The Probability that a Convex Body Intersects the Integer Lattice in a k -dimensional Set

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Abstract Let K be a convex body in $\mathbb{R}^d$. It is known that there is a constant C_0 depending only on d such that the probability that a random copy $\rho(K)$ of K does not intersect $\mathbb{Z}^d$ is smaller than $\frac{C_0}{|K|}$ and this is best possible. We show that for every $k < d$ there is a constant C such that the probability that $\rho(K)$ contains a subset of dimension k is smaller than $\frac{C}{|K|}$. This is best possible if $k = d - 1$. We conjecture that this is not best possible in the rest of the cases; if $d = 2$ and $k = 0$ then we can obtain better bounds. For $d = 2$, we find the best possible value of C_0 in the limit case when $\text{width}(K) \rightarrow 0$ and $|K| \rightarrow \infty$.

Keywords Lattices · Convex bodies · Geometric probability

1 Introduction

Let $\mathbb{Z}^d$ denote the lattice of integer points in $\mathbb{R}^d$. An isometry ρ of $\mathbb{R}^d$ is a pair (r, t) given by a rotation $r \in SO(d)$ and a translation vector $t \in \mathbb{R}^d$. For a given $K \subset \mathbb{R}^d$ and $\rho = (r, t)$, we write $\rho(K) = r(K) + t$. We are interested in the number of points or the dimension of the set $\rho(K) \cap \mathbb{Z}^d$. If $t' \in \mathbb{Z}^d$, then $(\rho(K) + t') \cap \mathbb{Z}^d$ has the same number of points and the same dimension as $\rho(K)$. Because of this it makes sense to consider exactly one vector in each equivalence class of $\mathbb{R}^d / \mathbb{Z}^d$. One way to do this is to consider only the vectors in the unit cube $[0, 1)^d$, however it will be useful not to fix this set yet.

The set $SO(d) \times (\mathbb{R}^d / \mathbb{Z}^d)$ has a natural probability measure given by the product of the normalised Haar measures in $SO(d)$ and $\mathbb{R}^d / \mathbb{Z}^d$. To any element $(r, \bar{t}) \in SO(d) \times (\mathbb{R}^d / \mathbb{Z}^d)$ we can assign an isometry $\rho = (r, t)$ such that $t \in \bar{t}$. In this way we

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can assign a probability measure to a subset of the isometries of $\mathbb{R}^d$. The probabilities we deal with in this paper are independent of this assignment.

We use Vinogradov $\ll$ notation, and write $f \ll g$ to mean that there exists a constant $C > 0$ such that $f < Cg$. In this paper, unless specified otherwise, this constant will depend only on the dimension d .

It was shown in [1, 2] that

$$\text{Prob}\{\rho(K) \cap \mathbb{Z}^d = \emptyset\} \ll \frac{1}{|K|}$$

for all convex bodies $K \subset \mathbb{R}^d$. Moreover,

$$\text{Prob}\{\rho(K) \cap \mathbb{Z}^d = \emptyset\} \gg \frac{1}{|K|}$$

for all convex bodies $K \subset \mathbb{R}^d$ with small enough width.

The next question is to compute the probability that $\rho(K)$ contains a certain amount of integer lattice points. This seems to be hard in dimensions $d > 3$, but for $d = 2$ we have the following theorem.

Theorem 1 *For every positive integer n and every convex body $K \subset \mathbb{R}^2$ with $|K| \gg n^{\frac{3}{2}}$*

$$\text{Prob}\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\} \ll \frac{n^2}{|K|^2}.$$

For every rectangle K with small enough width and $|K| > n$ we have

$$\text{Prob}\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\} \gg \frac{1}{|K|^2}.$$

This tells us that for a fixed n , this probability is comparable with $|K|^{-2}$. It would be interesting to know exactly how big $|K|^2 \text{Prob}\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\}$ can be. We have to be careful here if we do not fix n , for example, there are families of convex bodies such that this probability is larger than $\frac{n^{0.3}}{|K|^{1.5}}$ for an infinite number of values of n . If we drop the hypothesis of $|K| \gg n^{\frac{3}{2}}$ in the first part of Theorem 1, then a bound of $\frac{n^3}{|K|^2}$ can be easily obtained using the same proof method.

