongs to G.

Let ℓ be the smallest positive index for which U_ℓ is a common vertex of triangles lying on different sides of XY. Then $i_0 < \ell \leq k$. It follows from Lemma 4.5 that for every $i_0 < i \leq \ell$, the length of the segment $U_{i-1}U_i$ is either a or b. Therefore, $\overline{XU_\ell} = i_0 c + ra$ with a positive rational r.

Let $d_1, \dots, d_\nu$ be the side lengths of the triangles supported by XU_ℓ and lying on the side opposite to the triangles T_i. Then either each d_i equals c or each is different from c. Indeed, otherwise an edge would start from an inner point of XY different from V which is impossible. Thus the length $\overline{XU_\ell}$ is either $k'c$ with a positive integer k' or sa with a positive rational s. Since both of the equation $i_0 c + ra = k'c$ and $i_0 c + ra = sa$ implies that c is a rational multiple of a, both cases are impossible. □

Lemma 4.7 *If a and b are commensurable and c is not a rational multiple of a, then the sides of A are pairwise commensurable.*

Proof Since Γ_c is empty, the situation described in Lemma 4.5 cannot happen. In particular, if XY is a side of A then either all triangles supported by XY have sides of length c on XY, or all have sides of length different from c on XY. Thus the length of each side of A is either an integer multiple of c or a rational multiple of a. We prove that only one of these cases can occur. It is enough to show that if X, Y, Z are consecutive vertices of A, then side lengths $\overline{XY}$ and $\overline{YZ}$ are commensurable.

Let $T_1, \dots, T_s$ denote the triangles having Y as a vertex listed counterclockwise. The equation at Y is of the form $r\gamma = \delta$ or $p(\alpha + \beta) = \delta$. If the equation is $r\gamma = \delta$, then $s = r$, and the sides of T_1 and T_r lying on the boundary of A are of length a or b. In this case the lengths of XY and of YZ are rational multiples of a, hence commensurable.

Suppose that equation at Y is $p(\alpha + \beta) = \delta$. Then $s = 2p$, and there are halflines h_i $(i = 0, \dots, 2p)$ starting from Y such that T_i lies in the angular domain bounded by h_{i-1} and h_i for every $i = 1, \dots, 2p$. Let α_i and β_i denote the angles of T_i at its vertices lying on the halflines h_{i-1} and h_i, respectively. Since no edge starts from Y, for every $i = 1, \dots, 2p - 1$ we have either $\beta_i = \alpha_{i+1} = \gamma$ or $\alpha_i = \beta_{i+1} = \gamma$. This implies that we have either $\alpha_1 = \beta_{2p} = \gamma$ or $\alpha_1 \neq \gamma,\ \beta_{2p} \neq \gamma$. In the first case the lengths of XY and of YZ are rational multiples of a, and in the second case they are integer multiples of c. In both cases, they are commensurable. □

Lemma 4.8 *Suppose that a and b are commensurable and c is not a rational multiple of a. Then the triangles that have their side of length c in the interior of A come in pairs. The triangles of each pair have a common side of length c, and thus they form a quadrilateral which is either a parallelogram of sides a, b and of angles $\gamma, \alpha + \beta$, or a kite of sides a, b and of angles $2\beta, \gamma, 2\alpha, \gamma$.*

Proof Let Δ be a triangle of the tiling such that its side UV of length c lies in the interior of A (except perhaps the endpoints). The side UV is contained in a segment XY with the following properties: XY belongs to the union of the boundaries of the triangles, X and Y are common vertices of triangles lying on different sides of XY, and no point in the interior of XY has this property. Then it follows from Lemma 4.5 and from the fact that Γ_c is empty that every triangle supported by XY and lying on the same side of XY as Δ has a side of length c on XY. Thus the length of XY is an integer multiple of c. Since c is not a linear combination of a and b with integer coefficients, it follows that the triangles supported by XY and lying on the other side of XY also have their sides of length c on XY. Therefore, there exists a triangle Δ' such that the sides of length c of Δ and of Δ' coincide. □

Let G be an additive subgroup of the reals such that $\pi \in G$. We shall denote by $\mathcal{P}_G$ the family of all simple, closed polygons satisfying the condition that for every side XY of P, the angle between the line going through the side XY and the x-axis belongs to G.

Suppose that $\chi : G \to \mathbb{C}$ is a multiplicative function; that is, χ satisfies the functional equation $\chi(\theta_1 + \theta_2) = \chi(\theta_1) \cdot \chi(\theta_2)$ $(\theta_1, \theta_2 \in G)$. Also, we assume that $\chi(\pi) = -1$. Then $\chi(\theta + \pi) = -\chi(\theta)$ and $\chi(\theta + 2\pi) = \chi(\theta)$ hold for every $\theta \in G$.

For a polygon $P \in \mathcal{P}_G$, let $X_0, \dots, X_{k-1}, X_k = X_0$ be the vertices of P listed counterclockwise, let h_i denote the halfline starting from X_{i-1} and going through X_i, and let θ_i denote the directed angle between the positive x-axis and h_i $(i = 1, \dots, k)$. Then we define

$$\Phi(P) = \sum_{i=1}^{k} \chi(\theta_i) \cdot \overline{X_{i-1}X_i}. \tag{14}$$

The definition makes sense, as the function χ is periodic mod 2π. Using the fact that $\chi(\theta + \pi) = -\chi(\theta)$ for every $\theta \in G$ it is easy to see that the function Φ is additive in the following sense: if P is decomposed into the nonoverlapping polygons $P_1, \dots, P_t \in \mathcal{P}_G$, then $\Phi(P) = \sum_{i=1}^{t} \Phi(P_i)$. It is clear that Φ is invariant under translations. Also, it follows from the multiplicative property of χ that if $P \in \mathcal{P}_G$ and P' is obtained from P by a rotation of angle $\theta \in G$, then $\Phi(P') = \chi(\theta) \cdot \Phi(P)$.

Let α_i be the angle of P at the vertex X_i $(i = 1, \dots, k)$, and put $\alpha_i' = \pi - \alpha_i$. Since the halfline h_{i+1} is obtained from h_i by a rotation of angle α_i', it follows that

$$\theta_i \equiv \theta_1 + \alpha_1' + \dots + \alpha_{i-1}' \pmod{2\pi}$$

for every $1 \le i \le k$. Thus $\chi(\theta_i) = \chi(\theta_1) \cdot \chi(\alpha_1') \cdots \chi(\alpha_{i-1}')$ for every $1 \le i \le k$, and

$$\Phi(P) = \chi(\theta_1) \cdot \left(\overline{X_0X_1} + \chi\left(\alpha_1'\right) \cdot \overline{X_1X_2} + \dots + \chi\left(\alpha_1'\right) \cdots \chi\left(\alpha_{k-1}'\right) \cdot \overline{X_{k-1}X_k}\right). \tag{15}$$

Now we turn to the proof of Theorem 2.1. We shall consider the following five cases separately: $\gamma > \pi/2$ and $\gamma \ne 2\pi/3$; $\gamma < \pi/2$ and $\gamma \ne \pi/3$; $\gamma = 2\pi/3$; $\gamma = \pi/3$; $\gamma = \pi/2$.

5 Case I: $\gamma > \pi/2$, $\gamma \ne 2\pi/3$

In this case, in each of equations (1), if $r_i > 0$, then the equation must be of the form $\gamma = \delta_i$. We claim that in each of equations (2) and (3) we have $r_i \le p_i$. Indeed, otherwise the reduced form of the equation in question would be $r\gamma = v\pi$, where r is a positive integer and $v = 1$ or 2. However, as $\pi/2 < \gamma < \pi$ and $\gamma \ne 2\pi/3$, no such equation is possible.

Let $P(\alpha + \beta) + R\gamma = (N - 2)\pi$ be the sum of equations (1). Then, by (4), we have $P \le R$. If $N = 3$ then $P(\alpha + \beta) + R\gamma = \pi$, and thus $P \le R$ gives $P = 0$ or $P = R = 1$. If $P = 0$ then $R\gamma = \pi$ which is impossible. If $P = R = 1$, then at one of the angles of A the equation is $\alpha = \delta$ which contradicts the regularity of the tiling. Thus $N = 3$ is impossible, and we have $N \ge 4$.

By $P(\alpha + \beta) + R\gamma = (N - 2)\pi$ and $P \le R$ we have $R \ge N - 2$, and thus the possible values of R are $N - 2$, $N - 1$ and N. If $R = N - 2$, then necessarily $P = N - 2$, and in each of the equations (2) and (3) we have $p_i = r_i$.

If $R = N - 1$, then $N - 1$ of the angles of A equal γ. The equation $P(\alpha + \beta) + (N - 1)\gamma = (N - 2)\pi$ gives $P(\alpha + \beta) + (N - 1)(\pi - \alpha - \beta) = (N - 2)\pi$ and $\pi = (N - 1 - P)(\alpha + \beta)$. There must exist an equation $p_i(\alpha + \beta) + r_i\gamma = v\pi$ with $r_i < p_i$ and $v = 1, 2$. The reduced form of this equation is $p(\alpha + \beta) = v\pi$, where we have $p = (N - 1 - P)v$. Since $\sum p_i = \sum r_i$, it follows that $p + P \le R$; that is, $(N - 1 - P)v + P \le N - 1$. Thus $v = 1$, and there is no other vertex with $r_j < p_j$ and $j > N$. Therefore, apart from the vertices of A, there is only one vertex of the tiling with $p_i \ne r_i$, and the reduced equation at this vertex equals $(N - 1 - P)(\alpha + \beta) = \pi$.

Finally, if $R = N$, then each angle of A equals γ. Thus $\alpha + \beta = \pi - \gamma = 2\pi/N$. There must exist an equation $p_i(\alpha + \beta) + r_i\gamma = v\pi$ with $r_i < p_i$ and $v = 1, 2$. The reduced form of this equation is $p(\alpha + \beta) = v\pi$, where we have $2p = Nv$. Since $\sum p_i = \sum r_i$, it follows that $p + P \le R$; that is, $N \cdot (v/2) \le N$. If $v = 2$, then there is no other vertex with $r_j < p_j$ and $j > N$. If $v = 1$, then there is one more such vertex, where the reduced equation is $(N/2) \cdot (\alpha + \beta) = \pi$. Therefore, apart from the vertices of A, there are at most two vertices with $p_i \ne r_i$. If there is one such vertex, then the equation at this vertex is $N(\alpha + \beta) = 2\pi$, and if there are two such vertices, then N is even and the corresponding reduced equations are $(N/2) \cdot (\alpha + \beta) = \pi$.

Summing up: there are three cases. In the first case $P = R = N - 2$, and $p_i = r_i$ for every $i > N$; that is, every vertex other than the vertices of A is normal. In this case $N - 2$ of the vertices of A equals γ, and the other two vertices equal $p(\alpha + \beta)$ and $p'(\alpha + \beta)$, where $p + p' = N - 2$.

