Counterexamples of the Conjecture on Roots of Ehrhart Polynomials

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Abstract On roots of Ehrhart polynomials, Beck et al. conjecture that all roots α of the Ehrhart polynomial of an integral convex polytope of dimension d satisfy $-d \leq \Re(\alpha) \leq d - 1$. In this paper, we provide counterexamples for this conjecture.

Keywords Integral convex polytope · Ehrhart polynomial · δ -vector

1 Introduction

Recently, in many research papers on convex polytopes, e.g., [1, 3–5, 9, 10], roots of Ehrhart polynomials have been studied. One of the most important problems is to solve the conjecture given in [1, Conjecture 1.4]. In this paper, we disprove this conjecture.

First of all, we review what the Ehrhart polynomial is. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\partial\mathcal{P}$ its boundary. Here an integral convex polytope is a convex polytope all of whose vertices have integer coordinates. Given a positive integer n , we write

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N| \quad \text{and} \quad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|,$$

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ denotes the cardinality of a finite set X . The systematic studies of $i(\mathcal{P}, n)$ originated in the work of Ehrhart [6], who established the following fundamental properties:

- $i(\mathcal{P}, n)$ is a polynomial in n of degree d .
- $i(\mathcal{P}, 0) = 1$.

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- (loi de réciprocité) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We call this polynomial $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of $\mathcal{P}$. We refer the reader to [2, Chap. 3], [8, Part II], or [13, pp. 235–241] for the introduction to the theory of Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \dots$ of integers by the formula

$$(1 - \lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n) \lambda^n = \sum_{j=0}^{\infty} \delta_j \lambda^j. \quad (1)$$

Since $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$, a fundamental fact on generating functions (see [13, Corollary 4.3.1]) guarantees that $\delta_j = 0$ for every $j > d$. The sequence $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ is called the δ -vector of $\mathcal{P}$. Alternate names of δ -vectors are, for example, *Ehrhart h -vector*, *Ehrhart δ -vector* or *h^* -vector*. By the reciprocity law, one has

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n) \lambda^n = \frac{\sum_{i=0}^d \delta_{d-i} \lambda^{i+1}}{(1 - \lambda)^{d+1}}. \quad (2)$$

The following properties on δ -vectors are well known:

- By (1), one has $\delta_0 = i(\mathcal{P}, 0) = 1$ and $\delta_1 = i(\mathcal{P}, 1) - (d + 1) = |\mathcal{P} \cap \mathbb{Z}^N| - (d + 1)$.
- By (2), one has $\delta_d = i^*(\mathcal{P}, 1) = |(\mathcal{P} \setminus \partial \mathcal{P}) \cap \mathbb{Z}^N|$. In particular, we have $\delta_1 \geq \delta_d$.
- Each δ_i is nonnegative [12].
- When $d = N$, the leading coefficient of $i(\mathcal{P}, n)$, which coincides with $\sum_{i=0}^d \delta_i / d!$, is equal to the usual volume of $\mathcal{P}$ (see [13, Proposition 4.6.30]). In general, the positive integer $\text{vol}(\mathcal{P}) = \sum_{i=0}^d \delta_i$ is called the *normalized volume* of $\mathcal{P}$.

For a complex number $a \in \mathbb{C}$, let $\Re(a)$ denote the real part of a . Beck et al. [1] suggest the following

Conjecture 1.1 [1, Conjecture 1.4] *All roots α of the Ehrhart polynomial of an integral convex polytope of dimension d satisfy*

$$-d \leq \Re(\alpha) \leq d - 1. \quad (3)$$

It is proved in [1] and [5] that this conjecture is true when $d \leq 5$ or roots are real numbers. Moreover, the norm bound of roots of Ehrhart polynomials is given with $O(d^2)$ (see [4]), and it is known that this bound is best possible [3, Theorem 1.7]. In addition, in order to provide evidence for this conjecture, roots of the Ehrhart polynomials of several integral convex polytopes arising from finite graphs are discussed in [9].

In this paper, we disprove Conjecture 1.1 (see Example 3.1). We obtain many possible counterexamples by Theorem 2.1, and we find a certain counterexample when $d = 15$.