The next theorem tells us that we cannot lower the upper bound too much.

Theorem 2 *For every $\varepsilon > 0$, $C > 0$ and $N > 0$ there is a rectangle K and $n > N$ such that*

$$\text{Prob}\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\} > C \frac{n^{1-\varepsilon}}{|K|^2}.$$

Returning to the general case, we consider the probability that $\rho(K)$ contains a set of dimension $k < d$.

Theorem 3 *Let $k < d$ be an integer, then*

$$\text{Prob}\{\dim\left(\rho(K) \cap \mathbb{Z}^d\right) = k\} \ll \frac{1}{|K|}.$$

If $k = d - 1$ then this is best possible.

We believe that the bound of the last theorem is not best possible if $k \neq d - 1$, this is indeed the case when $d = 2$ as Theorem 1 implies.

Finally, we return to the probability that $\rho(K)$ does not contain any lattice points. If $d = 2$ we can obtain a very precise bound.

Theorem 4 *For every $\varepsilon > 0$, there exist constants k_0 and w_0 such that if K is a plane convex body with $k_0 < |K|$ and width $w < w_0$, then*

$$\text{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\} < \frac{1}{4|K|}(1 + \varepsilon).$$

Furthermore, the constant $\frac{1}{4}$ is best possible.

We conjecture that $\frac{1}{4}$ is actually the best possible value C_0 can have for all bodies K with big enough area.

Theorem 4 can be stated as

$$\lim_{\substack{w \rightarrow 0 \\ |K| \rightarrow \infty}} \sup_K \{|K| \text{ Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\}\} = \frac{1}{4},$$

where the supremum is taken over all convex bodies K . If we take the supremum over the family of ellipses we still have equality, but if we take it over the family of rectangles we obtain $\frac{2}{\pi^2}$.

From the proof we can see that the reason for this is the Blaschke–Santaló inequality. If we take a family $\mathcal{F}$ of convex bodies that are not similar to the ellipse (in the sense that $|K||K^P| < \pi^2 - \varepsilon$ for all $K \in \mathcal{F}$), then by taking the supremum over $\mathcal{F}$ the result is smaller than $\frac{1}{4}$ (see Sect. 2 for statements and definitions).

Bárány asked in [1] for which convex body of fixed volume is this probability largest. In dimension 2, the proof of the last theorem shows that thin ellipses have relatively high probability of not containing lattice points. It may be possible that the body that maximises this probability does not exist but with a thin enough ellipse we can get arbitrarily close.

2 Preliminaries

The *lattice-width* of a convex body K is defined as

$$W(K) = \min_{z \in \mathbb{Z}^d} \max\{z \cdot (x - y) : x, y \in K\}.$$

A vector $z \in \mathbb{Z}^d$ that minimises this quantity is called a *lattice-width vector* of K . The lattice-width vectors come in pairs, if z is one then so is $-z$.

The set of *primitive vectors* $\mathbb{P}^d$ is the set of vectors in $\mathbb{Z}^d$ such $\frac{1}{m}z$ is not an integer lattice point for any positive integer m . This set is sometimes referred to as the set of lattice points *visible from the origin*. If z is a lattice-width vector of K , then clearly z must be a primitive vector.

If $z \in \mathbb{P}^d$, we call z -lattice hyperplanes the hyperplanes perpendicular to z that pass through a point in $\mathbb{Z}^d$. Note that the distance between two consecutive z -lattice hyperplanes is $\frac{1}{|z|}$. If z is a lattice-width vector of K , then the number of z -lattice hyperplanes that K intersects is at most $W(K) + 1$.

An essential tool for our results is a generalisation of the Flatness theorem which can be found in [4].

Theorem (Generalised Flatness theorem) *If $L \subset \mathbb{R}^d$ is a d -dimensional lattice and $K \subset \mathbb{R}^d$ is a convex body with $\#(K \cap \mathbb{Z}^d) \leq n$, then there exists a nonzero element z of the dual L^* such that $\max\{z \cdot (x - y) : x, y \in K\} \ll (n + 1)^{\frac{1}{d}} d^2$.*

This is stronger than the usual Flatness theorem (see [5]) which only deals with $n = 0$.

