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Non-periodic Tilings of $\mathbb{R}^n$ by Crosses

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Abstract An n-dimensional cross consists of $2n + 1$ unit cubes: the "central" cube and reflections in all its faces. A tiling by crosses is called a Z-tiling if each cross is centered at a point with integer coordinates. Periodic tilings of $\mathbb{R}^n$ by crosses have been constructed by several authors for all $n \in N$. No non-periodic tiling of $\mathbb{R}^n$ by crosses has been found so far. We prove that if $2n + 1$ is not a prime, then the total number of non-periodic Z-tilings of $\mathbb{R}^n$ by crosses is $2^{\aleph_0}$ while the total number of periodic Z-tilings is only $\aleph_0$. In a sharp contrast to this result we show that any two tilings of $\mathbb{R}^n, n = 2, 3$, by crosses are congruent. We conjecture that this is the case not only for $n = 2, 3$, but for all n where $2n + 1$ is a prime.

Keywords Tiling by n-cross · Non-periodic tilings · Enumeration of tilings

1 Introduction

The n-cube in $\mathbb{R}^n$ centered at $X = (x_1, \ldots, x_n)$, denoted $\mathcal{C}_X$, is the set $\{(y_1, \ldots, y_n);$ $y_i = x_i + \alpha_i$, where $-\frac{1}{2} \le \alpha_i \le \frac{1}{2}\}$. A tiling $\mathcal{T}$ of $\mathbb{R}^n$ by cubes is lattice-like if the centers of cubes in $\mathcal{T}$ form a group under vector addition.

Interest in tilings of $\mathbb{R}^n$ by cubes and by clusters of cubes goes back to a conjecture of Minkowski [14]. In 1907 he asked whether in a lattice-like tiling of $\mathbb{R}^n$ by cubes there must be a pair of cubes that share a complete $(n - 1)$-dimensional face. This conjecture was related to Minkowski's work on positive definite quadratic forms. In

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1930, when Minkowski's conjecture was still open, Keller [9] suggested that the lattice condition in the conjecture is redundant, that the nature of the problem is purely a geometric one, and not algebraic as assumed by Minkowski. Thus he conjectured that each tiling of $\mathbb{R}^n$ by unit cubes contains twin cubes. In 1940 Peron [12, 13] verified Keller's conjecture for $n \leq 6$. However, in 1992, Lagarias and Shor [10] showed that Keller's conjecture is false for each $n \geq 10$. This remarkable result is on one hand surprising, while on the other hand intuitive. The surprising part is that there is a tiling of $\mathbb{R}^{10}$ by unit cubes not containing twins. However, once we have such a tiling, it is expected that a tiling with this property exists for all higher dimensions. The higher the dimension of the space, the more freedom we get. Mackey [11] showed that the conjecture is false for $n = 8, 9$ as well. In 2011 Debroni et al. showed [2], providing a computer based proof, that there is no counterexample to Keller's conjecture for $n = 7$ if each coordinate of the center of cubes is either an integer or an integer $+0.5$.

As to Minkowski's conjecture, in 1938 Hajós translated it into a conjecture on finite abelian groups. Three years later this reformulation enabled him to answer Minkowski's conjecture in the affirmative, see [6].

After Minkowski' conjecture was settled, tilings of $\mathbb{R}^n$ by various clusters of cubes were considered, see e.g. [4, 15, 19–22, 24]. Thanks to a connection to coding theory, most attention has been paid to tilings of $\mathbb{R}^n$ by crosses and their generalizations. The cross centered at X, denoted by $\mathcal{K}_X$, is the set of $2n+1$ cubes, $\mathcal{K}_X = \{\mathcal{C}_T; T = X \pm \mathbf{e}_i$ or $T = X\}$, where, as usual, $\mathbf{e}_i = (e_1, \ldots, e_n)$, with $e_i = 1$, and $e_j = 0$ for $j \neq i$. The origin of tilings of $\mathbb{R}^n$ by crosses can be traced to several independent sources. It seems that Kárteszi [8] was the first who asked whether there exists a tiling of $\mathbb{R}^3$ by crosses. Such a tiling was constructed by Freller in 1970; Korchmaros about the same time treated $n > 3$. Golomb and Welch showed the existence of these tilings in terms of error correcting codes [4]. After the existence question has been answered, the enumeration of tilings has been studied. In [15] Molnar proved that the number of pair-wise non-congruent lattice-like $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses equals the number of pair-wise non-isomorphic commutative groups of order $2n+1$. We recall that two tilings $\mathcal{T}$ and $\mathcal{S}$ of $\mathbb{R}^n$ are congruent if there exists a linear, distance preserving bijection of $\mathbb{R}^n$ which maps $\mathcal{T}$ on $\mathcal{S}$, and a tiling $\mathcal{T}$ of $\mathbb{R}^n$, where the center of every cross in $\mathcal{T}$ has integer coordinates, is called a $\mathbb{Z}$-tiling.

It turns out that all tilings of $\mathbb{R}^n$ by crosses mentioned above are lattice-like. A tiling $\mathcal{T}$ of $\mathbb{R}^n$ by crosses with their centers forming a set Σ is periodic if there exist numbers d_i, $i = 1, \ldots, n$, such that if $P \in \Sigma$ then $P \pm d_i \mathbf{e}_i \in \Sigma$ as well for $i = 1, \ldots, n$. A tiling which is not periodic will be called primitive or simply non-periodic. Obviously, each lattice-like tiling is periodic. Szabo [22] showed that if $2n + 1$ is not a prime number, then there exists a periodic $\mathbb{Z}$-tiling of $\mathbb{R}^n$ by crosses that is not lattice-like. When infinitely many tiles are needed for tiling a space $\mathcal{S}$ usually it is most convenient to use either an algebraic approach, or to find a tiling $\mathcal{T}$ of a subspace of $\mathcal{S}$ that comprises finitely many tiles and then to extend $\mathcal{T}$ periodically to a tiling of $\mathcal{S}$. However, the two methods cannot be used directly for constructing a non-periodic tiling of $\mathcal{S}$, and hence in general it is most difficult to find such a tiling. To the best of our knowledge so far no non-periodic tiling of $\mathbb{R}^n$ by crosses has been constructed. When working on a revised version of this paper we learned that Etzion [3] asked about the existence of this type of tiling.

The main result of this paper settles the question of the existence of a non-periodic tiling of $\mathbb{R}^n$ by crosses for values of n when $2n+1$ is not a prime. In fact, in this case we are able to enumerate non-periodic tilings of $\mathbb{R}^n$ by crosses as well as periodic ones.

Theorem 1 *If* $2n+1$ *is not a prime*, *then* (i) *the total number of non-congruent non-periodic tilings of* $\mathbb{R}^n$ *by crosses as well as non-periodic* $\mathbb{Z}$*-tilings of* $\mathbb{R}^n$ *by crosses is* $2^{\aleph_0}$; (ii) *the total number of non-congruent periodic* $\mathbb{Z}$*-tilings of* $\mathbb{R}^n$ *by crosses is* $\aleph_0$.

Not much is known about the number of tilings of $\mathbb{R}^n$ by crosses in the case when $2n+1$ is prime. From Molnar's result we know that there is the unique, up to a congruency, lattice-like $\mathbb{Z}$-tiling of $\mathbb{R}^n$ by crosses. It is obvious that there is only one tiling, up to a congruency, of $\mathbb{R}^n$ by crosses for $n=1$. We will prove that

Theorem 2 *For* $n=2$ *and* $n=3$, *any two tilings of* $\mathbb{R}^n$ *by crosses are congruent.*

After submitting the original version of this paper we learned that Szabo [23], by using a computer extensively, has also proved Theorem 2 for $n=3$.

We believe, see also [1], that the statement can be extended to all n when $2n+1$ is a prime. Therefore

Conjecture 3 *If* $2n+1$ *is a prime number*, *then there exists*, *up to a congruency*, *only one* $\mathbb{Z}$*-tiling of* $\mathbb{R}^n$ *by crosses.*

The above conjecture, if true, would go totally against our intuition that suggests the higher the dimension of $\mathbb{R}^n$ the more freedom we get; see also a comment on the Lagarias–Shor result on Keller's conjecture. There are $2^{\aleph_0}$ tilings by crosses of $\mathbb{R}^4$ but there would be only ONE tiling of $\mathbb{R}^5$ by crosses. Yet, we believe that we have some evidence that supports the conjecture. In addition, a result of Redei [17] implies that when $2n+1$ is a prime then each lattice-like tiling of $\mathbb{R}^n$ by crosses is congruent to a $\mathbb{Z}$-tilings. Combining it with the above mentioned result of Molnar we find that there exists a unique, up to a congruency, lattice-like tiling of $\mathbb{R}^n$ by crosses if $2n+1$ is a prime.

As mentioned above each $\mathbb{Z}$-tiling of $\mathbb{R}^n$ by crosses can be seen as a perfect 1-error correcting Lee code. We have been attracted to the area of tilings by crosses via the Golomb–Welch conjecture. This conjecture is in fact a generalization of the Kárteszi's question mentioned above. In order not to have to introduce a lot of notions and notation we will reformulate the conjecture in terms of tilings of $\mathbb{R}^n$. By the n-dimensional Lee sphere of radius r centered at a point X, denoted $L(X,r)$, we understand the set of cubes $\{\mathcal{C}_T,\ T=X+\mathbf{v}$, where $\mathbf{v}=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+\cdots+\alpha_n\mathbf{e}_n$, $\alpha_i\in\mathbb{Z}$, and $\sum_{i=1}^{n}|\alpha_i|\le r\}$. Thus the n-cross is the n-dimensional Lee sphere of radius 1.

Conjecture 4 [4] *For* $n\ge 3$, *there is no tiling of* $\mathbb{R}^n$ *by Lee spheres of radius* $r\ge 2$.

Golomb and Welch proved the conjecture for $n=3$, and $r=2$. They have also proved that for each $n>4$ there is an r_n, r_n not specified, so that the conjecture is true for all pairs (n,r) with $r \geq r_n$. Later Post [16] showed that there is no *periodic* tiling of $\mathbb{R}^n$ by Lee spheres of radius r for $3 \leq n \leq 5, r \geq n-2$, and for $n \geq 6$, and $r \geq \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2}-2)$.

Further, Gravier et al. [5] settled the Golomb–Welch conjecture for $n=3$ and all r, while Špacapan [18], whose proof is computer aided, proved it for $n=4$ and all r. Horak [7] provided an algebraic proof of the result for $n=4$, and also proved the conjecture for $n=5$ and all r.

The proof of Theorem 1 is based on a modification of a tiling of $\mathbb{R}^n$ by crosses given by Szabo in [22]. Therefore in the next section we introduce the necessary part of his work.