Remark 1.2 In a recent paper [10], another counterexample is provided. On the one hand, integral convex polytopes given there are so-called *smooth Fano polytopes* and

a certain counterexample is found when the dimension is 124. On the other hand, our counterexamples are simpler polytopes, which are integral simplices, and we obtain them of small dimension 15.

2 A significant family of integral simplices

This section is devoted to proving the following

Theorem 2.1 *Let $m, d, k \in \mathbb{Z}_{>0}$ be arbitrary positive integers satisfying*

$$m \geq 1, d \geq 2 \quad \text{and} \quad 1 \leq k \leq \lfloor (d+1)/2 \rfloor. \quad (4)$$

Then there exists an integral convex polytope whose Ehrhart polynomial coincides with

$$\binom{d+n}{d} + m \binom{d+n-k}{d}. \quad (5)$$

Before proving the theorem, we recall the well-known combinatorial technique to compute the δ -vector of an integral simplex.

Given an integral simplex $\mathcal{F} \subset \mathbb{R}^N$ of dimension d with the vertices $v_0, v_1, \dots, v_d \in \mathbb{Z}^N$, we set

$$S = \left\{ \sum_{i=0}^d r_i (v_i, 1) \in \mathbb{R}^{N+1} : 0 \leq r_i < 1 \right\} \cap \mathbb{Z}^{N+1}.$$

We define the degree of an integer point $(\alpha, n) \in S$ with $\deg(\alpha, n) = n$, where $\alpha \in \mathbb{Z}^N$ and $n \in \mathbb{Z}_{\geq 0}$. Let $\delta_i = |\{\alpha \in S : \deg \alpha = i\}|$. Then we have

$$\delta(\mathcal{F}) = (\delta_0, \delta_1, \dots, \delta_d).$$

We also recall the following

Lemma 2.2 [2, Theorem 2.4] *Suppose that $(\delta_0, \delta_1, \dots, \delta_d)$ is the δ -vector of an integral convex polytope of dimension d . Then there exists an integral convex polytope of dimension $d+1$ whose δ -vector is $(\delta_0, \delta_1, \dots, \delta_d, 0)$.*

Note that the required δ -vector is obtained by forming the pyramid over the integral convex polytope.

Proof of Theorem 2.1 We show that there exists an integral convex polytope of dimension d whose δ -vector is

$$\delta_i = \begin{cases} 1 & \text{if } i = 0, \\ m & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

When $k = 1$, it is obvious that $(1, m, 0, \dots, 0)$ is a δ -vector. Thus, we assume that $k \geq 2$. By Lemma 2.2, it is enough to construct an integral convex polytope of dimension d with its δ -vector

$$\delta_i = \begin{cases} 1 & \text{if } i = 0, \\ m & \text{if } i = (d+1)/2, \\ 0 & \text{otherwise,} \end{cases}$$

for any positive integer m and any odd number d with $d \geq 3$.

Let $d \geq 3$ be an odd number and $c = (d-1)/2$. We define the integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d by setting the convex hull of the integer points $v_0, v_1, \dots, v_d \in \mathbb{Z}^d$, which are of the form

$$v_i = \begin{cases} \mathbf{e}_i & \text{if } i = 1, \dots, d-1, \\ \sum_{j=1}^c \mathbf{e}_j + \sum_{j=c+1}^{2c} m\mathbf{e}_j + (m+1)\mathbf{e}_d & \text{if } i = d, \\ (0, 0, \dots, 0) & \text{if } i = 0, \end{cases}$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ denote the unit coordinate vectors of $\mathbb{R}^d$. In other words, for $i = 1, \dots, d$, v_i is equal to the i th row vector of the $d \times d$ lower triangular integer matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & \cdots & 1 & m & \cdots & m & m+1 \end{pmatrix}, \quad (6)$$

where there are c 1's and c m 's in the d th row. Then we notice that $\text{vol}(\mathcal{P}) = m+1$, which coincides with the determinant of (6).