If $L = \mathbb{Z}^d$ then the Flatness theorem states that

$$W(K) \ll (n + 1)^{\frac{1}{d}} d^2.$$

In the case $d = 2$, if K contains no lattice points and its area is larger than some fixed constant c , then K can only have one pair lattice-width vectors. Otherwise, the Flatness theorem implies that K is contained in a parallelogram of fixed area with sides perpendicular to two of the lattice-width vectors. If $d > 2$ this is not the case in general. However, if the projection of K into any 2-dimensional plane has area larger than c , then clearly K can only have one pair of lattice-width vectors.

If $d = 2$, we can significantly strengthen the conclusion of the Flatness theorem by making $|K|$ big.

Lemma 1 *Let n be a non-negative integer. There exists a constant C such that if $K \subset \mathbb{R}^2$ is a convex body that contains n integer lattice points and $|K| > C(n + 1)^{\frac{3}{2}}$, then there is a unique pair $\{z, -z\} \subset \mathbb{P}$ such that K intersects at most 2 z -lattice hyperplanes (or z -lattice lines).*

Proof By the Generalised Flatness theorem there exists a constant c and $z \in \mathbb{P}^2$ such that K intersects at most $c\sqrt{n+1}$ z -lines. Let s be a section of K perpendicular to z with largest length. Note that $|K| \leq |s| \frac{c\sqrt{n+1}}{|z|}$, therefore if $|K| > C(n + 1)^{\frac{3}{2}} > \frac{c\sqrt{n+1}(2n+4)}{|z|}$ then $|s| > 2n + 4$. From here it easily follows that if K intersects more than one z -lattice line on the same side of s , then K must contain more than n lattice points in some z -lattice line. Therefore, K intersects at most two z -lattice lines.

For the uniqueness, if there are two independent lattice-width directions, then K is contained in a parallelogram of area 9. $\square$

In the proofs of Lemma 4.4 in [2] and Theorem 1.1 in [1] what is basically proved is the following.

Lemma 2 Let $W_0 > 0$ and $K \subset \mathbb{R}^d$ be a convex body. Then

$$\text{Prob}\{W(\rho(K) \cap \mathbb{Z}^d) < W_0\} \ll \frac{1}{|K|}$$

where the implicit constant depends on W_0 and d .

We also need some additional properties about the distribution of $\mathbb{P}^2$ in $\mathbb{Z}^2$, several of these are described in [3]. The following lemma is well known, but we could not find a reference for it, so we include a sketch of the proof.

Lemma 3 If $m \geq -1$ is an integer and $R > 1$, then

$$\sum_{\substack{z \in \mathbb{P}^2 \\ |z| \leq R}} |z|^m = \frac{12 R^{m+2}}{\pi(m+2)} \left(1 + O\left(\frac{\log(R)}{R}\right) \right).$$

Here the implicit constant depends on m .

Proof Let μ be the Möbius function. If $z \in \mathbb{Z}^2$ and d is a positive integer, we write $d|z$ if d divides both coordinates of z . All the implicit constants in this proof will depend on m . Using standard arguments we have

$$\begin{aligned} \sum_{\substack{z \in \mathbb{P}^2 \\ |z| \leq R}} |z|^m &= \sum_{\substack{z \in \mathbb{P}^2 \\ |z| \leq R}} \sum_{d|z} \mu(d) |z|^m = \sum_{d=1}^R \sum_{\substack{w \in \mathbb{Z}^2 \\ |dw| \leq R}} \mu(d) |dw|^m \\ &= \sum_{d=1}^R d^m \mu(d) \sum_{\substack{w \in \mathbb{Z}^2 \\ |w| \leq R/d}} |w|^m. \end{aligned} \quad (1)$$

To deal with the last term, let $B(R) \subset \mathbb{R}^2$ be the disc centred at 0 with radius R and set $c = \sqrt{2}/2$. If $R > 1$,

$$\begin{aligned} \sum_{\substack{z \in \mathbb{Z}^2 \\ |z| \leq R}} |z|^m &= \int_{B(R)} |x|^m dx + O\left(\int_{B(R+c) \setminus B(R-c)} |x|^m dx\right) \\ &= \frac{2\pi R^{m+2}}{(m+2)} + O(R^{m+1}). \end{aligned}$$