2 Szabo's Construction

Let E^n be the n-dimensional vector space over the real numbers. For a set X, by $\mathcal{C}_X$ and $\mathcal{K}_X$ we denote the set of cubes and crosses centered at vertices in X, respectively. When $P = O + \mathbf{v}$ we will write $\mathcal{C}_\mathbf{v}$ and $\mathcal{K}_\mathbf{v}$ instead of $\mathcal{C}_P$ and $\mathcal{K}_P$ and say that $\mathcal{C}_\mathbf{v}$ ($\mathcal{K}_\mathbf{v}$) is centered at $\mathbf{v}$. If $\mathbf{X}$ is a set of vectors, $\mathcal{C}_\mathbf{X}$ and $\mathcal{K}_\mathbf{X}$ will be used in the same sense as well. A tiling $\mathcal{T}$ of $\mathbb{R}^n$ by the set $\mathcal{K}_\mathbf{X}$ of crosses is lattice-like if $\mathbf{X}$ is a lattice; a tiling $\mathcal{T}$ by crosses $\mathcal{K}_\mathbf{X}$ is periodic with the block of size $d'_1 \times \cdots \times d'_n$ if for each $\mathbf{v} \in \mathbf{X}$ it is $\mathbf{v} \pm d'_i \mathbf{e}_i \in \mathbf{X}$, $i = 1, \ldots, n$. Further, $\mathcal{T}$ is periodic with the base block of size $d_1 \times \cdots \times d_n$ if $\mathbf{v} \in \mathbf{X}$ then $\mathbf{v} \pm d'_i \mathbf{e}_i \in \mathbf{X}$ implies $d'_i = t_i d_i$, $t_i \in \mathbb{Z}$, for $i = 1, \ldots, n$. In addition, it is obvious that if a tiling $\mathcal{T}$ is lattice-like then $\mathcal{T}$ is a periodic tiling. In this case the size of the base block is $d_1 \times \cdots \times d_n$ where d_i is the smallest positive number so that $d_i \mathbf{e}_i \in \mathbf{X}$.

Let $\mathbf{M}$ be a set of vectors, $\mathbf{M} = \{\frac{c_1}{2}\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n; c_i \in \mathbb{Z}\}$. Then $\mathbf{M}$ is a lattice and it is spanned by vectors $\mathbf{m}_1 = \frac{1}{2}\mathbf{e}_1$, and $\mathbf{m}_i = \mathbf{e}_i$, for $i \neq 1$. In what follows we consider tilings of $\mathbb{R}^n$ by crosses $\mathcal{K}_\mathbf{L}$ so that $\mathbf{L} \subset \mathbf{M}$. The following two theorems, which constitute a special case of results proved in [22], will be our main tool in proving Theorem 1.

Theorem 5 *If there exists a factorization of an Abelian group G of the form $G = \{0, 2g_1, g_2, \ldots, g_n, -2g_1, -g_2, \ldots, -g_n\} + \{0, g_1\}$ then there exists a lattice-like tiling of $\mathbb{R}^n$ by crosses $\mathcal{K}_\mathbf{L}$, where $\mathbf{L}$ is a lattice, $\mathbf{L} \subset \mathbf{M}$. Further, $\mathbf{L}$ is the kernel of the homomorphism $\Phi : \mathbf{M} \to G$ given by $(c_1\mathbf{m}_1 + c_2\mathbf{m}_2 + \cdots + c_n\mathbf{m}_n)\Phi = c_1g_1 + c_2g_2 + \cdots + c_ng_n, c_i \in \mathbb{Z}$.*

Theorem 6 *Let $2n+1 = uv$, where $u, v \in \mathbb{Z}, u \neq 1 \neq v$. Then the cyclic group C_{2uv} of order $2uv$ has a factorization of the above form. This factorization is given by $C_{2uv} = \{(j + 2kv)g; j = 0, \pm 1, \ldots, \pm\frac{v-1}{2}, k = 0, \pm 1, \ldots, \pm\frac{u-1}{2})\} + \{0, vg\}$, where g is a generator of C_{2uv}.*

Combining the two theorems we get the following.

Corollary 7 *Let $2n+1 = uv$, $u, v \in \mathbb{Z}, u \neq 1 \neq v$, and let $\{0, \pm 2vg, \pm g, \pm a_3 g, \ldots, \pm a_n g\} + \{0, vg\}$ be the above factorization of C_{2uv}. Then the set of crosses $\mathcal{K}_\mathbf{L}$,* *where* $\mathbf{L}$ *is spanned by* $\mathbf{l}_1 = -\frac{1}{2}\mathbf{e}_1 + v\mathbf{e}_2, \mathbf{l}_2 = 2uv\mathbf{e}_2, \mathbf{l}_3 = a_3\mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{l}_n = a_n\mathbf{e}_2 - \mathbf{e}_n$, *constitutes a tiling of* $\mathbb{R}^n$.

Proof By Theorems 5 and 6 we need only to show that $\mathbf{L}$ is the kernel of the homomorphism $(\frac{c_1}{2}e_1 + c_2e_2 + \dots + c_ne_n)\Phi = c_1vg + c_2g + c_3a_3g + \dots + c_na_ng, c_i \in \mathbb{Z}$. It is easy to check that $\mathbf{l}_i\Phi = \mathbf{0}$, for all $i = 1, \dots, n$. Moreover, $|\mathbf{M}/\mathbf{L}| = |\frac{\det(\mathbf{l}_1,\dots,\mathbf{l}_n)}{\det(\mathbf{m}_1,\dots,\mathbf{m}_n}| = \frac{uv}{\frac{1}{2}} = |C_{2uv}|$. □

Two vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} - \mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \dots + a_n\mathbf{e}_n$, are said to be in relation δ if a_1 is an integer. Szabo proved

Theorem 8 [22] *Let* $\mathcal{T}$ *be a tiling of* $\mathbb{R}^n$ *by crosses* $\mathcal{K}_{\mathbf{L}}$. *If* $\mathbf{T}$ *is a* δ *class of* $\mathbf{L}$, *then* $\mathcal{K}_{\mathbf{T}}$ *is an infinite prism along* $\mathbf{e}_1$. *In other words, if* $A \in \mathbb{R}^n$ *belongs to a cross from* $\mathcal{K}_{\mathbf{T}}$ *and* $\lambda \in \mathbb{R}$, *then* $P = A + \lambda\mathbf{e}_1$ *belongs to a cross from* $\mathcal{K}_{\mathbf{T}}$ *as well.*

Consider a tiling $\mathcal{T}$ of $\mathbb{R}^n$ by $\mathcal{K}_{\mathbf{L}}$ where $\mathbf{L}$ is a lattice. Let $\mathbf{X}$ be the δ class containing the vector $\mathbf{0}$. Clearly, $\mathbf{X}$ is a subgroup of $\mathbf{L}$. The cosets of the quotient group $\mathbf{L}/\mathbf{X}$ are δ classes of $\mathbf{L}$. Let $\mathbf{Y}$ be a coset of $\mathbf{L}/\mathbf{X}$. Then the set of crosses $\mathcal{K}_{\mathbf{L}'}$ where $\mathbf{L}' = (\mathbf{L} - \mathbf{Y}) \cup (\mathbf{Y} + \lambda\mathbf{e}_1)$ constitutes a tiling of $\mathbb{R}^n$. In other words, any translation of crosses centered in $\mathbf{Y}$ along $\mathbf{e}_1$ yields a tiling of $\mathbb{R}^n$. This idea has been used in [22] to construct a periodic tiling of $\mathbb{R}^n$ by crosses $\mathcal{K}_{\mathbf{L}}$, where $\mathbf{L}$ is not a lattice.

3 Proof of Theorems 1 and 2

In the rest of the paper we will use the following notation. We consider only dimensions $n \in \mathbb{N}$ so that $2n + 1$ is not a prime. Let $u, v \in \mathbb{Z}, u \neq 1 \neq v$, so that $2n + 1 = u \cdot v$. Further, $\mathcal{T}$ will stand for the tiling of $\mathbb{R}^n$ by crosses $\mathcal{K}_{\mathbf{L}}$ given by Corollary 7. That is, $\mathbf{L}$ is the lattice spanned by vectors $\mathbf{l}_1 = -\frac{1}{2}\mathbf{e}_1 + v\mathbf{e}_2, \mathbf{l}_2 = 2uv\mathbf{e}_2, \mathbf{l}_3 = a_3\mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{l}_n = a_n\mathbf{e}_2 - \mathbf{e}_n$, and $C_{2uv} = \{0, \pm 2vg, \pm g, \pm a_3g, \dots, \pm a_ng\} + \{0, vg\}$ is the factorization of C_{2uv} from Theorem 6. Clearly, the δ class $\mathbf{X}$ containing the vector $\mathbf{0}$ is the subgroup of $\mathbf{L}$, which is spanned by vectors $2\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n$. The other δ class $\mathbf{Y}$ is obtained by a translation of $\mathbf{X}$ by $\mathbf{l}_1$, that is, $\mathbf{Y} = \mathbf{X} + \mathbf{l}_1$. So, in this case, the order of $\mathbf{L}/\mathbf{X}$ equals 2.

The main idea of the proof is as follows. As stated above, both $\mathcal{K}_{\mathbf{X}}$ and $\mathcal{K}_{\mathbf{Y}}$ are infinite prisms along $\mathbf{e}_1$. We will prove that both $\mathcal{K}_{\mathbf{X}}$ and $\mathcal{K}_{\mathbf{Y}}$ consist of infinitely many connected components. Obviously, each connected component is a prism along $\mathbf{e}_1$ as well. Therefore, when translating $\mathcal{K}_{\mathbf{Y}}$, we do not have to translate it as one block but we can translate each connected component of $\mathcal{K}_{\mathbf{Y}}$ by a different vector parallel to $\mathbf{e}_1$. To construct the required tilings of $\mathbb{R}^n$ by crosses we will translate the connected components of $\mathcal{K}_{\mathbf{X}}$ and $\mathcal{K}_{\mathbf{Y}}$ in a suitable way.

As we do not use here the notion of a connected component in the standard way we start with a definition.

Definition 9 Let $\mathcal{C}_{\mathbf{A}}$ be a set of cubes in $\mathbb{R}^n$. We will say that $\mathcal{C}_{\mathbf{A}}$ forms a connected set if for any two cubes $\mathcal{C}_{\mathbf{u}}$ and $\mathcal{C}_{\mathbf{v}}$ in $\mathcal{C}_{\mathbf{A}}$ there is a sequence of cubes

$\mathcal{C}_{\mathbf{u}_o=\mathbf{u}}, \mathcal{C}_{\mathbf{u}_1}, \dots, \mathcal{C}_{\mathbf{u}_t=\mathbf{v}}$ in $\mathcal{C}_{\mathbf{A}}$ so that any two consecutive cubes $\mathcal{C}_{\mathbf{u}_i}$ and $\mathcal{C}_{\mathbf{u}_{i+1}}$ in the sequence are adjacent (= join each other along entire $(n-1)$-dimensional face; $|\mathbf{u}_i - \mathbf{u}_{i+1}| = 1$). Further, if $\mathcal{C}_{\mathbf{L}}$ is a set of cubes, then $\mathcal{C}_{\mathbf{B}}$, where $\mathbf{B} \subset \mathbf{L}$, is a connected component of $\mathcal{C}_{\mathbf{L}}$ if $\mathcal{C}_{\mathbf{B}}$ is connected, and there is no $\mathbf{D}, \mathbf{B} \subset \mathbf{D} \subset \mathbf{L}$, so that $\mathcal{C}_{\mathbf{D}}$ is connected.