In the second case $R = N - 1$, every vertex other than the vertices of A is normal with one exception. In the exceptional vertex the reduced equation is $(N - 1 - P)(\alpha + \beta) = \pi$. In this case $N - 1$ of the vertices of A equals γ, and one equals $P(\alpha + \beta)$.

In the third case $R = N$, every vertex other than the vertices of A is normal with at most two exceptions. If there is only one exceptional vertex, then the corresponding equation is $N(\alpha + \beta) = 2\pi$; and if there are two exceptional vertices, then their reduced equations are $(N/2) \cdot (\alpha + \beta) = \pi$. In this case each angle of A equals γ.

5.1 Subcase Ia

First we assume that a is not a linear combination of b and c with nonnegative rational coefficients. Then we may consider the directed graph Γ_a.

Suppose that $P = R = N - 2$. Then there is no exceptional point other than the vertices of A and thus, by Lemma 4.2, the out-degree is not smaller than the in-degree at each vertex. Therefore, they are equal everywhere, and thus the graph Γ_a is the union of disjoint directed cycles. Since the in-degree is zero at each vertex belonging to the boundary of A, it follows that no edge starts from the boundary of A. Therefore, by (i) of Lemma 4.3, A does not have two consecutive vertices with angle γ. However, we know that A has $N - 2$ vertices where its angle is γ. Since $N - 2 \ge 2$, it follows that $N = 4$. Then $P = 2$, and thus two equations at the vertices of A are of the form $\alpha + \beta = \delta$. In other words, A is a quadrilateral with angles $\gamma, \alpha + \beta, \gamma, \alpha + \beta$ in this order. Therefore, A is a parallelogram.

Let X be a vertex of A with angle γ, and let T be the unique triangle of the tiling having X as a vertex. Let U be the vertex of T with angle β, and suppose that U is on the side XY. Then, by (ii) of Lemma 4.3, the length of the side XY equals ka, where k is a positive integer. Therefore, by Lemma 4.1, (i) of Theorem 2.1 holds in this case.

Next suppose that $R = N - 1$. Let V denote the exceptional vertex where the reduced equation is $(N - 1 - P)(\alpha + \beta) = \pi$. If, at the vertex V, the out-degree is not smaller than the in-degree then, as this is also true at every other vertex, it follows that the out-degree is equal to the in-degree everywhere, and Γ_a is the union of disjoint directed cycles. Since the in-degree is zero at each vertex belonging to the boundary of A, it follows that no edge starts from the boundary of A. We know that

$N - 1 \geq 3$ vertices of A equals γ, and thus there are at least $N - 2 \geq 2$ sides of A such that A has angle γ at each endpoint. However, the exceptional point V can be an inner point of at most one of these sides, so there is a side without exceptional point and outgoing edge such that A has angle γ at each endpoint. By (i) of Lemma 4.3, this is impossible.

Therefore, the out-degree is smaller than the in-degree at the vertex V. This means that there is an edge arriving at V, but no edge starts from V. Then V is an inner point of A, and thus there is no exceptional point on the boundary of A. Since the out-degree is not smaller than the in-degree at the vertices different from V, it follows that Γ_a is the union of disjoint directed cycles and one path arriving at V. Therefore, on the boundary of A there is at most one vertex with positive out-degree. Then we can find again a side of A without exceptional points and without outgoing edge such that A has angle γ at each endpoint, which is impossible.

Next we consider the case $R = N$. Then all angles of A equal γ, and then there are $N \geq 4$ sides such that the angles of A at the endpoints equal γ. We know that there are at most two exceptional points, where the out-degree can be smaller than the in-degree. The deficit can be at most two, and thus Γ_a is the union of disjoint directed cycles and at most two paths arriving at the exceptional point(s). Considering the possible cases according to the number of exceptional points on the boundary of A, we can check that in each case there is a side of A without exceptional points and without outgoing edge such that A has angle γ at each endpoint. But this is a contradiction again. This completes the proof in the subcase when a is not a linear combination of b and c with nonnegative rational coefficients.

5.2 Subcase Ib

Suppose that b is not a linear combination of a and c with nonnegative rational coefficients. Since the roles of a and b are symmetric in the conditions as well as in the statements of the theorem, this subcase can be treated similarly to Subcase Ia.

Therefore, we may assume that a is a linear combination of b and c with nonnegative rational coefficients, and b is a linear combination of a and c with nonnegative rational coefficients. This implies that either a, b, c are pairwise commensurable, or a and b are commensurable and c is not a rational multiple of a.

If a, b, c are pairwise commensurable, then $\sin\alpha, \sin\beta, \sin\gamma$ are pairwise commensurable, and so are the sides of A. We show that this implies $P = R = N - 2$, and thus in this case statement (ii) of the theorem holds. Indeed, if $R = N - 1$ or $R = N$, then α and β satisfy an equation of the form $p(\alpha + \beta) = v\pi$, and thus γ is a rational multiple of π. Since $\cos\gamma = (c^2 - a^2 - b^2)/(2ab)$ is rational as well, it follows that $\gamma = \pi/2, \pi/3$ or $2\pi/3$ (see [5, Corollary 3.12]), which is not the case.

5.3 Subcase Ic

Thus we are left with the case when a and b are commensurable and c is not a rational multiple of a. Then the graph Γ_c is empty by Lemma 4.6.

If $P = R = N - 2$, then using (i) of Lemma 4.4 and the corresponding argument above, we can see that A is a parallelogram. Since A is rational by Lemma 4.7, we find that the case (i) of Theorem 2.1 holds.

The cases $R=N-1$ and $R=N$ are impossible. Indeed, in these cases there is a side XY of A such that the angle of A at the vertices X and Y equals γ, and XY does not contain exceptional points. This, together with the fact that Γ_c is empty, contradicts (i) of Lemma 4.4.

6 Case II: $\gamma<\pi/2$, $\gamma\neq\pi/3$

In this case, in each of equations (1), if $p_i>0$, then the equation must be of the form $\alpha+\beta=\delta_i$. We claim that in each of equations (2) and (3) we have $r_i\geq p_i$. Indeed, otherwise the reduced form of the equation in question would be $p(\alpha+\beta)=v\pi$, where p is a positive integer and $v=1$ or 2. However, as $\pi/2<\alpha+\beta<\pi$ and $\alpha+\beta\neq 2\pi/3$, no such equation is possible.

Let $P(\alpha+\beta)+R\gamma=(N-2)\pi$ be the sum of equations (1). Then, by (4), we have $P\geq R$. If $N=3$ then $P(\alpha+\beta)+R\gamma=\pi$, and thus $P\geq R$ gives $R=0$ or $P=R=1$. If $R=0$ then $P(\alpha+\beta)=\pi$ which is impossible. If $P=R=1$, then at one of the angles of A the equation is $\alpha=\delta_i$ which contradicts the regularity of the tiling. Thus $N=3$ is impossible, and we have $N\geq 4$.

By $P(\alpha+\beta)+R\gamma=(N-2)\pi$ and $P\geq R$ we have $P\geq N-2$, and thus the possible values of P are $N-2$, $N-1$ and N. If $P=N-2$, then necessarily $R=N-2$, and in each of the equations (2) and (3) we have $p_i=r_i$.

If $P=N-1$, then $N-1$ of the angles of A equal $\alpha+\beta$. The equation $(N-1)(\alpha+\beta)+R\gamma=(N-2)\pi$ gives $(N-1-R)\gamma=\pi$. There must exist an equation $p_i(\alpha+\beta)+r_i\gamma=v\pi$ with $r_i>p_i$ and $v=1,2$. The reduced form of this equation is $r\gamma=v\pi$, where we have $r=(N-1-R)v$. Since $\sum p_i=\sum r_i$, it follows that, apart from the vertices of A, there is only one vertex of the tiling with $p_i\neq r_i$, and that the reduced equation at this vertex is $(N-1-R)\gamma=\pi$.

Finally, if $P=N$, then each angle of A equals $\alpha+\beta$. Thus $\gamma=\pi-(\alpha+\beta)=2\pi/N$. There must exist an equation $p_i(\alpha+\beta)+r_i\gamma=v\pi$ with $r_i>p_i$ and $v=1,2$. The reduced form of this equation is $r\gamma=v\pi$, where we have $2r=Nv$. Since $\sum p_i=\sum r_i$, it follows that, apart from the vertices of A, there are at most two vertices with $p_i\neq r_i$. If there is one such vertex, then the reduced equation at this vertex is $N\gamma=2\pi$, and if there are two such vertices, then N is even and the corresponding reduced equations are $(N/2)\gamma=\pi$.

Summing up: there are three cases. In the first case $P=R=N-2$, and $p_i=r_i$ for every $i>N$. In this case $N-2$ of the vertices of A equals $\alpha+\beta$, and the other two vertices equal $r\gamma$ and $r'\gamma$, where $r+r'=N-2$.

In the second case $P=N-1$, and $p_i=r_i$ for every $i>N$ with exactly one exception, where the reduced equation is $(N-1-R)\gamma=\pi$. In this case $N-1$ of the vertices of A equals $\alpha+\beta$, and one equals $R\gamma$.

In the third case $P=N$, and $p_i=r_i$ for every $i>N$ with at most two exceptions. If there is only one exceptional vertex, then the corresponding equation is $N\gamma=2\pi$, and if there are two exceptional vertices, then their reduced equations are $(N/2)\gamma=\pi$. In this case each angle of A equals $\alpha+\beta$.

6.1 Subcase IIa

First we assume that a is not a linear combination of b and c with nonnegative rational coefficients, and consider the graph Γ_a. If $R = N - 2$, then there are no exceptional points apart from the vertices of A, and the in-degree equals the out-degree at every vertex. Thus no vertex starts from the boundary of A. By (iii) of Lemma 4.3 this implies that there are no adjacent vertices of A with angle $\alpha + \beta$. As in case I, we can infer that A is a parallelogram, and using an analogous argument, we can check that in this case the statement (i) of the theorem holds. Repeating the analogous argument of case I, we can also see that the cases $R = N - 1$ and $R = N$ are impossible.

6.2 Subcase IIb

Since the roles of a and b are symmetric, we have the same conclusion if b is not a linear combination of a and c with nonnegative rational coefficients.

Therefore, we may assume that a is a linear combination of b and c with nonnegative rational coefficients, and b is a linear combination of a and c with nonnegative rational coefficients. This implies that either a, b, c are pairwise commensurable, or a and b are commensurable and c is not a rational multiple of a. If a, b, c are pairwise commensurable, then (iii) of the theorem is true. Indeed, if $R = N - 1$ or $R = N$, then γ is a rational multiple of π, which is impossible (see the analogous argument in case I).

6.3 Subcase IIc

In this subcase we assume that a and b are commensurable and c is not a rational multiple of a. Then the graph Γ_c is empty by Lemma 4.6.