Using this in (1) we obtain

$$\begin{aligned} \sum_{\substack{z \in \mathbb{P}^2 \\ |z| \leq R}} |z|^m &= \frac{2\pi R^{m+2}}{m+2} \sum_{d=1}^R \left(\frac{\mu(d)}{d^2} + \frac{1}{d} O\left(\frac{1}{R}\right) \right) \\ &= \frac{12 R^{m+2}}{\pi(m+2)} \left(1 + O\left(\frac{\log(R)}{R}\right) \right). \end{aligned}$$

□

The *polar reciprocal* of a convex body K relative to a point P is defined as

$$K^P = \{y : x \cdot (y - P) \leq 1 \text{ for all } x \in K\}.$$

The *Santaló point* of a convex body K is the point that minimises $|K^P|$. If K is centrally symmetric then its centre coincides with its Santaló point. The properties of K^P have been widely studied, a result we shall need later on can be found in [7] and is the following.

Theorem (Blaschke–Santaló inequality) *If K is a convex body in $\mathbb{R}^2$ with Santaló point P , then*

$$|K||K^P| \leq \pi^2,$$

with equality if and only if K is an ellipse.

Let $K \subset \mathbb{R}^d$ be a convex body and let $\mathcal{P}$ be a property of K which is invariant under translations by vectors in $\mathbb{Z}^d$. Most proofs in the following sections involve estimating some probability of the form

$$p = \text{Prob}\{\rho(K) \text{ has property } \mathcal{P}\}.$$

For example, Theorem 1 gives bounds for p when $\mathcal{P}$ is the property “intersects $\mathbb{Z}^2$ in n points”.

We now describe a general method that will be used several times in what follows.

Assume that for every isometry ρ , $\rho(K)$ has only one pair of lattice-width vectors of $\rho(K)$. This is done by considering K with big enough area when $d = 2$.

For a fixed $z \in \mathbb{P}^d$, let $\mathcal{P}_z$ be the property “has property $\mathcal{P}$ and has z as a lattice-width vector” and define

$$p_z = \text{Prob}\{\rho(K) \text{ has property } \mathcal{P}_z\}.$$

In the cases we consider, $p_z = 0$ if $\frac{1}{|z|}$ is small compared to the width of K . To compute p_z more easily, we fix a starting isometry ρ_0 . We define the set

$$\mathcal{A} = \{\alpha \in SO(d) : \exists t \text{ such that } \alpha(\rho_0(K)) + t \text{ satisfies } \mathcal{P}_z\}.$$

If $d = 2$, we also think of α as a angle.

For every $\alpha \in \mathcal{A}$ we define

$$T(\alpha) = \{t \in \mathbb{R}^d / \mathbb{Z}^d : \alpha(\rho_0(K)) + t \text{ satisfies } \mathcal{P}_z\}.$$

Then we have

$$p_z = \int_{\mathcal{A}} |T(\alpha)| d\alpha.$$

Since every $\rho(K)$ has exactly two lattice-width vectors (z and $-z$), we may compute p by adding p_z over all primitive vectors and dividing the result by 2. We do this to

obtain

$$\text{Prob}\{\rho(K) \text{ has property } \mathcal{P}\} = \frac{1}{2} \sum_{z \in \mathbb{P}^d} p_z.$$

In the proofs where this method is used we give bounds for $|\mathcal{A}|$ and $|T(\alpha)|$.

3 Proof of Theorems 1 and 2

In this section everything is in $\mathbb{R}^2$, so we refer to the use z -lattice hyperplanes as z -lattice lines.

We first prove Theorem 1, but the proof we give also yields another useful fact.

Lemma 4 *Let $K \subset \mathbb{R}^2$ be a convex body. For every isometry ρ fix a lattice-width vector z_ρ of $\rho(K)$. Then*

$$\text{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset \text{ and } \rho(K) \text{ intersects a } z_\rho\text{-lattice line}\} \ll \frac{1}{|K|^2}.$$

Note that it is enough to prove this for K with large enough area.