The above definition of the connected component does not coincide in general with the standard definition but it will serve our purpose. Let $\mathcal{K}_{\mathbf{A}}$ and $\mathcal{K}_{\mathbf{B}}$ be two sets of crosses so that each of them is an infinite prism along $\mathbf{e}_1$. Then we can shift them independently (= we can shift them by different vectors parallel to $\mathbf{e}_1$) if there is no cross $\mathcal{K}$ so that $\mathcal{K} \cap \mathcal{K}_{\mathbf{A}} \neq \emptyset$, and $\mathcal{K} \cap \mathcal{K}_B \neq \emptyset$ (we recall that $\mathcal{K}$ is a set of cubes). Thus if we prove that $\mathcal{K}_{\mathbf{A}}$ and $\mathcal{K}_{\mathbf{B}}$ are infinite prisms along $\mathbf{e}_1$, which form distinct connected components in the above sense, then $\mathcal{K}_{\mathbf{A}}$ and $\mathcal{K}_{\mathbf{B}}$ can be shifted independently of each other. The following theorem claims that $\mathcal{K}_{\mathbf{X}}$, where $\mathbf{X}$ is the δ class containing the vector $\mathbf{0}$, has infinitely many connected components. In fact these connected components are connected components also in the standard sense. However, to show this would make our proof even longer.

Theorem 10 will provide the key ingredient for the proof of Theorem 1. As defined above, for a set X, $\mathcal{K}_X$ is the set of crosses centered at vertices in X. Each cross is a set of $2n+1$ cubes. Abusing terminology slightly, we will understand $\mathcal{K}_X$ also as a set of cubes.

Theorem 10 *The set of cubes $\mathcal{K}_{\mathbf{X}}$ consists of infinitely many connected components $B_k, k \in \mathbb{Z}$. In particular, the cube centered at $(0, 2kv, 0, \dots, 0)$, $k \in \mathbb{Z}$, belongs to B_k.*

Proof We start with a technical claim that will be frequently used.

Claim 11 (i) *Let $\mathbf{w} = (a, b, 0, \dots, 0) \in \mathbf{L}$. Then $v|b$. In particular, if $a = 0$, i.e., $\mathbf{w} = (0, b, 0, \dots, 0)$, then $2uv|b$.* (ii) *If a vector $\mathbf{w} \in \mathbf{L}$ then $\mathbf{w} \pm u\mathbf{e}_1 \in \mathbf{L}$.*

Proof (i) Let $\mathbf{w} = (a, b, 0, \dots, 0) \in \mathbf{L}$. For $i \geq 3$, $\mathbf{l}_i = a_i \mathbf{e}_2 - \mathbf{e}_i$. Hence $\mathbf{w} = s\mathbf{l}_1 + r\mathbf{l}_2$, where $s, r \in \mathbb{Z}$, and consequently $\mathbf{w} = -\frac{s}{2}\mathbf{e}_1 + v(s + r2v)\mathbf{e}_2$; i.e., $v|b$. For $\mathbf{w} = b\mathbf{e}_2$, it is $\mathbf{w} = s\mathbf{l}_2 = s2uv\mathbf{e}_2$, i.e., $2uv|b$. (ii) As $\mathbf{L}$ is a lattice it suffices to show that $u\mathbf{e}_1 \in \mathbf{L}$. By (i), the vector $-\frac{s}{2}\mathbf{e}_1 + v(s + r2u)\mathbf{e}_2 \in \mathbf{L}$. Setting $r = 1$ and $s = -2u$, we get that $u\mathbf{e}_1 \in \mathbf{L}$. □

To facilitate our discussion we introduction more notation. Let $b_3, \dots, b_n \in \mathbb{Z}$ be fixed. We denote by $P_2(b_3, \dots, b_n)$ the set of cubes centered at vertices $(0, t, b_3, \dots, b_n)$, where $t \in \mathbb{Z}$. Clearly $P_2(b_3, \dots, b_n)$ forms an infinite prism along the axis $\mathbf{e}_2$. The following claims will be needed:

Claim 12 *The set of cubes $P_2(0, \dots, 0) \cap \mathcal{K}_X$ comprises infinitely many connected components $D_k, k \in \mathbb{Z}$. D_0 consists of v consecutive cubes centered at vertices $(0, t, 0, \dots, 0)$, where $t = 0, \pm 1, \dots, \pm\frac{v-1}{2}$. The component $D_k, k \in \mathbb{Z}$, is obtained by translating D_0 by $2kv\mathbf{e}_2$. In addition, if $\mathbf{v} \in \mathbf{X}$ belongs to $P_2(0, \dots, 0) \cap \mathcal{K}_X$, then $\mathcal{C}_{\mathbf{v}}$ is the middle cube of D_k for some $k \in \mathbb{Z}$.*

Proof First we show that v consecutive cubes centered at vertices $(0,t,0,\dots,0)$, where $t=0,\pm1,\dots,\pm\frac{v-1}{2}$ belong to $P_2(0,\dots,0)\cap\mathcal{K}_X$. Since $\mathbf{0}\in\mathbf{X}$, it is $\mathcal{C}_A\in P_2(0,\dots,0)\cap\mathcal{K}_{\mathbf{X}}$ for $A=(0,t,0,\dots,0)$, where $t=-1,0,1$. By Theorem 6 the elements jg, $j=2,\dots,\frac{v-1}{2}$ belong to the first set of the factorization of the cyclic group C_{2uv}. Therefore, by Corollary 7, for each $j\in\{2,\dots,\frac{v-1}{2}\}$ there exists $i\ge3$, so that $\mathbf{l}_i=j\mathbf{e}_2-\mathbf{e}_i$. As $\mathbf{l}_i\in\mathbf{X}$ and the cube $\mathcal{C}_A$, $A=(0,j,0,\dots,0)$, belongs to $\mathcal{K}_{\mathbf{l}_i}$, we have $\mathcal{C}_A\in P_2(0,\dots,0)\cap\mathcal{K}_{\mathbf{X}}$. $\mathbf{X}$ is a lattice, thus $-\mathbf{l}_i\in\mathbf{X}$ as well. Thus D_0 belongs to $P_2(0,\dots,0)\cap\mathcal{K}_X$. Further, because $2k\mathbf{l}_1=-k\mathbf{e}_1+2kv\mathbf{e}_2\in\mathbf{X}$, and $\mathcal{K}_{\mathbf{X}}$ is an infinite prism along $\mathbf{e}_1$, each translation of D_0 by the vector $2kv\mathbf{e}_2$ belongs to $P_2(0,\dots,0)\cap\mathcal{K}_{\mathbf{X}}$. On the other hand, as $\mathbf{Y}=\mathbf{X}+\mathbf{l}_1$, and also $\mathbf{Y}=\mathbf{X}+(2k+1)\mathbf{l}_1$, for each $k\in\mathbb{Z}$, the translation of D_0 by the vector $(2k+1)\mathbf{l}_1=-\frac{2k+1}{2}\mathbf{e}_1+(2k+1)v\mathbf{e}_2$ does not belong to $\mathcal{K}_{\mathbf{X}}$. Further, $\mathcal{K}_{\mathbf{Y}}$ is an infinite prism along $\mathbf{e}_1$. Therefore, D_0 and each translation of D_0 by $2kv\mathbf{e}_2$ constitutes a connected component in $P_2(0,\dots,0)\cap\mathcal{K}_{\mathbf{X}}$. The last part of the statement follows from Claim 11(i). □

Claim 13 *For each $(n-2)$-tuple $(b_3,\dots,b_n)$, the set of cubes $P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ consists of infinitely many components E_k, $k\in\mathbb{Z}$. Let $\mathbf{l}=-(b_3\mathbf{l}_3+\cdots+b_n\mathbf{l}_n)$. Then the component E_0 comprises v consecutive cubes, the cube $\mathcal{C}_{\mathbf{l}}$ is the middle cube of E_0; the component E_k is obtained by a translation of E_0 by the vector $2kv\mathbf{e}_2$. In addition, if $\mathbf{v}\in\mathbf{X}$ belongs to $P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ then $\mathcal{C}_{\mathbf{v}}$ is the middle cube of E_k for some $k\in\mathbb{Z}$.*

Proof As $\mathbf{l}=(0,t,b_3,\dots,b_n)\in\mathbf{X}$ for some $t\in\mathbb{Z}$, the cube $\mathcal{C}_{\mathbf{l}}\in P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$. To prove the rest of the statement it suffices to replace in the above proof of Claim 12 the vectors $\mathbf{0}$ and $\mathbf{l}_i$ by the vectors $\mathbf{l}$ and $\mathbf{l}+\mathbf{l}_i$, respectively. □

Claim 13 describes connected components of the prism $P_2(b_3,\dots,b_n)$ in $\mathcal{K}_{\mathbf{X}}$. Now we are going to investigate connected components of the union of two neighboring prisms. More precisely, let $3\le i\le n$. Set $\mathbf{e}_i'=(e_3,\dots,e_n)$, where $e_i=1$, otherwise $e_j=0$. We will describe connected components of $(P_2(b_3,\dots,b_n)\cup P_2((b_3,\dots,b_n)+\mathbf{e}_i'))\cap\mathcal{K}_{\mathbf{X}}$. By Claim 13, for any tuple $(b_3,\dots,b_n)$, a connected component of $P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ consists of v consecutive cubes. These v cubes are followed by v consecutive cubes that belong $\mathcal{K}_{\mathbf{Y}}$. The next v consecutive cubes form a connected component of $P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$, etc. Therefore, connected components of $P_2((b_3,\dots,b_n)+\mathbf{e}_i'))\cap\mathcal{K}_{\mathbf{X}}$ can be seen as a shift of by $P(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ by the vector $-\mathbf{l}_i=-a_i\mathbf{e}_2+\mathbf{e}_i$. It follows from Theorem 6 that a_i is not an odd multiple of v. Thus, each connected component of $(P_2(b_3,\dots,b_n)\cup P_2((b_3,\dots,b_n)+\mathbf{e}_i'))\cap\mathcal{K}_{\mathbf{X}}$ is a union of a connected component of $(P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ and a connected component of $P_2((b_3,\dots,b_n)+\mathbf{e}_i')\cap\mathcal{K}_{\mathbf{X}}$. If a_i were of the form $(2k+1)v$ for some integer k, then each connected component of $(P_2(b_3,\dots,b_n)\cup P_2((b_3,\dots,b_n)+\mathbf{e}_i'))\cap\mathcal{K}_{\mathbf{X}}$ would coincide with either a connected component of $(P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ or a connected component of $P_2((b_3,\dots,b_n)+\mathbf{e}_i')\cap\mathcal{K}_{\mathbf{X}}$. Let $T=E_k$ (see Claim 13 for notation) with the middle cube $\mathcal{C}_{\mathbf{u}}$ and $R=E_{k'}$ with the middle cube $\mathcal{C}_{\mathbf{w}}$ be connected components of $(P_2(b_3,\dots,b_n)\cap\mathcal{K}_{\mathbf{X}}$ and $P_2((b_3,\dots,b_n)+\mathbf{e}_i')\cap\mathcal{K}_{\mathbf{X}}$, respectively, so that $T\cup R$ is a connected component of $(P_2(b_3,\dots,b_n)\cup P_2((b_3,\dots,b_n)+\mathbf{e}_i'))\cap\mathcal{K}_{\mathbf{X}}$. By

Claim 13 $\mathbf{u} = \mathbf{l}+2kv\mathbf{e}_2$. Further, $\mathbf{w} = \mathbf{e}_i + p_i\mathbf{e}_2 + 2kv\mathbf{e}_2$, where p_i is determined in the unique way by $-\frac{v-1}{2} \le p_i \le \frac{v-1}{2}$, $p_i \in \mathbb{Z}$, and $-a_i = t2v + p_i, t \in \mathbb{Z}$. We recall that $a_i \neq (2k+1)v$ and that the connected components of $P_2((b_3,\dots,b_n)+\mathbf{e}'_i)) \cap \mathcal{K}_{\mathbf{X}}$ can be seen as a shift of connected components of $P(b_3,\dots,b_n) \cap \mathcal{K}_{\mathbf{X}}$ by the vector $-\mathbf{l}_i = -a_i\mathbf{e}_2 + \mathbf{e}_i$. The following claim summarizes our discussion.