Lemma 6.1 *If a and b are commensurable and c is not a rational multiple of a, then either* (i) *of Theorem 2.1 holds, or $P = N$; that is, each angle of A equals $\alpha + \beta$.*

Proof Let X, Y, Z be consecutive vertices of A, and suppose that the equation at X is $r\gamma = \delta$. We show that if there are no exceptional points on the sides XY and YZ, then the equation at Z is also of the form $r\gamma = \delta$. Indeed, by (i) of Lemma 4.4, the equation at Y must be $\alpha + \beta = \delta$. The proof of Lemma 4.4 also shows that there is a triangle T_k supported by the side XY, and there is a point U_{k-1} in the interior of XY such that $U_{k-1}Y$ is a side of T_k, and the angle of T_k at U_{k-1} is γ.

Let $T'_1, \dots, T'_m$ be the triangles supported by the side YZ, and let $Y = V_0, \dots, V_m = Z$ be a division of YZ such that $V_{i-1}V_i$ is a side of T'_i for every $i = 1, \dots, m$. Then the angle of T'_1 at the vertex Y equals α or β. Since no edge of Γ_c starts from Y and the angle of T_k at U_{k-1} equals γ, it follows that the angle of T'_1 at V_1 must be γ. Now the vertices V_i are normal by assumption, and no edge of Γ_c starts from any of them. Therefore, each T'_i has angle γ at the vertex V_i (see Fig. 4). In particular, T'_m has angle γ at $V_m = Z$, and thus the equation at Z is of the form $r\gamma = \delta$.

Suppose $R = N - 2$. Then there are no exceptional points, and thus it follows from what we proved above that the equation at every second vertex of A is of the form

$r\gamma = \delta$. Then $N = 4$ and A is a parallelogram. By Lemma 4.7, the sides of A are commensurable. Since a/b is rational, it follows that (i) of Theorem 2.1 holds.

Next suppose $P = N - 1$. Then there is one single exceptional point. Let Z_1, Y_1, X, Y_2, Z_2 be consecutive vertices of A such that the equation at the vertex X is of the form $r\gamma = \delta$. (The vertices Z_1 and Z_2 may coincide.) Since only one of the sides Z_1Y_1, Y_1X, XY_2, Y_2Z_2 can contain the exceptional point, it follows that the equation at either Z_1 or Z_2 is of the form $r\gamma = \delta$. This, however, contradicts the assumption $P = N - 1$. Therefore, the only remaining possibility is $P = N$, which completes the proof. □

Therefore, we are left with the case when $P = N$. If $N = 4$, then A is a rectangle. Since $\alpha + \beta = \pi/2$, $\sin\alpha/\sin\beta = a/b$ is rational and A is rational by Lemma 4.7, it follows that (i) of Theorem 2.1 holds. Thus we may assume that $N \geq 5$. Our next aim is to show that in this case (v), (vi) or (vii) of Theorem 2.1 holds. The rest of the section is devoted to the proof of this statement.

In Lemma 4.8 we proved that those triangles of the tiling that have their side of length c in the interior of A come in pairs, and each pair forms a quadrilateral which is either a parallelogram of sides a, b and of angles $\gamma, \alpha + \beta$, or a kite of sides a, b and of angles $2\beta, \gamma, 2\alpha, \gamma$.

Lemma 6.2 *Suppose that a and b are commensurable, c is not a rational multiple of a, and $P = N$. Then among the pairs described above only parallelograms can occur. More precisely, there are no kites, unless $\alpha = \beta$ when each kite is a parallelogram.*

Proof Suppose this is not true; that is, $\alpha \neq \beta$ and there exists at least one kite. We shall denote by Γ_{kite} the set of those directed segments $\overrightarrow{XY}$ for which the segment XY is the common side of two triangles having angles α at X and β at Y. Then the line going through the edge $\overrightarrow{XY}$ of Γ_{kite} is the axis of symmetry of a kite described in Lemma 4.8.

If $\overrightarrow{XY}$ is an edge of Γ_{kite}, then Y is in the interior of A. Indeed, the left hand side of the equation at Y is of the form $p(\alpha + \beta) + r\gamma$ with $p \geq 2$. Since $r \geq p \geq 2$, we have $p = r = 2$, and the equation at Y is $2\alpha + 2\beta + 2\gamma = 2\pi$.

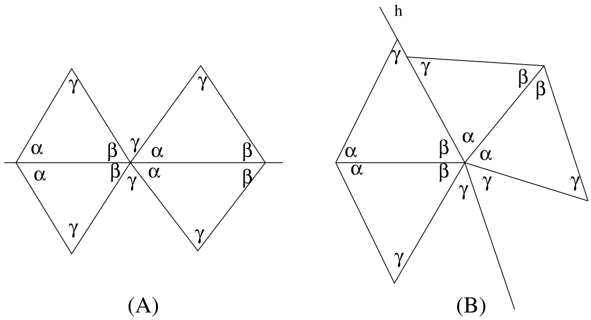
By Lemma 4.8, the triangles having Y as a vertex and having angle α or β at Y come in pairs. Each pair forms a quadrilateral which is either a parallelogram having angle $\alpha + \beta$ or γ at Y, or a kite having angle 2α or 2β at Y.

Since $p = r = 2$, it follows that the arrangement of the triangles around Y must be one of two cases presented in Fig. 5.

Since there is exactly one kite having angle 2α at Y, it follows that there is exactly one edge of Γ_{kite} starting from Y. Therefore, Γ_{kite} is the union of disjoint cycles, and every vertex of Γ_{kite} is in the interior of A. Note also that each cycle of Γ_{kite} is a simple polygon of angles $\alpha + \beta$ or $\alpha + 2\gamma + \beta$.

We shall need another graph on the set of vertices. Let XZ be a maximal segment contained by the union of the boundaries of the triangles Δ_i and such that XZ belongs to the interior of A except perhaps the endpoints X and Z. There are divisions $X = U_0, U_1, \ldots, U_k = Z$ and $X = V_0, V_1, \ldots, V_\ell = Z$ of the segment XZ such that each subinterval $U_{i-1}U_i$ $(i = 1, \ldots, k)$ is a side of a triangle T_i of the tiling supported by

Fig. 5



XZ and lying on the same side of the segment XZ, and each subinterval $V_{j-1}V_j$ ($j = 1, \dots, \ell$) is a side of a triangle T'_j of the tiling supported by XZ and lying on the other side of XZ. Suppose that the angle of T_1 at the vertex U_1 and the angle of T'_1 at the vertex V_1 both equal γ. Then there is a maximal index $1 \le i_0 \le k$ such that the angle of T_i at the vertex U_i equals γ for every $i = 1, \dots, i_0$. Similarly, there is a maximal index $1 \le j_0 \le \ell$ such that the angle of T'_j at the vertex V_j equals γ for every $j = 1, \dots, j_0$. Then we connect the vertices X and Y by a directed edge $\overrightarrow{XY}$ where $Y = U_{i_0}$ if U_{i_0} is closer to X than V_{j_0}, $Y = V_{j_0}$ if V_{j_0} is closer to X than U_{i_0}, and $Y = U_{i_0} = V_{j_0}$ otherwise. We denote by Γ_{except} the set of these edges. (The notation will be justified later, when we show that each edge of Γ_{except} starts from a vertex of A and arrives at an exceptional point.)

Let $\overrightarrow{XY}$ be an edge of Γ_{except}. We show that (i) *no inner point of the segment XY can be a vertex of Γ_{kite}, and* (ii) *Y is a vertex of Γ_{kite} or Y is an exceptional point.*

In the proof of these statements we shall use the notation of the definition of the graph Γ_{except}. By symmetry, we may assume that $Y = U_{i_0}$. Let V be an inner point of XY, and suppose that V is a vertex of Γ_{kite}. Then the arrangement of the triangles around V cannot be as shown by (B) of Fig. 5 since $2\alpha + \gamma \ne \pi$ and $2\beta + \gamma \ne \pi$. But it cannot be as shown by (A) of Fig. 5 either. Indeed, if i and j are such that T_i, T'_j have V as a vertex, then they are supported by the segment XZ and have angles γ at V. However, those two triangles in (A) of Fig. 5 which have angle γ at the given point are not supported by a common segment. This proves (i).

Next suppose that Y is not a vertex of Γ_{kite} and that Y is normal. Since $Y = U_{i_0}$, we have either $U_0 = V_{j_0}$ or U_0 is closer to X than V_{j_0}. This implies that either Y is in the interior of a side of one of the triangles T'_j, or Y is the vertex of a triangle T'_j such that the angle of T'_j at Y equals γ. In the first case the equation at Y is $\alpha + \beta + \gamma = \pi$, and the angle of T_{i_0} at Y equals γ. By Lemma 4.6, no edge of Γ_c starts from Y. This implies that the angle of the triangle T_{i_0+1} at the point U_{i_0+1} must be γ. (Indeed, by considering the possible arrangements of the angles around Y we can see that otherwise an edge of Γ_c would start from Y.) This, however, contradicts the choice of i_0, since i_0 was the largest index such that T_i has angle γ at U_i for every $i \le i_0$.

In the second case there are two triangles supported by the continuation of the segment XY and having angle γ at Y. Since Y is normal, there are four other triangles having Y as a vertex, and their angles at Y are $\alpha, \alpha, \beta, \beta$. The two triangles with angle α at V cannot be adjacent, because in that case, depending on the location of their angle γ, either there would be an edge of Γ_c starting from Y, or Y would be a vertex of Γ_{kite}. Thus the angles of the four triangles at V are either $\alpha, \beta, \alpha, \beta$ or $\beta, \alpha, \beta, \alpha$ in this order. Thus Y is an inner point of the segment XZ. Then we can check, the same way as in the previous case, that the angle of T_{i_0+1} at the point U_{i_0+1} must be γ. This, again, contradicts the maximality of i_0, which completes the proof of (ii).

If P is a simple polygon, then the bounded component of $\mathbb{R}^2 \setminus P$ will be denoted by P°.

Let C be a cycle of Γ_{kite}. We shall also denote by C the polygon formed by the edges belonging to C. Let X be a vertex of C such that the angle of C at X is convex; that is, equals $\alpha + \beta$. (For short, in this case we shall say that X is a convex vertex of C.) Let h denote the halfline starting from X as in (B) of Fig. 5. Then there is a point Y on h such that $\overrightarrow{XY}$ is an edge of Γ_{except}. Since h intersects C°, and no interior point of XY can be a vertex of Γ_{kite}, it follows that one of the following must hold: (i) *Y is a vertex of C*; (ii) *$Y \in C^\circ$ and Y is a vertex of Γ_{kite}*; or (iii) *$Y \in C^\circ$ and Y is exceptional.*

Note that in case (i) the angle of C at Y is concave; that is, equals $\alpha + 2\gamma + \beta$. Indeed, Y must be a vertex of C as in (B) of Fig. 5. Since the halfline h arrives at Y from C°, it follows that the angle of C at Y equals $\alpha + 2\gamma + \beta$.

Our next aim is to show that *for every cycle C of Γ_{kite}, C° contains all exceptional points.* In order to prove this we may assume that C° does not contain any vertex of Γ_{kite}. Indeed, if $C^\circ \cap \Gamma_{\text{kite}} \neq \emptyset$, then we take a cycle C_1 of Γ_{kite} belonging to C° and having minimal area. Then C_1° does not contain any vertex of Γ_{kite}, and if C_1° contains all exceptional points then so does C°.