Proof of Lemma 4 and the first part of Theorem 1 Let $n \geq 0$ be an integer, Lemma 1 gives us $z \in \mathbb{P}^2$ such that $\rho(K)$ intersects at most $2z$ -lines for every isometry ρ .

Let R be a rectangle containing K with smallest possible width and such that all of its sides touch K . Let w and l be the lengths of the sides of R with $w \leq l$. Note that $\frac{1}{2}wl \leq |K| \leq wl$.

Now we use the method above with property $\mathcal{P}$ being “contains at most n lattice points of any z_ρ -lattice line”. If $n = 0$ this is exactly what we need for Lemma 4 and if $n \geq 1$ it is a weaker condition than the one needed in Theorem 1.

Fix $z \in \mathbb{P}^2$ and choose ρ_0 such that the long side of $\rho_0(R)$ is parallel to z . We may assume that $w < \frac{3}{|z|}$, otherwise $p_z = 0$. If $\alpha \in \mathcal{A}$, then $|\sin(\alpha)| \leq \frac{3}{\sqrt{l^2 + w^2}|z|} < \frac{3}{l|z|}$. Therefore, we can think of $\mathcal{A}$ as a subset of

$$\left[-\frac{3\pi}{2l|z|}, \frac{3\pi}{2l|z|}\right] \cup \left[\pi - \frac{3\pi}{2l|z|}, \pi + \frac{3\pi}{2l|z|}\right].$$

Now fix $\alpha \in \mathcal{A}$ and consider sections of $\alpha(\rho_0(K))$ parallel to z . Let s be the length of the longest of these sections, then we have $|K| \leq \frac{3}{|z|}s$. Since the area of K is big, this implies $s \geq \frac{|K||z|}{3} > (n+1)|z|$.

To measure $T(\alpha)$ if $n > 0$, it is easier to think of $\alpha(\rho_0(K))$ as being fixed and translating $\mathbb{Z}^2$. We can bound this by the area of the region where O can be translated to. This region must be contained in the union of the caps of $\rho(K)$ cut off by sections perpendicular to z of length $(n+1)|z|$ (see Fig. 1). If O were outside these caps, then the section parallel to z through O would have length larger than $(n+1)|z|$ and therefore would contain at least $n+1$ points. If $n = 0$ the same holds by a similar argument.

Fig. 1 The caps of $\rho(K)$

Fig. 2 Picture for Lemma 5

Lemma 5 *The area of any of these caps is at most $(n+1)^2|z|^2(\alpha + \frac{l}{w}\alpha^2)$.*

This lemma is proved below. With this bound for $|T(\alpha)|$ we have

$$p_z \ll n^2 \int_0^{\frac{\pi}{2|z|}} \left(|z|^2 \alpha^2 \frac{l}{w} + |z|^2 \alpha \right) d\alpha \ll n^2 \left(\frac{1}{|z|wl^2} + \frac{1}{l^2} \right).$$

Summing over $z \in \mathbb{P}$ with $|z| \leq \frac{3}{w}$ and using Lemma 3 we obtain

$$\text{Prob}\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\} \ll n^2 \sum \left(\frac{1}{|z|wl^2} + \frac{1}{l^2} \right) \ll \frac{n^2}{w^2l^2} \ll \frac{n^2}{|K|^2}. \quad \square$$

Proof of Lemma 5 Let D be one of the caps and let A be the point of D farthest away from the line determined by the section of D perpendicular to z with length $(n+1)|z|$. Set h as the distance between A and this line.

Note that the convexity of K implies that any section perpendicular to z inside the cap has length at most $(n+1)|z|$, therefore the area of the cap is at most $(n+1)|z|h$.

Now we will bound h . Choose new coordinates such that the vertices of $\rho(R)$ are $(0,0)$, $(l,0)$, (l,w) , $(0,w)$, and $z = |z|(\sin(\alpha), \cos(\alpha))$. Let $A = (a,b)$ (see Fig. 2).