Claim 14 *Let T be a connected component of the prism $P_2(b_3,\dots,b_n)$, with the middle cube $\mathcal{C}_{\mathbf{u}}$. Further, let $(b'_3,\dots,b'_n) = (b_3,\dots,b_n) \pm \mathbf{e}'_j$, and let $\mathbf{l}^*_i = \mathbf{e}_i + p_i\mathbf{e}_2$ Then T forms a connected component in $(P_2(b_3,\dots,b_n) \cup P_2(b'_3,\dots,b'_n)) \cap \mathcal{K}_{\mathbf{X}}$ with exactly one connected component E of $P_2(b'_3,\dots,b'_n) \cap \mathcal{K}_{\mathbf{X}}$; this connected component E has the cube $\mathcal{C}_{\mathbf{u}+\mathbf{l}^*_i}$ as its middle cube.*

Now we define sets of cubes B_k, $k \in \mathbb{Z}$, and show that B_ks are connected components of $\mathcal{K}_{\mathbf{X}}$. Let $k \in \mathbb{Z}$. Then, for each $b_3,\dots,b_n \in \mathbb{Z}$, B_k contains exactly one connected component of $P_2(b_3,\dots,b_n) \cap \mathcal{K}_{\mathbf{X}}$. It is the component with the middle cube $\mathcal{C}_{\mathbf{u}}$, where $\mathbf{u} = b_3\mathbf{l}^*_3 + \cdots + b_n\mathbf{l}^*_n + k2v\mathbf{e}_2 = ((\sum_{i=3}^n p_ib_i)\mathbf{u}_2 + b_3\mathbf{e}_3 + \cdots + b_n\mathbf{e}_n) + k2v\mathbf{e}_2$ (see Claim 14). Finally, B_k forms an infinite prism along $\mathbf{e}_1$; that is, if $\mathcal{C}_{\mathbf{v}} \in B_k$, then $\mathcal{C}_{\mathbf{v}\pm\mathbf{e}_1} \in B_k$.

It is obvious that $\bigcup_{k\in\mathbb{Z}} B_k = \mathcal{K}_{\mathbf{X}}$ and that the set B_k can be obtained by a translation of B_0 by the vector $2kv\mathbf{e}_2$. To see that B_0 is a connected set it suffices to show that, for each $b_3,\dots,b_n \in \mathbb{Z}$, the cube $\mathcal{C}_{\mathbf{u}}$, $\mathbf{u} = b_3\mathbf{l}^*_3 + \cdots + b_n\mathbf{l}^*_n$, is in the same connected component as the cube centered at $\mathbf{O}$. The relation “to be in the same connected component” is transitive. So the statement follows by induction using Claim 14 in the inductive step. As B_k is a translation of B_0, B_k is a connected set as well.

To prove that, for each k, B_k is a connected component we show first B_ks are disjoint. Indeed, if there were a cube $\mathcal{C}_A, \mathcal{C}_A \in B_z \cap B_w$, where $z \neq w$, then, by definition of B_ks, there exists a prism $P_2(b_3,\dots,b_n)$ and its component T which belongs to $B_z \cap B_w$. However, this implies that the middle cube of T is centered at $b_3\mathbf{l}^*_3 + \cdots + b_n\mathbf{l}^*_n + 2wv\mathbf{e}_2$, but at the same time it is centered at $b_3\mathbf{l}^*_3 + \cdots + b_n\mathbf{l}^*_n + 2zv\mathbf{e}_2$ as well. But this is possible only if $z = w$, a contradiction. Therefore, $B_z \cap B_w = \emptyset$ for $z \neq w$. If $B_z \cup B_w$ were a connected set then there would be a cube $\mathcal{C}_Z \in B_z$ and a cube $\mathcal{C}_W \in B_w$ so that vertices Z and W were at the distance 1. Let $\mathcal{C}_Z$ belong to a component D of the prism $P_2(b_3,\dots,b_n)$, and $\mathcal{C}_W$ belongs to a component D' of the prism $P_2(b'_3,\dots,b'_n)$. Thus, $D \subset B_z$ and $D' \subset B_w$. However, by definition of B_ks, there is in B_z a connected component D'' from $P_2(b'_3,\dots,b'_n) \cap \mathcal{K}_{\mathbf{X}}$, $D'' \neq D'$. This in turn implies that D forms a connected component in $(P_2(b_3,\dots,b_n) \cup P_2(b'_3,\dots,b'_n)) \cap \mathcal{K}_{\mathbf{X}}$ with two different components of $P_2(b'_3,\dots,b'_n) \cap \mathcal{K}_{\mathbf{X}}$, a contradiction with Claim 14. The proof of the theorem is complete. □

For the coset $\mathbf{Y}$ of the quotient group $\mathbf{L}/\mathbf{X}$ we have $\mathbf{Y} = \mathbf{X} + \mathbf{l}_1$. Hence, as an immediate consequence of Theorem 10 we get the following.

Corollary 15 *The set of cubes $\mathcal{K}_{\mathbf{Y}}$ consists of infinitely many components D_k, $k \in \mathbb{Z}$. In particular, the cube centered at $(\pm\frac{1}{2}, (2k+1)v, 0,\dots,0)$, $k \in \mathbb{Z}$, belongs to D_k.*

Now we are ready to prove the main result of the paper.

Proof of Theorem 1 First of all we show that the total number of tilings of $\mathbb{R}^n$ by crosses is at most $2^{\aleph_0}$. To see this it suffices to note that any tiling of $\mathbb{R}^n$ by crosses $\mathcal{K}_{\mathbf{T}}$ comprises $\aleph_0$ crosses. Indeed, consider the tiling of $\mathbb{R}^n$ by cubes $\mathcal{C}_{\mathbf{N}}$, where $\mathbf{N}$ consists of all vectors with integer coordinates. Each cube in $\mathcal{C}_{\mathbf{N}}$ contains only finitely many centers of crosses in $\mathcal{K}_{\mathbf{T}}$. Since $|\mathbf{N}| = \aleph_0$, it is $|\mathbf{T}| = \aleph_0$ as well. This in turn implies that the total number of tilings of $\mathbb{R}^n$ by crosses is at most the number of ways how to choose $\aleph_0$ points in $\mathbb{R}^n$; thus it is at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Therefore, there are at most $2^{\aleph_0}$ tilings of $\mathbb{R}^n$ by crosses.

We pointed out that $\mathcal{K}_{\mathbf{Y}}$ constitutes an infinite prism along axis $\mathbf{e}_1$. Therefore the set of crosses $\mathcal{K}_{\mathbf{L}'}$ where $\mathbf{L}' = \mathbf{X} \cup (\mathbf{Y} + \lambda\mathbf{e}_1)$ constitutes a tiling of $\mathbb{R}^n$ for each $\lambda \in R$. In other words, any translation of all crosses centered at vertices in $\mathbf{Y}$ along axis $\mathbf{e}_1$ yields a tiling of $\mathbb{R}^n$. As pointed above, because $\mathbf{Y}$ consists of connected components $D_k, k \in \mathbb{Z}$, instead of translating all crosses in $\mathbf{Y}$ by the same vector we can translate connected components D_k independently of each other. We recall that $\mathbf{Y} = \mathbf{X} + \mathbf{l}_1 = \mathbf{X} + v\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_1$. Thus a cross in $\mathcal{T}$ is centered at a vector in $\mathbf{Y}$ if and only if the first coordinate of its center is of the form $t + 0.5$, where $t \in \mathbb{Z}$.

Now we are ready to prove part (i) of Theorem 1. As there are at most $2^{\aleph_0}$ tilings of $\mathbb{R}^n$ by crosses, to prove this part of the statement it suffices to show that there are $2^{\aleph_0}$ non-periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses. Let $r \in (0, 1)$ be an irrational number, $r = 0.r_1 \dots r_k \dots$ be its binary representation. The tiling $\mathcal{T}_r$ will be obtained from the tiling $\mathcal{T}$ by translating connectivity components D_k along $\mathbf{e}_1$. Let $k \in \mathbb{Z}$, and let $q = k \pmod u$. If $k > 0$ and $r_k = 1$, then the connected component D_k is translated by the vector $\frac{2q+1}{2}\mathbf{e}_1$. For $k \le 0$, and for $k > 0, r_k = 0$, D_k is translated by the vector $-\frac{1}{2}\mathbf{e}_1$ for $q = 0$, and by $\frac{1}{2}\mathbf{e}_1$ otherwise. It is obvious that $\mathcal{T}_r$ is a $\mathbb{Z}$-tiling since each connected component D_k of $\mathbf{Y}$ has been translated by a vector $t_k\mathbf{e}_1$, where $t_k = m + 0.5$, where $m \in \mathbb{Z}$. Let $\mathbf{L}_r$ be centers of crosses in $\mathcal{T}_r$. In order to prove that $\mathcal{T}_r$ is not a periodic tiling, we show that $\mathcal{T}_r$ is not periodic along $\mathbf{e}_2$. Consider vectors $\mathbf{v} \in \mathbf{L}_r$ parallel to $\mathbf{e}_2$, that is, $\mathbf{v} = t\mathbf{e}_2$, $t \in \mathbb{Z}$. There are two types of those vectors. In the first case $\mathbf{v} \in \mathbf{L}$ as well. Then, by Claim **11**(i), $\mathbf{v} = s2uv\mathbf{e}_2, s \in \mathbb{Z}$, which in turn implies $\mathbf{v} \in \mathbf{X}$, and therefore $\mathbf{v} \in \mathbf{L}_r$ as only crosses with centers in $\mathbf{Y}$ have been translated to construct $\mathcal{T}_r$ from $\mathcal{T}$. In the other case, $\mathbf{v} \in \mathbf{L}_r$ but $\mathbf{v} \notin \mathbf{L}$. Then $\mathbf{v}$ is a translation of a vector $\mathbf{v}^*$ in $\mathbf{Y}$. This can happen only if a vector $\mathbf{v}^* = (a, b, 0, \dots, 0)$ in $\mathbf{Y}$ is translated by $-a\mathbf{e}_1$. From Claim 11(i), $v|b$, and by Claim 12 and Corollary 15, b is an odd multiple of v. Thus $\mathbf{v}^*$ has to be of the form $(a, (2k+1)v, 0, \dots, 0)$. By Corollary 15, the cube centered at $(\pm\frac{1}{2}, (2k + 1)v, 0, \dots, 0)$ belongs to the connected component D_k, thus $\mathbf{v}^* \in D_k$. Finally, cf. Claim 11(i), $\mathbf{v}^* = (-q - \frac{1}{2} + tu, (2k + 1)v, 0, \dots, 0)$, where $q = k \pmod u$, and $t \in \mathbb{Z}$. By the construction of $\mathcal{T}_r$, the connected component D_k is translated by $(-q - \frac{1}{2})\mathbf{e}_2$ if and only if $k > 0$ and $r_k = 1$. To summarize our discussion, if $\mathbf{v} = t\mathbf{e}_2, t \in \mathbb{Z}$, then the cross $\mathcal{K}_{\mathbf{v}} \in \mathcal{T}_r$ if and only if either $t = 2kuv, k \in \mathbb{Z}$, or $t = (2k + 1)v$, $k > 0$, and $r_k = 1$. Clearly, this set of vectors is not periodic as r is an irrational number and therefore $r_k = 1$ at least for one k. Hence $\mathcal{T}_r$ is not periodic along $\mathbf{e}_2$, which in turn implies that $\mathcal{T}_r$ is a non-periodic tiling.