As we proved above, if V is a convex vertex of C, then there is an edge $\overrightarrow{VW}$ of Γ_{except} such that either W is a concave vertex of C or $W \in C^\circ$ and W is exceptional. (Here we used the fact that C° does not contain any vertex of Γ_{kite}.) If u and v denote the number of convex and concave vertices of C, respectively, then we have

$$u(\alpha + \beta) + v(\alpha + \beta + 2\gamma) = (u + v - 2)\pi.$$

Since $\gamma = 2\pi/N$ and $\alpha + \beta = \pi(1 - (2/N))$, this implies $u = v + N$. Let V and V' be distinct convex vertices of C, and let $\overrightarrow{VW}$, $\overrightarrow{V'W'}$ be the corresponding edges of Γ_{except}. It is clear that if W and W' are concave vertices of C then they must be distinct (see (B) of Fig. 5). Since $u = v + N$, it follows that there are at least N convex vertices of C such that the corresponding edge a$\overrightarrow{VW}$ arrives at an exceptional point belonging to C°. This proves the claim if there is only one exceptional point. Suppose there are two exceptional points: E and F. Then N is even, and the equation at E is either $(N/2)\gamma = \pi$ or $\alpha + \beta + ((N/2) + 1)\gamma = 2\pi$. In both cases, the number of edges of Γ_{except} ending at E is at most $N/2$; this follows from the fact that if an edge arrives at E then there are two triangles having E as a vertex and such that they are supported by the same segment, and their angles at E equal γ. The same is true for F. Since there are at least N edges arriving at exceptional points belonging to C°, it follows that both E and F must be in C°. This proves our claim.

Let C_1 and C_2 be disjoint cycles of Γ_{kite}. Since both of C_1° and C_2° contain the exceptional points, we have $C_1^\circ \cap C_2^\circ \neq \emptyset$, and thus we have either $C_2 \subset C_1^\circ$ or $C_1 \subset C_2^\circ$. Therefore, we may list the cycles of Γ_{kite} as $C_1, \dots, C_s$, where $C_{i+1} \subset C_i^\circ$ for every

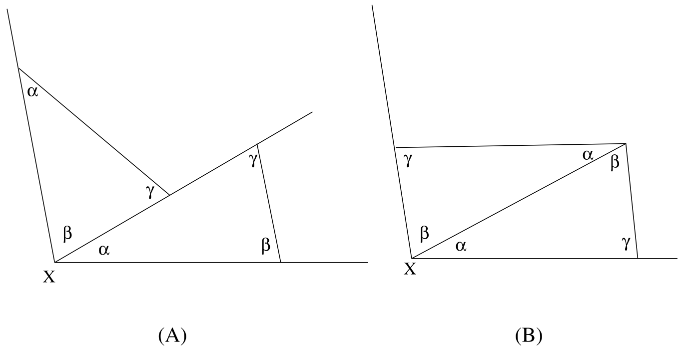


Fig. 6

$i = 1, \ldots, s-1$, and C_s° contains all exceptional points. In particular, there are no exceptional points in $A \setminus C_1^\circ$.

We prove that C_1 is convex. Suppose this is not true, and let V be a concave vertex of C_1. Then there is an edge $\overrightarrow{VW}$ of Γ_{except} starting from V (see (B) of Fig. 5). Since the edge starts from $A \setminus C_1^\circ$ and $A \setminus C_1^\circ$ contains no vertex of Γ_{kite} and no exceptional points, it follows that W belongs to the boundary of $A \setminus C_1^\circ$. That is, either W is in the boundary of A, or W is a vertex of C_1. But W cannot be a boundary point of A, since W is either a vertex of Γ_{kite} or is exceptional. Hence W is a vertex of C_1. Clearly, the segment VW is in the exterior of C_1 (except the points V and W). The argument above proves that for every concave vertex V of C_1 there is another vertex W of C_1 such that the segment VW is in the exterior of C_1. These segments VW are pairwise disjoint, since any intersection would be an exceptional point.

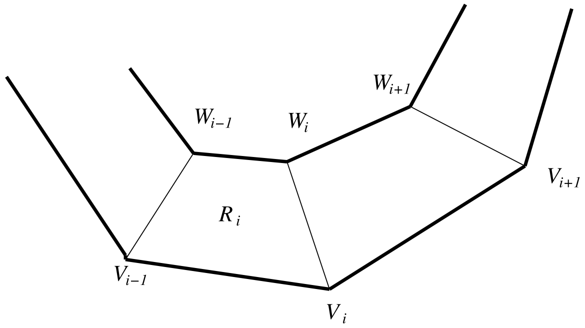
For every concave vertex V, the vertices V and W divide C_1 into the subarcs σ_1^V and σ_2^V. Let P_i^V denote the simple polygon $(VW) \cup \sigma_i^V$ $(i = 1, 2)$. We may choose the indices in such a way that $(P_1^V)^\circ$ is disjoint from C_1°, and $(P_2^V)^\circ$ contains C_1°. Let V be a concave vertex of C_1 for which the area of $(P_1^V)^\circ$ is minimal. Since $(P_1^V)^\circ \cap C_1^\circ = \emptyset$, it follows that there exists a concave vertex V' in the subarc σ_1^V. If $\overrightarrow{V'W'}$ is the edge of Γ_{except} starting from V' then, as the segments VW and $V'W'$ disjoint, W' belongs to σ_1^V. Then the area of $(P_1^{V'})^\circ$ is smaller than that of $(P_1^V)^\circ$, contradicting the choice of V. This contradiction proves that C_1 is convex. Then every angle of C_1 equals $\alpha + \beta = (1 - (2/N))\pi$, and hence C_1 has N vertices.

Let X be a vertex of A. Since no edge of Γ_c starts from X, the triangles having X as a vertex must be arranged as in Fig. 6.

Suppose that they are arranged as in (B) of Fig. 6. Let XY be a side of A. Let $X = U_0, \ldots, U_k = Y$ be a division of XY such that each $U_{i-1}U_i$ is the side of a triangle T_i of the tiling. Since the angle of T_1 at the vertex U_1 is γ and every vertex on XY is normal, it follows that the angle of T_i at the vertex U_i is γ for every i (see Fig. 4). In particular, the angle of T_k at the vertex $U_k = Y$ is γ, which is impossible.

Therefore, the triangles T and T' having X as a vertex must be arranged as in (A) of Fig. 6. Then the angles of T and of T' opposite to the sides on the boundary of A equal γ, which means that there exists an edge $\overrightarrow{XW}$ of Γ_{except} starting from X.

Fig. 7



Since $A \setminus C_1^\circ$ contains no vertex of Γ_{kite} and no exceptional points, it follows that W belongs to the boundary of $A \setminus C_1^\circ$. We can check, using a previous argument, that W is a vertex of C_1. Clearly, the segment XW is in the exterior of C_1 (except the point W).

Let $V_1, \dots, V_N = V_0$ denote the vertices of A such that V_{i-1} and V_i are adjacent for every $i = 1, \dots, N$. As we saw above, there are vertices W_i of C_1 such that $\overrightarrow{V_i W_i}$ is a vertex of Γ_{except} for every $i = 1, \dots, N$. The segments $V_i W_i$ are pairwise disjoint, since an intersection of two of them would be an exceptional point.

Since C_1 is an N-gon, it has no vertices other than $W_1, \dots, W_N$. Put $W_0 = W_N$. Then W_{i-1} and W_i are adjacent vertices of C_1 for every $i = 1, \dots, N$. Indeed, suppose that $W_1, W_i, W_j, \dots, W_k$ is a list of the consecutive vertices of C_1, where $2 < i < N$. Then $P = \{V_1 V_2 \dots V_i W_i W_1 V_1\}$ and $Q = \{V_i V_{i+1} \dots V_N, V_1, W_1, W_i, V_i\}$ are simple polygons. If $W_j \in P^\circ$, then all vertices of C_1 other than W_1 and W_i belong to P°, since the sides of C_1, with the exception of $W_k W_1$, $W_1 W_i$ and $W_i W_j$, are disjoint from the boundary of P. Since $W_N \notin P^\circ$, this is impossible. If $W_j \in Q^\circ$, then we obtain a contradiction by $W_2 \in P^\circ$. This shows that W_1 and W_i can be adjacent vertices of C_1 only if $i = 0$ or $i = 2$. We find, in the same way, that W_{i-1} and W_i are adjacent for every $i = 1, \dots, N$. Therefore, the quadrilaterals $R_i = \{V_{i-1}, V_i, W_i, W_{i-1}\}$ $(i = 1, \dots, N)$ form a ring along the boundary of A as shown in Fig. 7.

Now we prove that this is impossible (assuming $\alpha \neq \beta$). Since the equations at the vertices of A are of the form $\alpha + \beta = \delta$, the angles of R_i at the vertices V_{i-1} and R_i are equal to either α or β. We shall assume that the angle of R_1 at the vertex V_0 equals α. (The same argument applies if it equals β.) Let T denote the triangle having V_0 as a vertex and lying in R_1. Since $\overrightarrow{V_0 W_0}$ is an edge of Γ_{except}, the angle of T opposite to the side supported by $V_0 V_1$ equals γ. By (ii) of Lemma 4.4, each triangle supported by $V_0 V_1$ is a translated copy of T. This implies that the angle of R_1 at the vertex V_1 equals β. Then the angle of R_2 at the vertex V_1 equals α, and (ii) of Lemma 4.4 gives that the angle of R_1 at the vertex V_2 equals β. Continuing the argument we find that for every i, the angle of R_i at the vertex V_{i-1} equals α, and its angle at the vertex V_i equals β.

The vertex W_0 is a vertex of Γ_{kite}, and it is the endpoint of the edge $\overrightarrow{V_0 W_0}$ of Γ_{except}. Therefore, the angle of R_1 at the vertex W_0 equals either $\gamma + \alpha$ or $\gamma + \beta$ (see (B) of Fig. 5). Suppose it is $\gamma + \alpha$. Then the angle of R_1 at the vertex W_1 equals $\gamma + \beta$, since $2\pi - \alpha - \beta - (\gamma + \alpha) = \gamma + \beta$. Thus the angle of R_2 at the vertex W_1

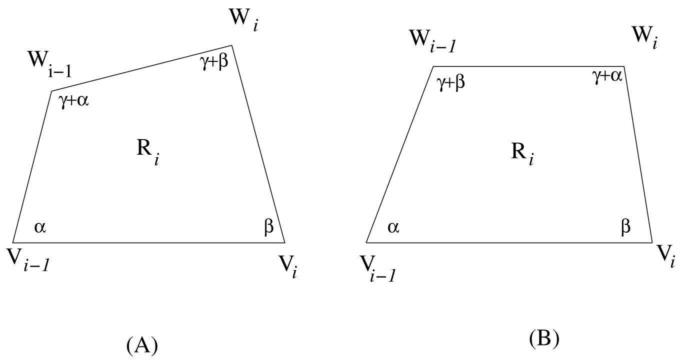


Fig. 8

equals $\gamma+\alpha$, and its angle at W_2 equals $\gamma+\beta$. Continuing the argument we find that for every i, the angles of R_i are as in (A) of Fig. 8. If the angle of R_1 at the vertex W_0 equals $\gamma+\beta$, then we find that for every i, the angles of R_i are as in (B) of Fig. 8.