For shortness, we will write $S = \sin(\alpha)$ and $C = \cos(\alpha)$. The line generated by z is the given by the equation $Sx + Cy = 0$. Now consider lines parallel to z that pass through A and $(l,0)$. The distance between these lines is $Sl - Sa - Cb$. Note that the line through A must separate the origin from the points $(l,0)$ and $(0,w)$.

Let $Q = (l,0) - P$, by measuring the area of the triangle with vertices $(l,0)$, (l,w) , (a,b) in two different ways we obtain $\frac{1}{2}|Q|(l-a, w-b) \cdot (S, C) = \frac{1}{2}w(l-a)$

and therefore

$$|Q| = \frac{w(l-a)}{Sl + Cw - Sa - Cb}.$$

Note that the convexity of K implies that h must be smaller than the distance between A and the marked line in the picture. This line is the section of the angle $\angle(l, w)(a, b)(l, 0)$ with length $(n+1)|z|$. This distance can be computed using similarity to obtain the following:

$$\begin{aligned} h &\leq (Sl - Sa - Cb) \frac{(n+1)|z|}{|Q|} \\ &= (n+1)|z| \frac{(Sl - Sa - Cb)(Sl + Cw - Sa - Cb)}{w(l-a)} \\ &\leq (n+1)|z| \frac{(Sl - Sa)(Sl + Cw)}{w(l-a)} < (n+1)|z| \frac{\alpha(\alpha l + w)}{w}. \end{aligned} \quad \square$$

Proof of the second part of Theorem 1 Let $z \in \mathbb{P}^2$ be a vector satisfying $|z| < \frac{1}{w}$.

Choose ρ_0 such that the width side of $\rho_0(K)$ is parallel to z . Let α be an angle such that $0 < \sin(\alpha) < \frac{1}{|z|l}$. We can rotate $\rho_0(K)$ by this angle and translate it in such way that it does not touch any z -lattice line. Therefore $\alpha \in \mathcal{A}$.

By thinking of $\alpha(\rho_0(K))$ as fixed and translating $\mathbb{Z}^2$, it is easy to see that $T(\alpha)$ contains a rectangle with sides parallel to those of K with lengths $|z|\sin(\alpha)$ and $|z|\cos(\alpha)$. This gives

$$p_z \geq \int_0^{\frac{1}{|z|l}} |z|^2 \sin(\alpha) \cos(\alpha) d\alpha \gg \frac{1}{l^2}.$$

Finally, we sum over $|z| < \frac{1}{w}$ and use Lemma 3 to obtain

$$\text{Prob}\{\#(\rho(K) \cap \mathbb{Z}^2) = n\} \gg \sum_{|z| < \frac{1}{w}} \frac{1}{l^2} \gg \frac{1}{|K|^2}. \quad \square$$

Proof of Theorem 2 It is well known that if $t \in [0, 1]$ is chosen with uniform probability, then the expected value of the number of integer lattice points in $K + t$ is $|K|$. This implies that the expected value of the number of integer lattice points in $\rho(K)$ is also $|K|$.

We may assume that C is large compared to $N^{1+\varepsilon}$, then Theorem 1 implies that

$$\text{Prob}\{\#(\rho(K) \cap \mathbb{Z}^2) = n\} \leq C \frac{n^{1-\varepsilon}}{|K|^2}$$

for every convex body K and $n \leq N$. Assume that this is also true for all $n > N$.

Let $l > 0$ be large and K be a rectangle with side lengths l and $l^{-\frac{\varepsilon}{6}}$. If $l^{-\frac{\varepsilon}{6}} < \frac{1}{\sqrt{2}}$ then every $\rho(K)$ intersects at most $l+1$ horizontal lines or $l+1$ vertical lines, assume that it intersects $l+1$ vertical lines. In each one of these lines there is at most one

lattice point inside K and therefore $\text{Prob}\{\#(\rho(K) \cap \mathbb{Z}^2) = n\} = 0$ for every $n > l + 1$. We then have

$$l^{3-\varepsilon} = |K|^3 = |K|^2 \sum_{n=1}^{\infty} n \text{Prob}\{\#(\rho(K) \cap \mathbb{Z}^2) = n\} \\ \ll \sum_{n=1}^{\lfloor l+1 \rfloor} n^{2-\varepsilon} \ll l^{3-\varepsilon},$$