Instead of proving that for any two irrational numbers r and s the corresponding tilings $\mathcal{T}_r$ and $\mathcal{T}_s$ are non-congruent it suffices to consider the family $\mathcal{S} = \{\mathcal{T}_r, r \in$

$(0, 1)$, r is irrational$\}$. Clearly any two tilings in $\mathcal{S}$ are distinct; we recall that a tiling $\mathcal{T}$ is a set of crosses, and two tilings $\mathcal{T}$ and $\mathcal{S}$ are called distinct if $\mathcal{T}$ and $\mathcal{S}$ are not equal sets. As each $\mathbb{Z}$-tiling is congruent to at most $\aleph_0$ distinct $\mathbb{Z}$-tilings, $\mathcal{S}$ has to contain $2^{\aleph_0}$ non-congruent tilings. The proof of the first part of the theorem is complete.

To prove part (ii) of Theorem 1 we first note that there are at most $\aleph_0$ periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses. Indeed, such a periodic tiling is obtained by a periodic repetition of the base block of a finite size $d_1 \times \cdots \times d_n$, $d_i \in \mathbb{Z}$. There are $\aleph_0$ blocks of different sizes of this type. Moreover, there are finitely many crosses whose centers are in the base block. Thus, the number of ways how to tile the base block is finite. In aggregate, there are at most $\aleph_0$ periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses. To finish the proof of (ii) it suffices to show that there are $\aleph_0$ periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses.

It was mentioned above that each lattice-like tiling $\mathcal{S}$ of $\mathbb{R}^n$ by crosses $\mathcal{K}_{\mathbf{S}}$ is periodic. It is easy to see that the size of the base block of $\mathcal{S}$ is $t_1 \times \cdots \times t_n$, where t_i is the smallest number so that $t_i\mathbf{e}_i \in \mathbf{S}$, $i = 1, \dots, n$. Therefore, $\mathcal{T}$ is a periodic tiling with the base block of size $d_1 \times \cdots \times d_n$, where $d_1 = u$, since $-2u\mathbf{l}_1 + \mathbf{l}_2 = u\mathbf{e}_1$ is the shortest vector in $\mathbf{L}$ parallel to $\mathbf{e}_1$. Further, the shortest vector $\mathbf{v}$ of the type $\mathbf{v} = k\mathbf{e}_2 \in \mathbf{L}$, is $2uv\mathbf{e}_2$, thus $d_2 = 2uv$. Finally, for $i \geq 3$, since $\mathbf{l}_i = a_i\mathbf{e}_2 - \mathbf{e}_i$ we get $d_i = \frac{2uv}{\text{g.c.d.}(2uv, a_i)}$ because $-d_i\mathbf{l}_i + \frac{a_i}{\text{g.c.d.}(2uv, a_i)}\mathbf{l}_2 = d_i\mathbf{e}_i$ is the shortest vector of the form $t\mathbf{e}_i \in \mathbf{L}$. As d_i is an integer we get that $d_i\mathbf{e}_i \in \mathbf{X}$ for all i, $1 \leq i \leq n$, As an immediate consequence we get the following.

Claim 16 *Let* $\mathbf{v} \in \mathbf{L}$. *If* $\mathbf{v} \in \mathbf{X}$ *then* $\mathbf{v} \pm d_i\mathbf{e}_i \in \mathbf{X}$, *and if* $\mathbf{v} \in \mathbf{Y}$ *then* $\mathbf{v} \pm d_i\mathbf{e}_i \in \mathbf{Y}$ *for* $i = 1, \dots, n$. *In particular, any translation of* $\mathbf{Y}$ *along* $\mathbf{e}_1$ *produces a periodic tiling with the same base block as in* $\mathcal{T}$.

Now we show that there are $\aleph_0$ non-congruent periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses. Let B_{α_i} be the component of $\mathcal{K}_{\mathbf{X}}$ so that $\mathcal{C}_{d_i\mathbf{e}_i} \in B_{\alpha_i}$ (we recall that $d_i\mathbf{e}_i \in \mathbf{X}$). Clearly, $\alpha_1 = 0$, as $d_1\mathbf{e}_1 = u\mathbf{e}_1 \in B_0$. Set

$$\alpha = \text{l.c.m.}\{\alpha_i, \alpha_i \neq 0, i = 1, \dots, n\}. \tag{1}$$

To produce $\aleph_0$ periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses we will modify the tiling $\mathcal{T}$. First, we produce a tiling $\mathcal{T}'$ by translating crosses in $\mathcal{K}_{\mathbf{Y}}$ by $\frac{1}{2}\mathbf{e}_1$. Clearly, $\mathcal{T}'$ is a $\mathbb{Z}$-tiling, and $\mathcal{T}'$ is a periodic tiling as well, cf. Claim 16, with the base block of the same size as $\mathcal{T}$ is. Set $\mathbf{Y}' = \mathbf{Y} + \frac{1}{2}\mathbf{e}_1$. Then $\mathcal{T}'$ is a tiling by crosses $\mathcal{K}_{\mathbf{L}'}$, where $\mathbf{L}' = \mathbf{X} \cup \mathbf{Y}'$. Further, for a prime number p, we construct a tiling $\mathcal{T}_p$ by translating connected components $B_{k\alpha p}$, $k \in \mathbb{Z}$, in $\mathcal{K}_{\mathbf{X}}$ by $\mathbf{e}_1$. We show that $\mathcal{T}_p$ is a periodic tiling and that the set $\mathcal{S} = \{\mathcal{T}_p;\ p$ is a prime$\}$ contains $\aleph_0$ non-congruent tilings. Although the construction of $\mathcal{T}_p$ is simple the proof that $\mathcal{S}$ has the desired property is quite technical and involved. We will need the following claims.

Claim 17 *Set* $\mathbf{X}_0 = \{\mathbf{v} = (v_1, \dots, v_n); \mathbf{v} \in \mathbf{X}, v_1 = 0$, *and* $\mathcal{C}_{\mathbf{v}} \in B_0\}$. *Then* $\mathbf{X}_0$ *is a sub-lattice of* $\mathbf{X}$.

Proof Suppose that $\mathbf{v}, \mathbf{w} \in \mathbf{X}_0$. To show that $\mathbf{X}_0$ is a lattice it is sufficient to prove that $\mathbf{v} - \mathbf{w} \in \mathbf{X}_0$. By definition of B_ks, (cf. Claim 13 as well), if a cube

$\mathcal{C}_{\mathbf{u}}$, $\mathbf{u} = (0, u_2, \ldots, u_n) \in P_2(u_3, \ldots, u_n) \cap \mathcal{K}_{\mathbf{X}}$ then $\mathcal{C}_{\mathbf{u}}$ is the middle of the connected component of $P_2(u_3, \ldots, u_n) \cap \mathcal{K}_{\mathbf{X}}$ that belongs to B_0. Therefore $u_2 = \sum_{i=3}^{n} p_i u_i$, where p_is are numbers independent on $\mathbf{u}$ (p_i are defined after Claim 13). Hence, if $\mathbf{v}, \mathbf{w} \in \mathbf{X}_0$, then $\mathcal{C}_{\mathbf{v}}$ and $\mathcal{C}_{\mathbf{w}}$ are cubes in the middle of their components, and $v_2 = \sum_{i=3}^{n} p_i v_i$ and $w_2 = \sum_{i=3}^{n} p_i w_i$. Since $\mathbf{X}$ is a sub-lattice of $\mathbf{L}$, $\mathbf{v} - \mathbf{w} \in \mathbf{X}$ as well. Further, $\mathcal{C}_{\mathbf{v}-\mathbf{w}}$ is the middle cube of a connected component in $P_2(v_3 - w_3, \ldots, v_n - w_3) \cap \mathcal{K}_{\mathbf{X}}$. As $\mathbf{v} - \mathbf{w} = (0, v_2 - w_2, \ldots, v_n - w_n)$, where $v_2 - w_2 = \sum_{i=3}^{n} p_i v_i - \sum_{i=3}^{n} p_i w_i = \sum_{i=3}^{n} p_i (v_i - w_i)$ we get $\mathcal{C}_{\mathbf{v}-\mathbf{w}}$ belongs to B_0. □

As an immediate consequence we get the following.

Claim 18 *If* $\mathbf{v} = (0, v_2, \ldots, v_n)$, $\mathbf{w} = (0, w_2, \ldots, w_n) \in \mathbf{X}$, *where* $\mathcal{K}_{\mathbf{v}} \in B_s, \mathcal{K}_{\mathbf{w}} \in B_t$ *then* $\mathcal{K}_{\mathbf{v}+\mathbf{w}} \in B_{s+t}$.