Now we prove that both cases are impossible if $\alpha \neq \beta$. First suppose that each R_i looks like (A) of Fig. 8. There is a point Y on the segment $V_{i-1}V_i$ such that $V_{i-1}W_{i-1}Y\measuredangle = \gamma$ and $W_iW_{i-1}Y\measuredangle = \alpha$. Then the segments YW_{i-1} and V_iW_i are parallel to each other. Thus the quadrilateral $YV_iW_iW_{i-1}$ is a trapezoid. If $\alpha > \beta$, then $(\gamma+\beta)+\beta < \pi$, and hence $\overline{V_iW_i} > \overline{YW_{i-1}}$. The angles of the triangle $V_{i-1}YW_{i-1}$ are α, β, γ, and thus the condition $\alpha > \beta$ implies $\overline{YW_{i-1}} > \overline{V_{i-1}W_{i-1}}$. Therefore, $\alpha > \beta$ implies $\overline{V_iW_i} > \overline{V_{i-1}W_{i-1}}$ for every i. Thus

$$\overline{V_1W_1} < \overline{V_2W_2} < \cdots < \overline{V_NW_N} < \overline{V_1W_1}, \tag{16}$$

which is impossible. If $\alpha < \beta$, then a similar argument gives

$$\overline{V_1W_1} > \overline{V_2W_2} > \cdots > \overline{V_NW_N} > \overline{V_1W_1}, \tag{17}$$

also impossible. Next suppose that the quadrilaterals R_i look like (B) of Fig. 8. Then the sides $W_{i-1}W_i$ and $V_{i-1}V_i$ are parallel to each other. There is a point Y on the segment $V_{i-1}V_i$ such that $V_{i-1}W_{i-1}Y\measuredangle = \gamma$ and $W_iW_{i-1}Y\measuredangle = \beta$. Then $YV_iW_iW_{i-1}$ is a parallelogram, and thus $\overline{V_iW_i} = \overline{YW_{i-1}}$. Suppose $\alpha > \beta$. Then, as the angles of the triangle $V_{i-1}YW_{i-1}$ are α, β, γ, we have $\overline{YW_{i-1}} > \overline{V_{i-1}W_{i-1}}$. Therefore, $\alpha > \beta$ implies $\overline{V_iW_i} > \overline{V_{i-1}W_{i-1}}$ for every i, which implies (16). If $\alpha < \beta$, then a similar argument gives (17). Since both are impossible, the proof is complete. □

Lemma 6.3 *Suppose that a and b are commensurable, c is not a rational multiple of a, and $P = N$; that is, each angle of A equals $\alpha+\beta$. Then one of* (v), (vi) *or* (vii) *of Theorem* 2.1 *holds.*

Proof We shall use the notation introduced in the proof of Lemma 6.2. We proved there that if $\overrightarrow{XY}$ is an edge of Γ_{except}, then Y is either a vertex of Γ_{kite}, or is an exceptional point. (Note that the argument proving this statement did not use the condition $\alpha \neq \beta$.) If $\alpha \neq \beta$, then Γ_{kite} is empty, and thus Y must be an exceptional point. The same conclusion holds if $\alpha = \beta$. Indeed, suppose that $\alpha = \beta$ and the endpoint Y is normal. Then Y is a vertex of Γ_{kite}, and the triangles around Y are arranged as in Fig. 5. It is clear that the case of (A) is impossible. In the case of (B), the halfline h is a continuation of the segment XY, since $\alpha + \alpha + \gamma = \alpha + \beta + \gamma = \pi$. Now the two triangles supported by h have angle γ at their vertices lying on h, which contradicts the fact that Y is an endpoint of an edge. Indeed, in the definition of Γ_{except} Y is defined as U_{i_0}, where i_0 is the maximal index such that the triangle T_i has angle γ at U_i for every $i \le i_0$, while, in (B) of Fig. 5, $i_0 + 1$ also has this property. This proves that Y must be exceptional.

In the proof of Lemma 6.2 it was also shown that each vertex of A is a starting point of an edge of Γ_{except}. (This argument was also independent of the condition $\alpha \ne \beta$.) Let $\overrightarrow{V_i W_i}$ be an edge of Γ_{except} for every $i = 0, \dots, N$, where $W_1, \dots, W_N = W_0$ are exceptional points.

As we saw earlier, we have either $W_{i-1}V_{i-1}V_i\measuredangle = \alpha$ and $V_{i-1}V_iW_i\measuredangle = \beta$ for every $i = 1, \dots, N$, or $W_{i-1}V_{i-1}V_i\measuredangle = \beta$ and $V_{i-1}V_iW_i\measuredangle = \alpha$ for every $i = 1, \dots, N$. Since the roles of α and β are symmetric, we may assume the former.

Suppose that there is one exceptional point, E. Then $W_i = E$ for every i, and the edges $\overrightarrow{V_i E}$ $(i = 1, \dots, N)$ decompose A into the nonoverlapping triangles $V_{i-1}V_iE$. Since $EV_{i-1}V_i\measuredangle = \alpha$ and $V_{i-1}V_iE\measuredangle = \beta$ for every $i = 1, \dots, N$, the triangles $V_{i-1}V_iE$ are similar to each other. If $\overline{V_0E} > \overline{V_1E}$, then $\overline{V_{i-1}E} > \overline{V_iE}$ for every i, and we obtain

$$\overline{V_0E} > \cdots > \overline{V_{N-1}E} > \overline{V_0E},$$

which is impossible. We get a similar contradiction if $\overline{V_0E} < \overline{V_1E}$. Thus $\overline{V_0E} = \overline{V_1E}$, $\alpha = \beta$, A is a regular N-gon, and we obtain (vii) of Theorem 2.1.

Next suppose that there are two exceptional points, E and F. Then N is even, $N = 2k$, and the equation at E is either $k\gamma = \pi$ or $\alpha + \beta + (k+1)\gamma = 2\pi$. It is clear that in the first case the number of edges of Γ_{except} with endpoint E is at most $k - 1$, and in the second case this number is at most k. The same is true for F. Since there are (at least) $N = 2k$ edges arriving at E and F, it follows that the equations at E and F must be $\alpha + \beta + (k+1)\gamma = 2\pi$. In particular, E and F are in the interior of A, and both of them are the endpoints of k edges. By shifting the indices we may assume that there is an index $1 \le j < N$ such that $W_i = E$ for every $i = 0, \dots, j-1$, and $W_{N-1} = W_j = F$.

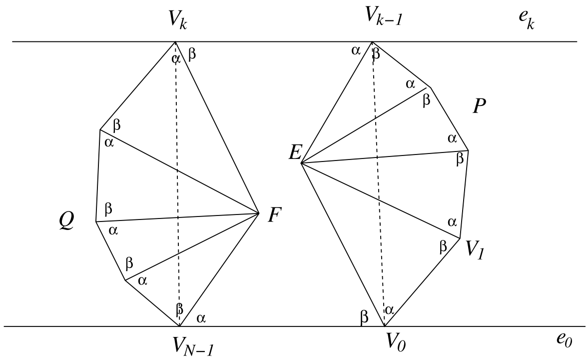
The edges starting from the vertices of A are pairwise disjoint, except the endpoints. This implies that E is in the interior of the simple polygon $\{V_{N-1}, \dots, V_j, F\}$, and thus $W_i = F$ for every $i = j, \dots, N-1$. Therefore, we have $j = k$, $W_0 = \cdots = W_{k-1} = E$ and $W_k, \dots, W_{N-1} = F$. (See Fig. 9.)

The polygons $P = \{V_0, \dots, V_{k-1}, E\}$ and $Q = \{V_k, \dots, V_{N-1}, F\}$ are convex, since their angle at E and F, respectively, equal $(k-1)\gamma = (1 - (1/k))\pi$. Put $q = \sin\alpha / \sin\beta = a/b$. Then, as a and b are commensurable, q is rational. The triangles $EV_{i-1}V_i$ $(i = 1, \dots, k-1)$ and $FV_{i-1}V_i$ $(i = k+1, \dots, N)$ are similar, and we have

$$\overline{V_iV_{i+1}}/\overline{V_{i-1}V_i} = q \quad (i \ne 0, k). \tag{18}$$

This implies that P and Q are similar polygons.

Fig. 9



Let D denote the middle point of the segment V_0V_k. We show that A is centrally symmetric with center D.

Let e_i denote the line going through the vertices V_{i-1} and V_i. Since each angle of A equals $\alpha+\beta=(1-(1/k))\pi$, it follows that the lines e_0 and e_k are parallel to each other.

Let ρ denote the reflection about the point D. Then $\rho(V_k)=V_0$ and $\rho(e_0)=e_k$. Since $V_{N-1}\in e_0$, it follows that $\rho(V_{N-1})\in e_k$. Now we have $V_{N-1}V_0E\measuredangle=V_{k-1}V_kF\measuredangle=\beta$, and thus $\rho(F)$ is on the halfline starting from V_0 and going through E. Since the triangles $\{V_0,V_{k-1},E\}$ and $\{V_k,V_{N-1},F\}$ are similar to each other and $\rho(V_{N-1})$ is on the line e_k, it follows that $\rho(\{V_k,V_{N-1},F\})$ coincides with $\rho(\{V_0,V_{k-1},E\})$. Thus $\rho(Q)=P$ and $\rho(A)=A$; that is, A is centrally symmetric.

By Lemma 4.7, A is a rational polygon. Then, it follows from (18) and $q=\sin\alpha/\sin\beta$ that (vi) of Theorem 2.1 holds if $N=6$, and (v) of Theorem 2.1 holds if $N\ge 8$. This completes the proof of the lemma. □

7 Case III: $\gamma=2\pi/3$

In this case each angle of A equals $\pi/3$ or $2\pi/3$. Thus A can be a regular triangle, a parallelogram, a trapezoid, a pentagon or a hexagon. If A is parallelogram or a trapezoid then two of its angles equal $\pi/3$ and the other two angles equal $2\pi/3$; if A is a pentagon then one of its angles equals $\pi/3$ and the other angles equal $2\pi/3$; if A is a hexagon, then all its angles equal $2\pi/3$.

7.1 Subcase IIIa

First we assume that a is not a linear combination of b and c with nonnegative rational coefficients. Then we consider the directed graph Γ_a.

Lemma 7.1 *Suppose that a is not a linear combination of b and c with nonnegative rational coefficients. Then*

(i) *the out-degree of Γ_a equals the in-degree at each vertex*;
(ii) *no edge of Γ_a starts from the boundary of A*; *and*

(iii) *every vertex lying on the boundary of A but different from the vertices of A is normal.*

Proof It is easy to check that the equation at an exceptional point must be one of $3\gamma = 2\pi$, $3\alpha + 3\beta = \pi$, $4\alpha + 4\beta + \gamma = 2\pi$, $6\alpha + 6\beta = 2\pi$.