a contradiction. $\square$

4 Proof of Theorem 3

Proof of the first part of Theorem 3 Let $K \subset \mathbb{R}^d$ be a convex body such that $\dim(K \cap \mathbb{Z}^d) = k$. There exists a lattice vector $z \in \mathbb{P}^d$ perpendicular to the affine space generated by $K \cap \mathbb{Z}^d$. Consider the family $\{H_n : n \in \mathbb{Z}\}$ of z -lattice hyperplanes ordered in the natural way such that all the points of $\mathbb{Z}^d$ in K are in H_0 . Let

$$L = \bigcup_{n \text{ odd}} (\mathbb{Z}^d \cap H_n).$$

The set L is the translate of a lattice of determinant 2 and K does not intersect any point of L . By using the Flatness theorem it can be shown that there is a constant W_k smaller than twice of the implicit constant in the Flatness theorem such that $W(K) \leq W_k$. We conclude by using Lemma 2. $\square$

Now we prove that this bound is best possible if $k = d - 1$.

Proof of the second part of Theorem 3 For $r, w > 0$ with w small and rw large, let K be a cylinder centred at the origin with radius r and height w . The symmetry of the cylinder is useful because instead of measuring in $SO(d)$, we only need to consider the sphere S^{d-1} . The natural way to do this is by identifying an element $\lambda \in S^{d-1}$ with the elements of $SO(d)$ that send K to a cylinder with its axis parallel to λ .

Since rw is large, every projection of K into a 2-dimensional space has large area. Therefore, if K contains no lattice points then it has a unique pair of lattice-width vectors. For every $z \in \mathbb{P}$ we define $L_z \subset \mathbb{Z}^d$ as the $d - 1$ dimensional lattice of vectors in $\mathbb{Z}^d$ perpendicular to z . The determinant of L_z is $|z|$.

Assume that the first coordinate of $z = (z_1, \dots, z_d)$ is not zero. If $\{e_i\}$ is the canonical basis of $\mathbb{R}^d$, then the family $\{z_i e_1 - z_1 e_i\}_{i=2}^d \subset L_z$ is linearly independent and all its elements have norm smaller than $|z|$. Therefore we can find a basis $\mathcal{B}$ of L_z such that all its elements have norm smaller than $|z|$.

Let $\lambda_0 = \frac{z}{|z|} \in S^{d-1}$. If $r > |z|$ and $w < \frac{1}{|z|}$, then $\lambda_0(K) \cap \mathbb{Z}^2$ has dimension $d - 1$. Note that if $|K|$ is large enough then $w < \frac{1}{|z|}$ implies $r > |z|$.

Let $\sin(\alpha) = \frac{1}{2|z|r}$. Consider the set $\mathcal{A} \subset S^{d-1}$ of elements that form an angle smaller than α from λ_0 , then

$$|\mathcal{A}| \gg \left(\frac{1}{|z|r} \right)^{d-1}.$$

At this point we choose a representative for every element of $\mathbb{R}^d/\mathbb{Z}^d$. The set of representatives is the parallelotope generated by $\mathcal{B}$ and $\frac{1}{|z|^2}z$.

By considering projections of K into 2 dimensional planes that contain $\lambda \in \mathcal{A}$, it can be easily seen that $\lambda(K)$ can be translated by a set of vectors with volume $\gg |z|w \cos(\beta) \gg |z|w$.

Using the method described in Sect. 2 with $\mathcal{P}$ being “intersects $\mathbb{Z}^d$ in a set of dimension $d-1$ ”, we obtain

$$\begin{aligned} p &= \sum_{|z| < \frac{1}{w}} p_z \gg \sum_{|z| < \frac{1}{w}} \frac{w}{r^{d-1}|z|^{d-2}} \\ &\gg \frac{1}{r^{d-1}w} \gg \frac{1}{|K|}. \end{aligned} \quad \square$$

5 Proof of Theorem 4

Proof of Theorem 4 First we express the probability as

$$\text{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\} = p + q,$$

where

$$p = \text{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset \text{ and } \rho(K) \text{ intersects no } z_\rho\text{-lattice line}\}$$

and

$$q = \text{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset \text{ and } \rho(K) \text{ intersects a } z_\rho\text{-lattice line}\}.$$

Lemma 4 states that

$$q \ll \frac{1}{|K|^2},$$

so we only need to bound p . Once again, we do this using the method described in Sect. 2.