Proof By definition of B_ks, there are $\mathbf{v}_0, \mathbf{w}_0 \in \mathbf{X}_0$ so that $\mathbf{v} = \mathbf{v}_0 + 2sv\mathbf{e}_2$ and $\mathbf{w} = \mathbf{w}_0 + 2tv\mathbf{e}_2$. This implies $\mathbf{v} + \mathbf{w} = (v_0 + \mathbf{w}_0) + 2(s+t)v\mathbf{e}_2$, which in turn implies $\mathcal{K}_{\mathbf{v}+\mathbf{w}} \in B_{s+t}$ since, by Claim 17, $\mathbf{v}_0 + \mathbf{w}_0 \in B_0$. □

Now we are ready to prove that $\mathcal{T}_p$ is a periodic tiling. We recall that $\mathcal{T}_p$ is obtained from $\mathcal{T}'$ by translating connected components $B_{k\alpha p}$, $k \in \mathbb{Z}$, by $\mathbf{e}_1$. Let $\mathbf{L}_p = \mathbf{X}_p \cup \mathbf{Y}'$ be the centers of crosses in $\mathcal{T}_p$. To show that $\mathcal{T}_p$ is a periodic tiling we prove that if $\mathbf{w} \in \mathbf{L}_p$ then $\mathbf{w} \pm d_i^p \mathbf{e}_i \in \mathbf{L}_p$ as well, for $i = 1, \ldots, n$, where $d_i^p = d_i$ for $\alpha_i = 0$, and $d_i^p = \frac{\alpha}{\alpha_i} p d_i$ for $\alpha_i \neq 0$. Set $\mathbf{u}_i = d_i^p \mathbf{e}_i$. Since $d_i \mathbf{e}_i \in \mathbf{X}$, and $\mathbf{X}$ is a sub-lattice of $\mathbf{L}$, we have $\mathbf{u}_i \in \mathbf{X}$ for all $i = 1, \ldots, n$. By definition of α_i, it is $\mathcal{K}_{d_i \mathbf{e}_i} \in B_{\alpha_i}$. Clearly, if $\alpha_i = 0$, then $\mathbf{u}_i = d_i \mathbf{e}_i \in B_0$. For $\alpha_i \neq 0$, by Claim 18, we get $\mathcal{K}_{\mathbf{u}_i} \in B_{\alpha p}$ as $\mathbf{u}_i = d_i^p \mathbf{e}_i = \frac{1}{\alpha_i} \alpha p d_i \mathbf{e}_i$. The tiling $\mathcal{T}'$ is a periodic tiling with the base block of size $d_1 \times \cdots \times d_n$. Further, since $d_i | d_i^p$ for all $i = 1, \ldots, n$, $\mathcal{T}'$ is periodic with the block of size $d_1^p \times \cdots \times d_n^p$ as well. In other words,

$$\text{if } \mathbf{w} \in \mathbf{L}' \quad \text{then } \mathbf{w} \pm \mathrm{u}_i \in \mathbf{L}' \text{as well.} \tag{2}$$

With respect to (2), to prove that $\mathcal{T}_p$ is periodic it is sufficient to show that, for each $\mathbf{w} \in \mathbf{L}'$, either both crosses $\mathcal{K}_{\mathbf{w}}$ and $\mathcal{K}_{\mathbf{w} \pm \mathbf{u}_i}$ have been shifted by $\mathbf{e}_1$ when constructing $\mathcal{T}_p$ from $\mathcal{T}'$ or neither of them was shifted. This is obvious in the case when $\mathbf{w} \in \mathbf{Y}'$ as in this case $\mathbf{w} \pm \mathbf{u}_i \in \mathbf{Y}'$ as well, and therefore neither of the two crosses were shifted. Suppose now that $\mathbf{w} \in \mathbf{X}$. Then, $\mathbf{w} \pm \mathbf{u}_i \in \mathbf{X}$ as well. Assume that $\mathcal{K}_{\mathbf{w}} \in B_z$ and $\mathcal{K}_{\mathbf{w} \pm \mathbf{u}_i} \in B_y$. To prove the statement in this case it suffices to prove that either αp divides both z and y, or αp divides neither of them. We will consider two cases.

(1) Let $\mathbf{w} = (0, w_2, \ldots, w_n) \in \mathbf{X}$, and let $\mathcal{K}_{\mathbf{w}} \in B_z$. By Claim 18, $\mathcal{K}_{\mathbf{w} \pm \mathbf{u}_i} \in B_y$, where $y = z \pm \alpha p$ for those i when $\alpha_i \neq 0$, and $y = z$ otherwise. Therefore, for all $i = 1, \ldots, n$, either both numbers z and y are divisible by αp or both numbers are not divisible by αp. Hence, we may conclude that in both cases if $\mathbf{v} \in \mathbf{L}_p$ then $\mathbf{v} \pm \mathbf{u}_i$ belongs to $\mathbf{L}_p$ as well.

(2) Suppose now $\mathbf{w} \in \mathbf{X}$, $\mathbf{w} = (w_1, \ldots, w_n)$, $w_1 \neq 0$. Put $\mathbf{w}^* = \mathbf{w} + 2w_1 \mathbf{l}_1 = (0, w_2 + 2vw_1, w_3, \ldots, w_n)$. Clearly, $\mathbf{w} \in \mathbf{X}$. Let $\mathbf{w}^* \in B_z$ and $\mathbf{w}^* \pm \mathbf{u}_i \in B_y$. Then, by (1), either $\alpha p | z$ and $\alpha p | y$ or $\alpha p \nmid z$ and $\alpha p \nmid y$. Set $\mathbf{r} = \mathbf{w}^* - 2vw_1 \mathbf{e}_2 =$

$(0, w_2, \dots, w_n)$. In general $\mathbf{r} \notin \mathbf{L}'$. By definition of B_ks, $\mathcal{C}_\mathbf{r} \in B_{z-w_1}$ and $\mathcal{C}_{\mathbf{r}\pm\mathbf{u}_i} \in B_{y-w_1}$.As B_ks are infinite prisms, $\mathbf{w} \in B_{z-w_1}$ and $\mathbf{w} \pm \mathbf{u}_i \in B_{y-w_1}$. Hence, also in this case we have proved that if $\mathbf{v} \in \mathbf{X}_p$ then $\mathbf{v} \pm \mathbf{u}_i \in \mathbf{X}_p$.

Thus we have proved that $\mathcal{T}_p$ is a periodic tiling with the block of size $d_1^p \times \cdots \times d_n^p$. Let $D_1 \times D_2 \times \cdots \times D_n$ be the size of the base block of $\mathcal{T}_p$. We will show that for each $p \geq 2$ it is $p|D_2$. In fact we prove $D_2 = 2v\alpha p$.

Consider the set of vectors $U \subset \mathbf{L}_p$, where $U = \{\mathbf{v};\ \mathbf{v} = \mathbf{e}_1 + t\mathbf{e}_2,\ t \in \mathbb{Z}\}$. Let $\mathbf{v} \in U$. Assume first $\mathbf{v} \notin \mathbf{L}'$. Then $\mathbf{v} - \mathbf{e}_1 \in \mathbf{L}'$, and consequently $\mathbf{v} - \mathbf{e}_1 = t\mathbf{e}_2 \in B_{k\alpha p}$ for some $k \in \mathbb{Z}$. Then, by definition of B_ks, in this case $\mathbf{v} \in U$ if and only if $t = k2\alpha vp, k \in \mathbb{Z}$. In the other case $\mathbf{v} \in \mathbf{L}'$. If $\mathbf{v} \in \mathbf{X}$, we get $\mathbf{v} = -2\mathbf{l}_1 + k\mathbf{l}_2 = \mathbf{e}_1 + (k2uv - 2v)\mathbf{e}_2$, for some $k \in \mathbb{Z}$. On the other hand, each vector of the form $\mathbf{v} = -2\mathbf{l}_1 + k\mathbf{l}_2 = \mathbf{e}_1 + (k2uv - 2v)\mathbf{e}_2 \in U$. That is, $\mathbf{v} \in \mathbf{X} \cap U$ if and only if $t = k2uv - 2v, k \in \mathbb{Z}$. Otherwise $\mathbf{v} \in \mathbf{Y}'$ and thus $\mathbf{v} = \frac{1}{2}\mathbf{e}_1 + \mathbf{u}, \mathbf{u} \in \mathbf{Y}$, which in turn implies $\mathbf{v} = \frac{1}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{l}_1 + k\mathbf{l}_2 = \mathbf{e}_1 + (k2uv - v)\mathbf{e}_2$ for some $k \in \mathbb{Z}$. As before this condition is also a sufficient condition. Hence, $\mathbf{v} \in \mathbf{Y}' \cap U$ iff and only if $t = kuv - v, k \in \mathbb{Z}$.

Thus, all vectors in U can be split into three periodic sequences: $\{\mathbf{v}'_k = \mathbf{e}_1 + k2\alpha vp\mathbf{e}_2, k \in \mathbb{Z}\}$, $\{\mathbf{v}''_k = \mathbf{e}_1 + (k2uv - 2v)\mathbf{e}_2, k \in \mathbb{Z}\}$, and $\{\mathbf{v}'''_k = \mathbf{e}_1 + (kuv - v)\mathbf{e}_2, k \in \mathbb{Z}\}$. The base period of these sequences is $2\alpha vp$, $2uv$, and uv, respectively. The three sequences are pair-wise disjoint, thus their union is periodic with the base period equal to the least common multiple M of their respective base periods. As α is a multiple of u, we have $M = 2\alpha vp$, hence $D_2 = 2v\alpha p$.

We have proved that for each prime p the tiling $\mathcal{T}_p$ is periodic and the second dimension D_2 of the base block of $\mathcal{T}_p$ is a multiple of p. Now we construct an infinite sequence of primes p_i so that the tilings $\mathcal{T}_{p_i}, i \in \mathbb{N}$, are pair-wise non-congruent. Set $p_1 = 2$. Assume that $p_i, i = 1, \dots, k-1$ have been selected. We choose the prime p_k so that p_k does not divide any dimension D_s of the base block $D_1 \times \cdots \times D_n$ of $\mathcal{T}_{p_i}$ for all $i < k$. As at least one dimension of the base block of $\mathcal{T}_{p_k}$ is a multiple of p_k, $\mathcal{T}_{p_k}$ is not congruent to $\mathcal{T}_{p_i}$ for all $i < k$. We have proved that there are $\aleph_0$ periodic $\mathbb{Z}$-tilings of $\mathbb{R}^n$ by crosses. The proof of part (ii) of Theorem 1 is complete. □

Proof of Theorem 2 First of all we note that each tiling of $\mathbb{R}^n, n = 2, 3$, by crosses is congruent to a $\mathbb{Z}$-tiling. It is not difficult to see that this is true for $n = 2$. For $n = 3$, the statement is proved in [5], Corollary 2.

Let $\mathcal{P}$ be a $\mathbb{Z}$-tiling of $\mathbb{R}^n, n = 2, 3$, by crosses with centers at $\mathbf{P}$. We assume wlog that $\mathbf{O} \in \mathbf{P}$ because any translation of a tiling is again a tiling. As mentioned in the introduction, Molnar [15] showed that if $2n + 1$ is a prime then there is only one, up to a congruency, lattice-like tiling of $\mathbb{R}^n$. So to prove the theorem it suffices to show that $\mathcal{P}$ has to be lattice-like.

Two crosses $\mathcal{K}_\mathbf{v}$ and $\mathcal{K}_\mathbf{w}$ in $\mathcal{P}$ will be called neighbors if $|\mathbf{v} - \mathbf{w}| = 3$. Further, for $\mathcal{K}_v$ in $\mathcal{P}$, we denote by $\mathbf{T}(\mathbf{v})$ the set of vectors $\{\mathbf{w} - \mathbf{v};\ \mathcal{K}_\mathbf{w}$ is a neighbor of $\mathcal{K}_\mathbf{v}\}$; the set will be called the neighborhood of $\mathcal{K}_\mathbf{v}$. The neighborhood $\mathbf{T}(\mathbf{v})$ is called symmetric if $\mathbf{w} \in \mathbf{T}(\mathbf{v})$ implies $-\mathbf{w} \in \mathbf{T}(\mathbf{v})$. In what follows we will show that all crosses in $\mathcal{P}$ have the same symmetric neighborhood. Then the theorem will follow from the claim below:

Claim 19 *A tiling $\mathcal{P}$ of $\mathbb{R}^n$ by crosses is lattice-like if and only if all crosses in $\mathcal{P}$ have the same neighborhood that is symmetric and* $\mathbf{O} \in \mathbf{P}$.