In order to prove (i) it is enough to show that if the in-degree of a vertex X is positive, then so is the out-degree of X. If X is normal, then the out-degree at X is positive by Lemma 4.2. If X is exceptional then the equation at X must be $3\alpha + 3\beta = \pi$, since X is an interior point of a side of a triangle. Let $T_1, \dots, T_6$ denote the triangles having X as a vertex listed counterclockwise, and let h_i denote the halfline starting from X and supporting the triangles T_i and T_{i+1} ($i = 1, \dots, 5$). Three of the triangles T_i have angle β at X, and at least two of them are of the same orientation in the sense that a rotation about the point X brings one of them onto the other. Let T_i and T_j be such triangles. If $j = i + 1$, then an edge of Γ_a starts from X along the halfline h_i. If $j > i + 1$, then an edge of Γ_a starts from X along one of the halflines h_i and h_{j-1}, depending on the location of the angle γ in T_i and T_j. This proves (i).

The argument above shows that if the equation at a vertex X is $3\alpha + 3\beta = \pi$, then an edge of Γ_a starts from X. Since no edge arrives at any boundary point, we obtain (ii) from (i). Therefore, if X is a boundary point and not a vertex of A, then the equation at X cannot be $3\alpha + 3\beta = \pi$. Since the right hand side of all other equations at exceptional points equals 2π, it follows that X must be normal, which proves (iii). □

It is easy to check that if the equation at a vertex is X is $3\gamma = 2\pi$, then the in-degree of X is zero and the out-degree of X is positive. Since this contradicts (i) of Lemma 7.1, it follows that the equation cannot be $3\gamma = 2\pi$ at any vertex. Therefore, we have $r_i \le p_i$ for every $i > N$. Let $P(\alpha + \beta) + R\gamma = (N - 2)\pi$ be the sum of equations at the vertices of A. Then we have $R \ge P$. In particular, we obtain $R > 0$, and thus $N = 3$ is impossible.

Suppose $N = 4$. By (i) of Lemma 4.3, there are no adjacent vertices of A at which the equation is $\gamma = \delta$. Thus $R \le 2$, and then $P \le R$ implies that $P = R = 2$, and that the angles of A must be $\alpha + \beta,\ \gamma,\ \alpha + \beta,\ \gamma$ in this order. Thus A is a parallelogram. Since $R > 0$, there is a side XY of A such that the equation at X is $\gamma = \delta$, and the triangle having X as a vertex has angles γ at X and has angle β at its vertex lying on the side XY. By (ii) of Lemma 4.3, the length of XY is $k \cdot a$ with a positive integer k. Therefore, by Lemma 4.1, (i) of Theorem 2.1 holds.

Next suppose $N = 5$. Since there are no adjacent vertices of A at which the equation is $\gamma = \delta$, we have $R \le 2$. Then $P \ge 3$, which contradicts $P \le R$.

Finally, if $N = 6$, then $R \le 3$, and thus there are at least three vertices of A where the equation is $2(\alpha + \beta) = \delta$. Thus $P \ge 6$, which contradicts $P \le R$ again. This completes the proof in the subcase when a is not a linear combination of b and c with nonnegative rational coefficients.

7.2 Subcase IIIb

Suppose that b is not a linear combination of a and c with nonnegative rational coefficients. Since the roles of a and b are symmetric in the conditions as well as in the statements of the theorem, this subcase can be treated similarly to Subcase IIIa.

Therefore, we may assume that a is a linear combination of b and c with nonnegative rational coefficients, and b is a linear combination of a and c with nonnegative rational coefficients. This implies that either a, b, c are pairwise commensurable, or a and b are commensurable and c is not a rational multiple of a.

If a, b, c are pairwise commensurable, then so are the sides of A. We know that A is an N-gon with $3 \le N \le 6$. If $4 \le N \le 6$, then A has $N-2$ vertices with angle $2\pi/3$, and the other two angles of A are integer multiples of $\pi/3$. Thus, in these cases, (ii) of Theorem 2.1 holds. If $N = 3$, then (iv) of Theorem 2.1 holds.

7.3 Subcase IIIc

Thus we are left with the case when a and b are commensurable and c is not a rational multiple of a.

If A is a parallelogram, then (i) of Theorem 2.1 holds, since A is rational by Lemma 4.7, and $\sin\alpha/\sin\beta = a/b \in \mathbb{Q}$. Therefore, we may assume that A is not a parallelogram; that is, A is a triangle, a trapezoid, a pentagon or a hexagon.

We show that α is not a rational multiple of π except when $\alpha = \pi/6$. Indeed, we have

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin((\pi/3)-\alpha)}{\sin\alpha} = \frac{\sqrt{3}}{2}\cdot\cot\alpha - \frac{1}{2},$$

and thus $\sqrt{3}\cdot\cot\alpha$ is rational. Thus $\tan^2\alpha$ is rational, and then so are $\cos^2\alpha = 1/(1+\tan^2\alpha)$ and $\cos 2\alpha$. By [5, Corollary 3.12]), this implies $\cos 2\alpha = 0, \pm 1, \pm 1/2$. Since $\alpha < \pi/3$, we have $0 < 2\alpha < 2\pi/3$, and thus $2\alpha = \pi/3$ or $\pi/2$, and $\alpha = \pi/6$ or $\pi/4$. If $\alpha = \pi/4$, then $\sqrt{3}\cdot\cot\alpha = \sqrt{3}$ is irrational, so the only possibility is $\alpha = \pi/6$.

If $\alpha = \pi/6$, then $\beta = \pi/6$ and, taking into consideration that A is rational by Lemma 4.7, we find that (viii) of Theorem 2.1 holds. Therefore, we may assume that α/π is irrational.

Let G denote the set of real numbers $n\cdot(\pi/3) + m\cdot\alpha$, where $n, m \in \mathbb{Z}$. Then G is an additive subgroup of the reals such that $\pi \in G$. Recall that we denote by $\mathcal{P}_G$ the family of all simple, closed polygons such that, for every side XY of P, the angle between the line going through the side XY and the x-axis belongs to G.

We may assume that the x-axis contains one of the sides of A. Then $A \in \mathcal{P}_G$ and $\Delta_i \in \mathcal{P}_G$ for every $i = 1, \ldots, t$. Indeed, let T be any of the triangles Δ_i, and let e be a line containing one of the sides of T. Then there is a sequence of triangles $T_0, \ldots, T_k$ and there is a sequence of lines $e_0, e_1, \ldots, e_k$ such that $T_k = T$, e_0 is the x-axis, $e_k = e$, and e_i contains a side of both T_{i-1} and T_i for every $i = 1, \ldots, k$. Then the angle θ_i of e_{i-1} and e_i is one of α, $\beta = (\pi/3) - \alpha$ and $2\pi/3$ for every $i = 1, \ldots, k$, and thus the angle between e_k and e_0 equals $\sum_{i=1}^{k} \theta_i \in G$.

If $\theta = n\cdot(\pi/3) + m\cdot\alpha$ where $n, m \in \mathbb{Z}$, then we put $\chi(\theta) = (-1)^n$. Then χ is well-defined on G. Indeed, if $n\cdot(\pi/3) + m\cdot\alpha = n'\cdot(\pi/3) + m'\cdot\alpha$ where $n, n', m, m' \in \mathbb{Z}$, then $n = n'$ by $\alpha/\pi \notin \mathbb{Q}$. Clearly, $\chi : G \to \{1, -1\}$ is a multiplicative function; that is, χ satisfies the functional equation $\chi(\theta_1 + \theta_2) = \chi(\theta_1)\cdot\chi(\theta_2)$ $(\theta_1, \theta_2 \in G)$. Also, we have $\chi(\pi) = -1$, and thus $\chi(\theta + \pi) = -\chi(\theta)$ and $\chi(\theta + 2\pi) = \chi(\theta)$ for every $\theta \in G$. We also have $\chi(-\theta) = \chi(\theta)$ for every $\theta \in G$. Let $\Phi : \mathcal{P}_G \to \mathbb{C}$ be defined by (14).

Let $V_1, \dots, V_N = V_0$ be the vertices of A listed counterclockwise. We may assume that the side X_0X_1 lies on the x-axis. Moreover, we shall assume that if $N = 4$, then X_0X_1 and X_2X_3 are parallel sides of A, and if $N = 5$, then the only acute angle of A is at the vertex X_1. We put $d_i = \overline{X_{i-1}X_i}$ for every $i = 1, \dots, N$.

Lemma 7.2 *If $N = 3$, then $\Phi(A) = 3d_1$. If $4 \le N \le 5$, then $\Phi(A) = 3d_4$. If $N = 6$, then*

$$\Phi(A) = d_1 - d_2 + d_3 - d_4 + d_5 - d_6. \tag{19}$$

Proof Using (15) and $\chi(\pm\pi/3) = -1,\ \chi(\pm 2\pi/3) = 1$ we find that if $N = 3$, then $\Phi(A) = d_1 + d_2 + d_3 = 3d_1$.

If $N = 4$, then A is a trapezoid, and thus $d_2 = d_4$ and $d_1 = d_3 + d_4$. Therefore, we get $\Phi(A) = d_1 + d_2 - d_3 + d_4 = 3d_4$.

If $N = 5$, then A is a pentagon, $d_1 = d_3 + d_4$ and $d_2 = d_4 + d_5$. Therefore, $\Phi(A) = d_1 + d_2 - d_3 + d_4 - d_5 = 3d_4$.

It is clear that if $N = 6$, then (19) holds. □

In Lemma 4.8 we proved that those triangles of the tiling that have their side of length c in the interior of A come in pairs, and each pair forms a quadrilateral which is either a parallelogram of sides a, b and of angles $\gamma, \alpha + \beta$, or a kite of sides a, b and of angles $2\beta, \gamma, 2\alpha, \gamma$. Let $Q_1, \dots, Q_u$ be a list of these quadrilaterals. We prove that $\Phi(Q_i) = 0$ for every $i = 1, \dots, u$. This is clear if Q_i is a parallelogram. Suppose Q_i is a kite. Since $\chi(2\alpha) = 1$, $\chi(2\beta) = \chi((2\pi/3) - 2\beta) = 1$ and $\chi(2\pi/3) = 1$, (15) gives $\Phi(Q_i) = \chi(\theta_1)(a - b + b - a) = 0$. Therefore, if the quadrilaterals $Q_1, \dots, Q_u$ tile A, then $\Phi(A) = 0$. By Lemma 7.2, this implies $N = 6$ and $d_1 - d_2 + d_3 - d_4 + d_5 - d_6 = 0$.