Let w be the width of K and assume $w < 1$. Clearly $p_z > 0$ if and only if $|z| < \frac{1}{w}$.

For a given z , let ρ_0 be a rotation such that the width of $\rho_0(K)$ is attained in the horizontal direction. Let $w(\alpha)$ be the width in direction α of $\rho_0(K)$, then $w = w(0)$. Set $l = w(\frac{\pi}{2})$. Note that

$$|T(\alpha)| = |z| \left(\frac{1}{|z|} - w(\alpha) \right).$$

A particular case of Proposition 2 in [6] states that

$$\int_0^{2\pi} \frac{1}{(w(\alpha)/2)^2} d\alpha \leq 2|K^P| \quad (2)$$

with equality if and only if K is centrally symmetric with centre P .

Now we can compute p very accurately. The following computations are explained below. For small enough w we have

$$\begin{aligned} p &= \frac{1}{4\pi} \sum_{|z| < \frac{1}{w}} \int_{\{\alpha: w(\alpha) < \frac{1}{|z|}\}} (1 - |z|w(\alpha)) d\alpha \\ &= \frac{1}{4\pi} \int_{\{\alpha: w(\alpha) < 1\}} \sum_{|z| < \frac{1}{w(\alpha)}} (1 - |z|w(\alpha)) d\alpha \\ &= \frac{1}{4\pi} \int_{\{\alpha: w(\alpha) < 1\}} \frac{2}{\pi} \frac{1}{w(\alpha)^2} + O\left(\frac{\log(w(\alpha))}{w(\alpha)}\right) d\alpha \end{aligned} \quad (3)$$

$$= \frac{1}{8\pi^2} \int_0^{2\pi} \frac{1}{(w(\alpha)/2)^2} d\alpha + O\left(\frac{w \log(w)^2}{|K|}\right) \quad (4)$$

$$\leq \frac{1}{4\pi^2} |K^P| + O\left(\frac{w \log(w)^2}{|K|}\right) \quad (5)$$

$$\leq \frac{1}{4|K|} + O\left(\frac{w \log(w)^2}{|K|}\right) \quad (6)$$

where P is the Santaló point of K . Equality (3) is by Lemma 3, (4) is easy to obtain by using the fact that $w + l\alpha \ll w(\alpha) \ll w + l\alpha$ for $0 \leq \alpha \leq \pi/2$. Inequality (5) is a direct consequence of (2) and (6) is the Blaschke–Santaló inequality.

From here it follows that for some constant C ,

$$\text{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\} \leq \frac{1}{4|K|} \left(1 + Cw \log(w)^2 + \frac{C}{|K|}\right).$$

To see that the $\frac{1}{4|K|}$ is best possible, notice that if K is an ellipse, then (5) and (6) are equalities. $\square$

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References

1. Bárány, I.: The chance that a convex body is lattice-point free: a relative of Buffon's needle problem. *Random Struct. Algorithms* **32**, 414–426 (2007)
2. Bárány, I., Matoušek, J.: On randomized integer convex hull. *Discrete Comput. Geom.* **32**, 135–142 (2005)

3. Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers. Clarendon Press, Oxford (1979)
4. Kannan, R., Lovász, L.: Covering minima and lattice point free convex bodies. *Ann. Math.* **128**, 577–622 (1988)
5. Khintchin, A.: A quantitative formulation of Kronecker's theory of approximation. *Izv. Akad. Nauk SSSR, Ser. Mat.* **12**, 113–122 (1948). (in Russian)
6. Lutwak, E.: Width-Integrals of Convex Bodies. *Proc. Am. Math. Soc.* **53**(2), 435–439 (1975)
7. Meyer, M., Pajor, A.: On the Blaschke–Santaló inequality. *Arch. Math.* **55**, 82–93 (1990)