Proof The necessity of the condition is straightforward. To prove the sufficiency, we need to show that if $\mathbf{v}, \mathbf{w} \in \mathbf{P}$ then $\mathbf{v} - \mathbf{w} \in \mathbf{P}$ as well. Consider a digraph $G = (\mathbf{P}, E)$ where two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbf{P}$ are connected by an arc $(\mathbf{v}, \mathbf{w})$ in G if $|\mathbf{v} - \mathbf{w}| = 3$. Thus, $(\mathbf{v}, \mathbf{w})$ is an arc in G iff $(\mathbf{w}, \mathbf{v})$ is. Clearly, G is strongly connected. Let $\mathbf{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be the joint symmetric neighborhood of all crosses in $\dot{\mathcal{P}}$. For $(\mathbf{v}, \mathbf{w}) \in E$ there is a vector $\mathbf{a}_i \in \mathbf{A}$ so that $\mathbf{v} + \mathbf{a}_i = \mathbf{w}$. We label the arc $(\mathbf{v}, \mathbf{w})$ with $\mathbf{a}_i$. Obviously, for any vector $\mathbf{v} \in \mathbf{P}$ and for any vector $\mathbf{a}_i \in A$ there is an arc with the initial vertex $\mathbf{v}$ labeled by $\mathbf{a}_i$. A vector $\mathbf{u} \in \mathbf{P}$ if and only if $\mathbf{u}$ is a linear combination of vectors in $\mathbf{A}$. Indeed, if $\mathbf{v} \in \mathbf{P}$, then there is a path S in G from $\mathbf{O}$ to $\mathbf{v}$, and $\mathbf{v}$ is a linear combination of vectors that label the arcs on S. On the other hand, if a vector $\mathbf{u}$ is a linear combination of vectors in $\mathbf{A}$, say $\mathbf{u} = \sum \alpha_i \mathbf{a}_i$, consider a path S from $\mathbf{O}$ so that for each i, $1 \le k \le i$, the arc labeled $\mathbf{a}_i$ occurs on S exactly α_i times. Then $\mathbf{u}$ is the endvertex of S, that is $\mathbf{u} \in \mathbf{P}$. Thus, if $\mathbf{v}, \mathbf{w} \in \mathbf{P}$ then both $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of vectors in $\mathbf{A}$. As $\mathbf{A}$ is symmetric, $-\mathbf{w}$ is a linear combination of vectors in $\mathbf{A}$ as well. This in turn implies $\mathbf{v} - \mathbf{w}$ is a linear combination of vectors in $\mathbf{A}$. □

To show that all crosses in $\mathcal{P}$ have the same neighborhood, and that this joint neighborhood is symmetric, it suffices to prove that any two neighbors in $\mathcal{P}$ have the same neighborhood. Let $\mathcal{K}_{\mathbf{u}}$ be an arbitrary but fixed cross in $\dot{\mathcal{P}}$. We show that each neighbor of $\mathcal{K}_{\mathbf{u}}$ has the same neighborhood as $\mathcal{K}_{\mathbf{u}}$. Let $\mathbf{v} \in \mathbf{T}(\mathbf{u})$. Since $|\mathbf{v}| = 3$, $\mathbf{v}$ is of type $[\pm 3^1]$, $[\pm 2^1, \pm 1^1]$, or $[\pm 1^3]$. These sets of vectors in $\mathbf{T}(\mathbf{u})$ of the given type will be denoted by A, B, and, C, respectively. We set $a = |A|, b = |B|, c = |C|$. Further, if, for a cube $\mathcal{C}_{\mathbf{w}}$, we have $|\mathbf{w} - \mathbf{u}| = 2$ then this cube belongs to a cross that is a neighbor of $\mathcal{K}_{\mathbf{u}}$. The vector $\mathbf{v} = \mathbf{w} - \mathbf{u}$ is of type $[\pm 2^1]$ or $[\pm 1^2]$. To simplify the language, we will also say that, with respect to $\mathcal{K}_{\mathbf{u}}$, the cube $\mathcal{C}_{\mathbf{v}}$ is of type $[\pm 2^1]$ or $[\pm 1^2]$. Each of $2n$ cubes of type $[\pm 2^1]$ with respect to $\mathbf{u}$ belongs to a cross $\mathcal{K}_{\mathbf{w}}$, where $\mathbf{w} - \mathbf{u} \in A \cup B$. Hence,

$$a + b = 2n. \tag{3}$$

Further, there are $4\binom{n}{2}$ cubes of type $[\pm 1^2]$ with respect to $\mathbf{u}$; hence

$$b + 3c = 4\binom{n}{2} \tag{4}$$

as each cross $\mathcal{K}_{\mathbf{w}}$, where $\mathbf{w} - \mathbf{u} \in C$, contains 3 cubes of type $[\pm 1^2]$. We make two more simple but useful observations. Both follow from the fact that if $\mathbf{v}, \mathbf{w} \in \mathbf{T}(\mathbf{u})$ then $|\mathbf{v} - \mathbf{w}| \ge 3$.

Observation 1 If $b = 2n$ then for each i, $1 \le i \le n$, there is a vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{T}(\mathbf{u})$ so that $x_i = 2$, and also a vector $\mathbf{x}$ so that $x_i = -2$.

Observation 2 Any two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbf{T}(\mathbf{u})$ have the same sign in at most one coordinate; that is, $v_i w_i > 0$ for at most one i, $1 \le i \le n$.

First let $n = 2$. Then $C = \emptyset$, and the only solution of (3) and (4) is $a = c = 0$, $b = 4$; i.e., each vector in $\mathbf{T}(\mathbf{u})$ is of type $[\pm 2^1, \pm 1^1]$

Claim 20 *Let* $\mathbf{v} = (x, y) \in \mathbf{T}(\mathbf{u})$. *Set* $\mathbf{w} = (-y, x)$. *Then* $\mathbf{T}(\mathbf{u}) = \{\mathbf{v}, -\mathbf{v}, \mathbf{w}, -\mathbf{w}\}$. *In particular,* $\mathbf{T}(\mathbf{u})$ *is symmetric, and any vector in* $\mathbf{T}(\mathbf{u})$ *determines* $\mathbf{T}(\mathbf{u})$ *in the unique way.*

Proof Let $\mathbf{v} = (x, y) \in \mathbf{T}(\mathbf{u})$. We show that then $-\mathbf{v} \in \mathbf{T}(\mathbf{u})$. Assume wlog that $|x| = 2$. As there are four vectors in $\mathbf{T}(\mathbf{u})$, by Observation 2, there has to be in $\mathbf{T}(\mathbf{u})$ a vector whose both coordinates have signs different from coordinates of $\mathbf{v}$; hence either $-\mathbf{v} = (-x, -y) \in \mathbf{T}(\mathbf{u})$, or $(-y, -x) \in \mathbf{T}(\mathbf{u})$ In the former case we are done, in the latter case, by Observation 1, $(-x, y) \in \mathbf{T}(\mathbf{u})$, and one of the vectors (y, x) or $(-y, x)$ has to be in $\mathbf{T}(\mathbf{u})$. This is a contradiction as both vectors (y, x) or $(-y, x)$ are at distance ≤ 2 from a vector in $\mathbf{T}(\mathbf{u})$. To see that $\mathbf{w} = (-\mathbf{y}, \mathbf{x}) \in \mathbf{T}(\mathbf{u})$, it suffices to notice that (y, x) would have the same sign in two coordinates with either $\mathbf{v}$ or $-\mathbf{v}$. Therefore, by Observation 1, $\mathbf{w} \in \mathbf{T}(\mathbf{u})$. ☐

To see that for a neighbor $\mathcal{K}_{\mathbf{w}}$ of $\mathcal{K}_{\mathbf{u}}$ it is $\mathbf{T}(\mathbf{w}) = \mathbf{T}(\mathbf{u})$ note that then $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{v} \in \mathbf{T}(\mathbf{u})$, but at the same time $\mathbf{u} = \mathbf{w} + (-\mathbf{v})$, i.e. $-\mathbf{v} \in \mathbf{T}(\mathbf{w})$. By Claim 20, $\mathbf{v} \in \mathbf{T}(\mathbf{w})$, and consequently $\mathbf{T}(\mathbf{w}) = \mathbf{T}(\mathbf{u})$.

The case $n = 3$ is much more involved. There are three solutions of (3) and (4): $a = 6$, $b = 0$, $c = 4$, and $a = 3$, $b = 3$, $c = 3$, and $a = 0$, $b = 6$, $c = 2$. First we show that for our tiling $\mathcal{P}$ it is $a = 0$. For the sake of simplicity we prove the statement only for $\mathbf{u} = \mathbf{O}$. For $\mathbf{u} \neq \mathbf{O}$ the proof would be nearly identical. Let $a \geq 3$. Then we can assume wlog that $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$ are in $\mathbf{P}$. The cube centered at $(1, 1, 0)$ belongs in $\mathcal{P}$ to the cross $\mathcal{K}_{\mathbf{w}}$, where $\mathbf{w} = (1, 1, \varepsilon)$, $\varepsilon \in \{-1, 1\}$, say wlog, $\mathbf{w} = (1, 1, -1)$. The cubes centered at $(2, 1, 0)$ and $(1, 2, 0)$ have to belong in $\mathcal{P}$ to the cross centered at $(2, 2, 0)$. This in turn implies that the cube centered at $(1, 1, 1)$ has to belong in $\mathcal{P}$ to the cross centered at $(1, 1, 2)$. Now, it is easy to check that each cross containing the cube centered at $(1, 2, 1)$ would have the distance <3 from one of the crosses centered at $(0, 3, 0)$, $(1, 1, 2)$, $(1, 1, -1)$, $(1, 2, 1)$, and $(2, 2, 0)$, a contradiction. Thus, $a = 0$, and consequently, $b = 6$, and $c = 2$.

As $c > 0$, there is in $\mathbf{T}(\mathbf{u})$ a vector $\mathbf{v} = (v_1, v_2, v_3)$ of type $[\pm 1^3]$. Let $\mathbf{x} = (x_1, x_2, x_3)$ be of type $[\pm 2^1, \pm 1^1]$. The vector $S_{\mathbf{v}}(\mathbf{x}) = (y_1, y_2, y_3)$, where $y_{i+1} = x_i \frac{v_{i+1}}{v_i}$, (the indices taken mod 3), will be called the shift of $\mathbf{x}$ with respect to $\mathbf{v}$. Since $|v_i| = 1$, the shift of $\mathbf{x}$ is also of type $[\pm 2^1, \pm 1^1]$. Further, set $S_{\mathbf{v}}^{(1)}(\mathbf{x}) = S_{\mathbf{v}}(\mathbf{x})$, and $S_{\mathbf{v}}^{(k)}(\mathbf{x}) = S_{\mathbf{v}}(S_{\mathbf{v}}^{(k-1)}(\mathbf{x}))$ for $k > 1$. The following claim provides the key ingredient of the rest of the proof.

Claim 21 *Let* $\mathbf{v}, \mathbf{x} \in \mathbf{T}(\mathbf{u})$, *where* $\mathbf{v}$ *be of type* $[\pm 1^3]$ *and let* $\mathbf{x} = (x_1, x_2, x_3)$ *be of type* $[\pm 2^1, \pm 1^1]$. *Then* $\mathbf{T}(\mathbf{u}) = \{\mathbf{v}, -\mathbf{v}, \mathbf{x}, -\mathbf{x}, S_{\mathbf{v}}(\mathbf{x}), -S_{\mathbf{v}}(\mathbf{x}), S_{\mathbf{v}}^{(2)}(\mathbf{x}), -S_{\mathbf{v}}^{(2)}(\mathbf{x})\}$. *In particular,* $\mathbf{T}(\mathbf{u})$ *is symmetric, and any vector of type* $[\pm 1^3]$ *together with any vector of type* $[\pm 2^1, \pm 1^1]$ *determines* $\mathbf{T}(\mathbf{u})$ *in the unique way since* $S_{\mathbf{v}}^{(3)}(\mathbf{x}) = S_{\mathbf{v}}(\mathbf{x})$.