It is easy to check, using the fact that each angle of A equals $2\pi/3$, that $d_1 + d_2 = d_4 + d_5$ and $d_2 + d_3 = d_5 + d_6$. Thus $d_1 - d_2 + d_3 - d_4 + d_5 - d_6 = 0$ implies $d_1 = d_4,\ d_2 = d_5$ and $d_3 = d_6$. Therefore, A is centrally symmetric. Since A is rational by Lemma 4.7 and $\sin\alpha/\sin\beta = a/b \in \mathbb{Q}$, it follows that (vi) of Theorem 2.1 holds.

Next we suppose that $Q_1, \dots, Q_u$ do not tile A. Then, by Lemma 4.8, there is a triangle of the tiling that has its side of length c on the boundary of A. Let $T_1, \dots, T_v$ be a list of all these triangles. Then we have

$$\Phi(A) = \sum_{i=1}^{u} \Phi(Q_i) + \sum_{j=1}^{v} \Phi(T_j) = \sum_{j=1}^{v} \Phi(T_j). \tag{20}$$

Since the graph Γ_c is empty by Lemma 4.6, it follows from Lemma 4.5 that there is a side XY of A which is covered by the sides of length c of the triangles T_j supported by the side XY. Since A is rational by Lemma 4.7, we find that each side of A has this property; that is, the boundary of A is covered by the sides of length c of the triangles $T_1, \dots, T_v$. Thus $d_i = k_i \cdot c$ $(i = 1, \dots, N)$, where $k_1, \dots, k_N$ are positive integers.

It is easy to check, using $\chi(\alpha) = 1,\ \chi(\beta) = -1$ and $\chi(\gamma) = 1$, that if a triangle T_j is supported by X_0X_1, then $\Phi(T_j) = c + a - b$. Then we find by checking each case that if $N=3$, then

$$\sum_{j=1}^{v}\Phi(T_j)=3k_1(c+a-b)=3d_1+3k_1(a-b),$$

and if $4\le N\le 5$, then

$$\sum_{j=1}^{v}\Phi(T_j)=3k_4(c+a-b)=3d_4+3k_4(a-b).$$

Since $a\ne b$ by $\alpha/\pi\notin\mathbb{Q}$, it follows from (20) and Lemma 7.2 that these cases are impossible. Thus we have $N=6$, when

$$\begin{aligned}\sum_{j=1}^{v}\Phi(T_j)&=(k_1-k_2+k_3-k_4+k_5-k_6)\cdot(c+a-b)\\&=(d_1-d_2+d_3-d_4+d_5-d_6)+k\cdot(a-b),\end{aligned}$$

where $k=k_1-k_2+k_3-k_4+k_5-k_6$. By (20) and Lemma 7.2 we have $k=0$, and thus $d_1-d_2+d_3-d_4+d_5-d_6=k\cdot c=0$. As we proved above, this implies that A is centrally symmetric, and that (vi) of Theorem 2.1 holds.

8 Case IV: $\gamma=\pi/3$

If $\gamma=\pi/3$ then, similarly to the previous case, each angle of A equals $\pi/3$ or $2\pi/3$. Thus A can be a regular triangle, a parallelogram, a trapezoid, a pentagon or a hexagon. If A is parallelogram or a trapezoid then two of its angles equals $\pi/3$ and the other two angles equal $2\pi/3$; if A is a pentagon then one of its angles equals $\pi/3$ and the other angles equal $2\pi/3$; if A is a hexagon, then all its angles equal $2\pi/3$.

8.1 Subcase IVa

First we assume that a is not a linear combination of b and c with nonnegative rational coefficients. Then we consider the directed graph Γ_a. Unfortunately, it can happen that the in-degree is different from the out-degree at certain points (unlike in the previous case). Therefore, we define another directed graph as well. We shall denote by Γ_k the set of those directed segments $\overrightarrow{XY}$ for which the segment XY is the common side of two triangles having angles γ at X and β at Y. We shall consider the set $\Gamma_a\cup\Gamma_k$ of all directed edges belonging to either Γ_a or Γ_k.

We show that the in-degree of $\Gamma_a\cup\Gamma_k$ is zero or one at each vertex. Indeed, if the in-degree of Γ_a is positive at a vertex V, then V is an inner point of a side of a triangle, and then either V is normal or the equation at V is $3\gamma=\pi$. The in-degree of Γ_k at the vertex V is zero in both cases. On the other hand, it is easy to see that the in-degree of Γ_k is at most one at every vertex.

Lemma 8.1 *Suppose that a is not a linear combination of b and c with nonnegative rational coefficients. Then*

(i) *the out-degree of $\Gamma_a \cup \Gamma_k$ equals the in-degree at each vertex*;
(ii) *no edge of $\Gamma_a \cup \Gamma_k$ starts from the boundary of A*; *and*
(iii) *every vertex lying on the boundary of A but different from the vertices of A is normal.*

Proof It is easy to check that the equation at an exceptional point must be one of $3\gamma = \pi$, $6\gamma = 2\pi$, $3\alpha + 3\beta = 2\pi$, $\alpha + \beta + 4\gamma = 2\pi$.

In order to prove (i) it is enough to show that if the in-degree of a vertex X is positive, then so is the out-degree of X.

First assume that X is the endpoint of an edge of Γ_a. If X is normal, then the out-degree at X is positive by Lemma 4.2. If X is exceptional then the equation at X must be $3\gamma = \pi$, since X is an interior point of a side of a triangle. It is easy to see that in this case either an edge of Γ_a or an edge of Γ_k starts from X.

Next assume that X is the endpoint of an edge $\overrightarrow{ZX}$ of Γ_k. Then the equation at X is $2\alpha + 2\beta + 2\gamma = 2\pi$ or $3\alpha + 3\beta = 2\pi$. Let $T_1, \dots, T_6$ denote the triangles having X as a vertex listed counterclockwise, and such that the common side of T_1 and T_6 is the segment ZX. Then the angles of T_1 and T_6 at Z and X equal γ and β, respectively. Let h_i denote the halfline starting from X and supporting the triangles T_i and T_{i+1} $(i = 0, \dots, 5)$, where we put $T_0 = T_6$.

Suppose that the equation at X is $3\alpha + 3\beta = 2\pi$. Then the angle between h_i and h_j is different from π for every $i \neq j$. Indeed, otherwise $p\alpha + q\beta = \pi$ would hold for some integers $0 \le p, q \le 3$. Since $\alpha + \beta = 2\pi/3$, this implies either $\alpha = \beta = \pi/3$ or $\{\alpha, \beta\} = \{\pi/6, \pi/2\}$. However, in each case we would have $a/b \in \mathbb{Q}$, which contradicts the assumption that a is not a linear combination of b and c with nonnegative rational coefficients. Therefore, no two of the halflines $h_0, \dots, h_5$ can form a line.

Since three of the triangles T_i have angle β at X, at least two of them are of the same orientation in the sense that a rotation about the point X brings one of them onto the other. Let T_i and T_j be such triangles. If $j = i + 1$, then an edge of Γ_a starts from X along the halfline h_i. If $j > i + 1$, then an edge of Γ_a starts from X along one of the halflines h_i and h_{j-1}, depending on the location of the angle γ in T_i and T_j.

Next suppose that the equation at X is $2\alpha + 2\beta + 2\gamma = 2\pi$. One can check, by considering the possible cases, that if the angle between h_i and h_j is different from π for every $i \neq j$ then an edge of $\Gamma_a \cup \Gamma_k$ must start from X. If there are $i \neq j$ such that the angle between h_i and h_j equals π, then $p\alpha + q\beta + r\gamma = \pi$ holds for some integers $0 \le p, q, r \le 2$. It is easy to see that the only possibility is $p = q = r = 1$. In this case the triangles T_1 and T_3 are supported by a line which is the union of the halflines h_0 and h_3. It is easy to check that in this case either an edge of Γ_a starts from X, or the common side of T_3 and T_4 is an edge of Γ_k starting from X. This proves (i).

Since no edge arrives at any boundary point, we obtain (ii) from (i). Therefore, if X is a boundary point and not a vertex of A, then the equation at X cannot be $3\gamma = \pi$. Since the right hand side of all other equations at exceptional points equals 2π, it follows that X must be normal, which proves (iii). □

The equations at the vertices of A must be of the form $\alpha+\beta=\delta$, $\gamma=\delta$ or $2\gamma=\delta$.

Suppose that the angle of A at the vertex V_1 equals $\pi/3$. Then the equation at V_1 is $\gamma=\delta$, and there is a triangle T_1 of the tiling such that V_1 is a vertex of T_1 and the angle of T_1 at V_1 is γ. We may assume that T_1 has angle β at its vertex lying on the side V_1V_2. By (ii) of Lemma 4.3, every triangle supported by V_1V_2 is a translated copy of T, and thus the length of V_1V_2 is an integer multiple of a.

There is a triangle T_k such that T_k is supported by V_1V_2, V_2 is a vertex of T_k, and T_k is a translated copy of T_1. Then the angle of T_k at V_2 is β, and thus the equation at V_2 is $\alpha+\beta=\delta_2$. Then there is a triangle T_1' supported by V_2V_3 such that V_2 is a vertex of T_1' and the angle of T_1' at V_2 equals α. Let $V_2=U_0,\dots,U_\ell=V_3$ be a division of V_2V_3 such that each $U_{i-1}U_i$ is the side of a triangle T_i' of the tiling. Then every triangle T_i' has angle α at the vertex U_{i-1} (see Fig. 3). In particular, T_ℓ' has angle α at $U_{\ell-1}$, and then the angle of T_ℓ' at V_3 is different from α. If it is β, then the equation at V_3 is $\alpha+\beta=\delta_2$. However, in this case an edge of Γ_a would start from V_3 which is impossible. Thus the angle of T_ℓ' at V_3 is γ. Then the equation at V_3 is $\gamma=\delta$ or $2\gamma=\delta$. The latter is impossible, because in that case an edge of $\Gamma_a\cup\Gamma_k$ would start from V_3. Thus the angle of A at V_3 equals $\pi/3$. Since the angle of A at V_1 is also $\pi/3$, it follows that A is a parallelogram such that the length of its side V_1V_2 is an integer multiple of a. Then, by Lemma 4.1, (i) of Theorem 2.1 holds.

Next suppose that no angle of A equals $\pi/3$. Then each angle of A equals $2\pi/3$, and the equations at the vertices of A are $2\gamma=\delta$ or $\alpha+\beta=\delta$.

Suppose that there is a vertex of A where the equation is $2\gamma=\delta$. Let V_1 be such a vertex. Since no edge of Γ_k starts from V_1, one of the triangles having V_1 as a vertex must have angle β at its vertex lying on the boundary of A. Let T_1 be such a triangle; we may assume that T_1 is supported by V_1V_2. By (ii) of Lemma 4.3, each triangle supported by V_1V_2 is a translated copy of T_1. Then the equation at V_2 is $\alpha+\beta=\delta_2$. Then there is a triangle T_1' supported by V_2V_3 such that V_2 is a vertex of T_1' and the angle of T_1' at V_2 equals α. Following the argument above we can see that in this case the angle of A at V_3 is $\pi/3$ which is impossible.