For $j = 1, 2, \dots, m$, since

$$\sum_{i=0}^c \frac{m+1-j}{m+1} (v_i, 1) + \sum_{i=c+1}^d \frac{j}{m+1} (v_i, 1) = (\underbrace{1, 1, \dots, 1}_c, \underbrace{j, j, \dots, j}_{c+1}, c+1) \in \mathbb{Z}^{d+1}$$

and

$$0 \leq \frac{m+1-j}{m+1} < 1, \quad 0 \leq \frac{j}{m+1} < 1,$$

we have $\delta_{c+1} \geq m$. Moreover, from $\text{vol}(\mathcal{P}) = m+1$ together with the non-negativity of δ -vectors, we obtain $\delta_{(d+1)/2} = m$. Therefore, we conclude that $\mathcal{P}$ has the required δ -vector. $\square$

3 Counterexamples of Conjecture 1.1

In this section, we consider the roots of the polynomial (5) given in Theorem 2.1.

Let $f(n)$ be the polynomial (5) in n of degree d . Since

$$f(n) = \frac{\prod_{j=1}^{d-k}(n+j)}{d!} \left(\prod_{j=d-k+1}^d (n+j) + m \prod_{j=0}^{k-1} (n-j) \right),$$

negative integers $-1, -2, \dots, -d+k$ are always the roots of $f(n)$. Let

$$g_{m,d,k}(n) = \prod_{j=d-k+1}^d (n+j) + m \prod_{j=0}^{k-1} (n-j)$$

be the polynomial in n of degree k . We consider the roots of $g_{m,d,k}(n)$.

Example 3.1 Let us consider the polynomial $g_{m,15,8}(n)$. When $1 \leq m \leq 8$, all its roots satisfy (3). On the other hand, when $m = 9$, its eight roots are approximately

$$\begin{aligned} &14.37537447 \pm 25.02096544\sqrt{-1}, \quad -0.77681486 \pm 10.23552765\sqrt{-1}, \\ &-2.56596317 \pm 4.52757516\sqrt{-1}, \quad \text{and} \quad -3.03259644 \pm 1.31223697\sqrt{-1}. \end{aligned}$$

By virtue of Theorem 2.1, this implies that there exists a counterexample of Conjecture 1.1. Moreover, it can be verified that for every $15 \leq d \leq 100$, $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$ possesses a root which violates (3), that is, there exists a counterexample of Conjecture 1.1 for each dimension $15 \leq d \leq 100$. It also seems to be true that there exists a counterexample when $d \geq 101$. In addition, we remark that when $d \geq 17$, we can verify that $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$ possesses a root whose real part is greater than d . (Those are computed by Maple and Maxima.)

These computational results are also supported theoretically. For example, on the roots of $g_{9,15,8}(n)$, by applying the *Routh–Hurwitz criterion* (see, e.g., [7, pp. 226–233]), we can check that $g_{9,15,8}(n + 14.3)$ possesses a root whose real part is non-negative but $g_{9,15,8}(n + 14.4)$ possesses no root whose real part is nonnegative. Of course, this means that $g_{9,15,8}(n)$ possesses a root α with $14.3 \leq \Re(\alpha) < 14.4$.

Remark 3.2 On the order of the largest real part of the non-real roots of $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$, the order seems not to be linear in d . For example, when $d = 30, 50, 100$, and 200 , the largest real parts of the non-real roots of $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$ are as follows:

d	approximate real part
30	60
50	174
100	722
200	2940

Thus, it is more natural to claim that the real parts of roots of Ehrhart polynomials are bounded with $O(d^2)$, which is known as the best possible norm bound of roots of Ehrhart polynomials.

Remark 3.3

- (a) When $m = 1$, the real parts of all the roots of $g_{1,d,k}(n)$ are $(-d + k - 1)/2$, which satisfies $-d < (-d + k - 1)/2 < -1/2$. In fact, since all the roots of $1 + \lambda^k$ are on the unit circle in the complex plane, we can apply the theorem of [11] to the polynomial $\binom{n+d}{d} + \binom{n+d-k}{d}$. On the other hand, when $m = 2$, we can obtain other counterexample of Conjecture 1.1 when $d = 37$ and $k = 19$.
- (b) When $k = 1$, one has $g_{m,d,1}(n) = (m + 1)n + d$. Thus, its root is $-d/(m + 1)$, which satisfies $-d < -d/(m + 1) < 0$. When $k = 2$, one has $g_{m,d,2}(n) = (m + 1)n^2 + (2d - m - 1)n + d(d - 1)$. If its discriminant is negative, then the real part of its roots is $-d/(m + 1) + 1/2$, which satisfies $-d + 1/2 < -d/(m + 1) + 1/2 < 1/2$.

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