Proof Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{T}(\mathbf{u})$ be a vector of type $[\pm 1^3]$, and let $\mathbf{w} = (w_1, w_2, w_3)$ be the other vector of type $[\pm 1^3]$ in $\mathbf{T}(\mathbf{u})$. We want to show that $\mathbf{w} = -\mathbf{v}$. Assume by contradiction that, say, $v_1 = w_1$. Then, by Observation 2, $v_2 = -w_2$, and $v_3 = -w_3$. However, then the vector $\mathbf{x} = (x_1, x_2, x_3)$ of type $[\pm 2, \pm 1]$ with $x_1 = 2v_1$, see Observation 1, would have distance 2 from one of $\mathbf{v}, \mathbf{w}$. Thus $\mathbf{w} = -\mathbf{v}$. Next we show that if $\mathbf{x} \in \mathbf{T}(\mathbf{u})$, $\mathbf{x} = (x_1, x_2, x_3)$ being of type $[\pm 2^1, \pm 1^1]$, then $-\mathbf{x}$ has to be in $\mathbf{T}(\mathbf{u})$ as well. We prove the statement for $\mathbf{x} = (2v_1, -v_2, 0)$ as in the other cases the proof would be nearly identical. Note that, by Observations 1 and 2, either, $\mathbf{x} = (2v_1, -v_2, 0)$ or $\mathbf{x} = (2v_1, 0, -v_3)$ is in $\mathbf{T}(\mathbf{u})$. Assume by contradiction that $-\mathbf{x} \notin \mathbf{T}(\mathbf{v})$; then, again by Observations 1 and 2, we must have $\mathbf{x}' = (-2v_1, 0, v_3) \in \mathbf{T}(\mathbf{u})$. This implies $\mathbf{y} = (0, -v_2, 2v_3)$ and $\mathbf{z} = (0, -2v_2, v_3)$ would have to be both in $\mathbf{P}$, which is a contradiction as $|\mathbf{z} - \mathbf{y}| = 2$. So $-\mathbf{x} \in \mathbf{T}(\mathbf{u})$. To finish the proof of claim we show that if $\mathbf{x} \in \mathbf{T}(\mathbf{u}), \mathbf{x} = (x_1, x_2, x_3)$ of type $[\pm 2, \pm 1]$ then $S_{\mathbf{v}}(\mathbf{x}) \in \mathbf{T}(\mathbf{u})$ as well. Again we will prove it only for the case $\mathbf{x} = (2v_1, -v_2, 0) \in \mathbf{T}(\mathbf{u})$. Suppose that $S_{\mathbf{v}}(\mathbf{x}) = (0, 2v_2, -v_3) \notin \mathbf{T}(\mathbf{u})$. Then $\mathbf{y} = (-v_1, 2v_2, 0)$ is in $\mathbf{T}(\mathbf{u})$ and, as proved above, $-\mathbf{y} = (v_1, -2v_2, 0) \in \mathbf{T}(\mathbf{u})$ which is a contradiction as $|\mathbf{x} - (-\mathbf{y})| = 2$. □

We are ready to show that if $\mathcal{K}_{\mathbf{w}}$ is a neighbor of $\mathcal{K}_{\mathbf{u}}$ then $\mathbf{T}(\mathbf{u}) = \mathbf{T}(\mathbf{w})$. Let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Then, as shown for $n = 2$, $\mathbf{v} \in \mathbf{T}(\mathbf{u})$ and $\mathbf{v} \in \mathbf{T}(\mathbf{w})$. We consider two cases. Let first $\mathbf{v}$ be of type $[\pm 1^3]$. By Claim 21, we only need to show that there is $\mathbf{x}$ of type $[\pm 2^1, \pm 1^1]$ so that $\mathbf{x} \in \mathbf{T}(\mathbf{u})$ and $\mathbf{x} \in \mathbf{T}(\mathbf{w})$. As argued above, exactly one of $(2v_1, -v_2, 0)$ and $(2v_1, 0, -v_3)$ is in $\mathbf{T}(\mathbf{u})$, and also one of $(2v_1, -v_2, 0)$ and $(2v_1, 0, -v_3)$ is in $T(w)$. Assume wlog that $(2v_1, -v_2, 0) \in \mathbf{T}(\mathbf{u})$. If $(2v_1, -v_2, 0) \in \mathbf{T}(\mathbf{w})$ as well, we are done. Otherwise, if $(2v_1, 0, -v_3) \in \mathbf{T}(\mathbf{w})$, then $-(2v_1, 0, -v_3) \in \mathbf{T}(\mathbf{w})$ and we arrive at a contradiction as we would have $\mathbf{u} + (-v_1, 0, 2v_3) \in \mathcal{P}$, $\mathbf{w} - (2v_1, 0, -v_3) \in \mathcal{P}$, but $|(\mathbf{w} - (2v_1, 0, -v_3)) - (\mathbf{u} + (-v_1, 0, 2v_3))| = |\mathbf{v} + (-v_1, 0, -v_3)| = 1$. Therefore, in this case $\mathbf{T}(\mathbf{u}) = \mathbf{T}(\mathbf{w})$. Finally, let $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{v}$ is of type $[\pm 2^1, \pm 1^1]$. As above, $\mathbf{v}$ is in both $\mathbf{T}(\mathbf{u})$ and $\mathbf{T}(\mathbf{w})$. By Claim 21, we only need to show that also $\mathbf{z} \in \mathbf{T}(\mathbf{w})$, where $\mathbf{z}$ is a vector of type $[\pm 1^3]$ in $\mathbf{T}(\mathbf{u})$. That is, we need to show that $\mathbf{w} + \mathbf{z} \in \mathcal{P}$. As $\mathbf{z}$ is of type $[\pm 1^3]$, we have $\mathbf{T}(\mathbf{u}) = \mathbf{T}(\mathbf{u} + \mathbf{z})$, see the previous case. Hence, from $\mathbf{v} \in \mathbf{T}(\mathbf{u})$, we get $\mathbf{v} \in \mathbf{T}(\mathbf{u} + \mathbf{z})$. This in turn implies $\mathbf{u} + \mathbf{v} + \mathbf{z} = \mathbf{w} + \mathbf{z} \in \mathcal{P}$, and the proof is complete. □

Remark We proved that, for $n = 2, 3$, there is the unique, up to a congruency, tiling of $\mathbb{R}^n$ by crosses. However, a slightly finer analysis of the proof allows one to determine the number of distinct tilings; two tilings $\mathcal{T}$ and $\mathcal{P}$ are considered to be distinct if $\mathcal{T} \neq \mathcal{P}$. Consider tilings of $\mathbb{R}^n$, $n = 2, 3$, by crosses so that $\mathcal{K}_{\mathbf{O}} \in \mathcal{P}$. By Claim 20, there are two distinct tiling of this type for $n = 2$, as exactly one of the two vectors $(2, 1)$ and $(1, 2)$ belongs to the neighborhood of each cross. By Claim 21, there are 8 distinct tilings for $n = 3$, as we have 4 ways how to choose the pair of vectors $\mathbf{v}$, $-\mathbf{v}$ of type $[\pm 1^3]$, and exactly one of the two vectors $(2v_1, -v_2, 0)$ and $(2v_1, 0, -v_3)$ of type $[\pm 2^1, \pm 1^1]$ is in the common neighborhood of each cross.

References

1. Al-Bdaiwi, B.F., Horak, P., Milazzo, L.: Perfect 1-error correcting Lee codes. Des. Codes Cryptogr. **52**, 155–162 (2009)
2. Debroni, J., Eblen, Aj.D., Langston, M.A., Myrvold, W., Shor, P.W., Weerapurage, D.: A complete resolution of the Keller maximum clique problem. In: SODA'2011, pp. 129–135
3. Etzion, T.: Product constructions for perfect Lee codes. IEEE Trans. Inf. Theory (to appear)

4. Golomb, S.W., Welsh, L.R.: Perfect codes in the Lee metric and the packing of polynomios. SIAM J. Appl. Math. **18**, 302–317 (1970)
5. Gravier, S., Mollard, M., Payan, Ch.: On the existence of three-dimensional tiling in the Lee metric. Eur. J. Comb. **19**, 567–572 (1998)
6. Hajós, G.: Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter. Math. Z. **47**, 427–467 (1942)
7. Horak, P.: On perfect Lee codes. Eur. J. Comb. **30**, 480–489 (2009)
8. Kárteszi, F.: Szemléletes Geometria. Gondolat, Budapest (1966)
9. Keller, O.H.: Über die lückenlose Einfü llung des Raumes mit Würfeln. J. Reine Angew. Math. **177**, 231–248 (1930)
10. Lagarias, J.F., Shor, P.W.: Keller's cube-tiling conjecture is false in high dimensions. Bull. Am. Math. Soc. **27**, 279–283 (1992)
11. Mackey, J.: A cube tiling of dimension eight with no facesharing. Discrete Comput. Geom. **28**, 275–279 (2002)
12. Peron, O.: Modulartige lückenlose Ausfüllung des $\mathbb{R}^n$ mit kongruenten Würfeln, I. Math. Ann. **117**, 415–447 (1940)
13. Peron, O.: Modulartige lückenlose Ausfüllung des $\mathbb{R}^n$ mit kongruenten Würfeln, II. Math. Ann. **117**, 609–658 (1941)
14. Minkowski, H.: Diophantische Approximationen. Teubner, Leipzig (1907)
15. Molnar, E.: Sui mosaici dello spazio di dimensione n. Atti R. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **51**, 177–185 (1971)
16. Post, K.A.: Nonexistence theorem on perfect Lee codes over large alphabets. Inf. Control **29**, 369–380 (1975)
17. Redei, L.: Neuer Beweis des Hajósschen Satzes über die endlichen Abelschen Gruppen. Acta Math. Acad. Sci. Hung. **6**, 27–40 (1955)
18. Špacapan, S.: Non-existence of face-to-face four dimensional tiling in the Lee metric. Eur. J. Comb. **28**, 127–133 (2007)
19. Schwartz, M.: Quasi-cross lattice tilings with applications to flash memory (submitted)
20. Stein, S.: Factoring by subsets. Pac. J. Math. **22**, 523–541 (1967)
21. Stein, S., Szabó, S.: Algebra and Tiling. The Carus Mathematical Monographs, vol. 25. The Mathematical Association of America, Washington (1994)
22. Szabó, S.: On mosaics consisting of multidimensional crosses. Acta Math. Acad. Sci. Hung. **38**, 191–203 (1981)
23. Szabó, S.: Low dimensional tilings by certain crosses and semi-crosses. Symmetry, Cult. Sci. **21**, 333–343 (2010)
24. Ulrich, W.: Non-binary error correcting codes. Bell Syst. Tech. J. 1341–1388 (1957)