Finally, suppose that the equation at every vertex of A is $\alpha+\beta=\delta$. Then, since no edge of Γ_a starts from V_1, one of the triangles having V_1 as a vertex must have angle β at V_1 and angle γ at its vertex lying on the boundary of A. Let T_1 be such a triangle; we may assume that T_1 is supported by V_1V_2. By (ii) of Lemma 4.3, each triangle supported by V_1V_2 is a translated copy of T_1. Then the equation at V_2 is $2\gamma=\delta_2$ which is impossible. This completes the proof in the subcase when a is not a linear combination of b and c with nonnegative rational coefficients.

8.2 Subcase IVb

Suppose that b is not a linear combination of a and c with nonnegative rational coefficients. Since the roles of a and b are symmetric in the conditions as well as in the statements of the theorem, this subcase can be treated similarly to Subcase IVa.

Therefore, we may assume that a is a linear combination of b and c with nonnegative rational coefficients, and b is a linear combination of a and c with nonnegative rational coefficients. This implies that either a,b,c are pairwise commensurable, or a and b are commensurable and c is not a rational multiple of a.

If a, b, c are pairwise commensurable, then so are the sides of A. We know that A is an N-gon with $3 \le N \le 6$. If $4 \le N \le 6$, then A has $N-2$ vertices with angle $2\pi/3$, and the other two angles of A are integer multiples of $\pi/3$. Thus, in these cases, (iii) of Theorem 2.1 holds. If $N=3$, then (iv) of Theorem 2.1 holds.

8.3 Subcase IVc

Thus we are left with the case when a and b are commensurable and c is not a rational multiple of a.

If A is a parallelogram, then (i) of Theorem 2.1 holds, since A is rational by Lemma 4.7, and $\sin\alpha/\sin\beta = a/b \in \mathbb{Q}$. Therefore, we may assume that A is not a parallelogram.

Suppose that $N \le 4$; that is, A is a triangle or a trapezoid. Then the angle of A equals $\pi/3$ at two consecutive vertices, say, X and Y, and the corresponding equations at X and Y must be $\gamma = \delta$. Since Γ_c is empty, it follows from (i) of Lemma 4.4 that there is an exceptional vertex in the interior of XY. Let $X = U_0, \dots, U_k = Y$ be a division of XY such that each $U_{i-1}U_i$ is the side of a triangle T_i of the tiling. Let i be the smallest positive index such that U_i is exceptional. Since T_1 has angle γ at U_0, it follows that T_j has angle γ at U_{j-1} for every $j = 1, \dots, i$ (see Fig. 4). In particular, T_i has angle γ at U_{i-1}, and thus the angle of T_i at U_i is different from γ. However, as U_i is exceptional, the equation at U_i must be $3\gamma = \pi$, which is a contradiction. Therefore, we have $N \ge 5$, and thus A is a pentagon or a hexagon.

We prove that A must be a centrally symmetric hexagon, and thus (vi) of Theorem 2.1 holds.

First we assume that α/π is irrational. Then we consider the group G, the multiplicative function $\chi : G \to \{-1, 1\}$ and the additive function $\Phi : \mathcal{P}_G \to \mathbb{R}$ as in the Subcase IIIc. Repeating the argument of IIIc we find that A must be a centrally symmetric hexagon.

Finally, we consider the case when α/π is rational. We prove that this happens only if $\alpha = \pi/6$ or $\alpha = \pi/2$. Indeed, we have

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin((2\pi/3)-\alpha)}{\sin\alpha} = \frac{\sqrt{3}}{2}\cdot\cot\alpha + \frac{1}{2},$$

and thus $\sqrt{3}\cdot\cot\alpha$ is rational. Thus $\tan^2\alpha$ is rational, and then so are $\cos^2\alpha = 1/(1+\tan^2\alpha)$ and $\cos 2\alpha$. By [5, Corollary 3.12]), this implies $\cos 2\alpha = 0, \pm 1, \pm 1/2$. Since $\alpha < 2\pi/3$, we have $0 < 2\alpha < 4\pi/3$, and thus $2\alpha \in \{\pi/3, \pi/2, 2\pi/3, \pi\}$ and $\alpha \in \{\pi/6, \pi/4, \pi/3, \pi/2\}$. If $\alpha = \pi/4$ then $\sqrt{3}\cdot\cot\alpha = \sqrt{3}$ is irrational, which is impossible. If $\alpha = \pi/3$, then $\alpha = \beta = \gamma$ and $a = b = c$, which contradicts the condition that c/a is irrational. Therefore, we have either $\alpha = \pi/6$ or is $\alpha = \pi/2$. If $\alpha = \pi/6$ then $\beta = \pi/2$. Since the roles of α and β are symmetric, we may assume that $\alpha = \pi/2$ and $\beta = \pi/6$. Then we have $a = 2b$.

Now we prove that A has to be a hexagon.

By Lemma 4.8, the triangles having their side of length c in the interior of A come in pairs. The triangles of each pair have a common side of length c, and thus they form a quadrilateral which is either a parallelogram of sides a, b and of angles $\gamma = \pi/3, \alpha+\beta = 2\pi/3$, or a kite of sides a, b and of angles $2\beta = \pi/3, \gamma = \pi/3, 2\alpha =$

$\pi, \gamma = \pi/3$. Note that the kites are, in fact, regular triangles of side $2b$. For every kite Q we denote by $V(Q)$ the vertex of Q with angle labeled with 2β, and by $W(Q)$ the middle point of the side opposite to $V(Q)$. Since the tiling is regular, it has the property that for every kite Q, we have $V(Q) = W(Q')$ and $W(Q) = V(Q'')$ for some kites Q' and Q''.

We divide each parallelogram into two rhombuses of side b and of angles $\pi/3, 2\pi/3$. Thus A is tiled with regular triangles of side $2b$ and rhombuses of side b and of angles $\pi/3, 2\pi/3$. In addition, each triangle Q of the tiling has a selected vertex $V(Q)$ and a point $W(Q)$ opposite to $V(Q)$ with the property described above. We show that the existence of such a tiling implies that A is either a parallelogram or a hexagon.

Suppose this is not true, and let A be a counterexample with a minimal number of pieces of the tiling. Since A is not a hexagon, it has a vertex X where the angle of A is $\pi/3$. Then X cannot be a vertex of a triangle Q, because in that case either $V(Q)$ or $W(Q)$ would be in the boundary of A, which is impossible. Thus X is the vertex of a rhombus R_1. Let Y be a vertex of A adjacent to X. Then R_1 has a vertex U_1 on the side XY. If $U_1 \neq Y$, then U_1 is also the vertex of a triangle Q_1 or a rhombus R_2. But U_1 cannot be the vertex of a triangle Q_1, because in that case either $V(Q_1)$ or $W(Q_1)$ would be a point of XY, which is impossible. Thus U_1 is the vertex of a rhombus R_2. Let U_2 be a vertex of R_2 lying on XY and different from U_1. If $U_2 \neq Y$ then, by repeating the argument we find that U_2 is the vertex of a rhombus R_3 etc. In this way we obtain a division $X = U_0, U_1, \ldots, U_k = Y$ of the segment XY and a sequence of rhombuses $R_1, \ldots, R_k$ such that $U_{i-1}U_i$ is a side of R_i for every $i = 1, \ldots, k$.

Then $P = R_1 \cup \cdots \cup R_k$ is a parallelogram. Since, by assumption, A is not a parallelogram, we have $P \neq A$. Then $A \setminus P$ is a convex polygon tiled with triangles and rhombuses satisfying the condition described above. Now, the number of pieces in the tiling of $A \setminus P$ is smaller than that of the tiling of A, and thus $A \setminus P$ is either a parallelogram or a hexagon. However, $A \setminus P$ has an angle $\pi/3$, and thus it cannot be a hexagon. Therefore, $A \setminus P$ is a parallelogram, and then so is A, which is a contradiction. This proves that A is a hexagon, and thus (ix) of Theorem 2.1 holds.

9 Case V: $\gamma = \pi/2$

Under this assumption each angle of A equals $\pi/2$, and thus A is a rectangle. Our aim is to prove that in this case (i) of Theorem 2.1 holds.

9.1 Subcase Va

First we assume that a is not a linear combination of b and c with nonnegative rational coefficients. Then we consider the directed graph Γ_a.

Lemma 9.1 *Suppose that $\gamma = \pi/2$ and a is not a linear combination of b and c with nonnegative rational coefficients. Then*

(i) *the out-degree of Γ_a equals the in-degree at each vertex*;
(ii) *no edge of Γ_a starts from the boundary of* A; *and*
(iii) *the length of one of the sides of A is an integer multiple of a.*

Proof In order to prove (i) it is enough to show that if the in-degree of a vertex Y is positive, then so is the out-degree of Y. If Y is normal, then the out-degree at Y is positive by Lemma 4.2. If Y is exceptional then the equation at Y must be $2\alpha + 2\beta = \pi$ or $2\gamma = \pi$, since Y is an interior point of a side of a triangle. An inspection of all possible cases shows that at least one edge of Γ_a starts from Y.

Since no edge arrives at any boundary point, (ii) follows from (i).

Let X be a vertex of A. In order to prove (iii), first we show that there is a triangle T such that X is a vertex of T and the side of T of length a is on the boundary of A. This is clear if the equation at X is $\gamma = \delta$. Therefore, we may assume that the equation at X is $\alpha + \beta = \delta$. Let T be the triangle having X as a vertex and having angle β at X. Since, by (ii), no edge of Γ_a starts from X, it follows that the vertex of T with angle α is not on the boundary of A. Then, the side of T of length a is on the boundary of A.

We proved that there is a side XY of A and there is a division $X = U_0, \dots, U_k = Y$ of XY such that each $U_{i-1}U_i$ is the side of a triangle T_i of the tiling, and $\overline{U_0U_1} = a$. We prove that $\overline{U_{i-1}U_i} = a$ for every i. Suppose this is not true. Then there is an $1 \le i < k$ such that $\overline{U_{i-1}U_i} = a$ and $\overline{U_iU_{i+1}} \neq a$. Then an edge of Γ_a must start from U_i; this can be shown by considering the same cases as in the proof of (i). This, however, contradicts (ii). Thus $\overline{U_{i-1}U_i} = a$ for every i, which proves (iii). □

Now Lemma 4.1 and (iii) of Lemma 9.1 imply that (i) of Theorem 2.1 holds, which completes the proof in the subcase when a is not a linear combination of b and c with nonnegative rational coefficients. The same argument applies if b is not a linear combination of a and c with nonnegative rational coefficients.

Therefore, we may assume that a is a linear combination of b and c with nonnegative rational coefficients, and b is a linear combination of a and c with nonnegative rational coefficients. This implies that either a, b, c are pairwise commensurable, or a and b are commensurable and c is not a rational multiple of a.

If a, b, c are pairwise commensurable, then so are the sides of A, and we find that (i) of Theorem 2.1 holds again.

Finally, if a and b are commensurable and c is not a rational multiple of a, then the sides of A are commensurable by Lemma 4.7, and we have the same conclusion. This completes the proof of Theorem 2.1.

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