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A Hyperplane Inequality for Measures of Convex Bodies in $\mathbb{R}^n$, $n \le 4$

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Abstract Let $2 \le n \le 4$. We show that for an arbitrary measure μ with even continuous density in $\mathbb{R}^n$ and any origin-symmetric convex body K in $\mathbb{R}^n$,

$$\mu(K) \le \frac{n}{n-1} \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|} \max_{\xi \in S^{n-1}} \mu\big(K \cap \xi^\perp\big) \operatorname{Vol}_n(K)^{1/n},$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to ξ, and $|B_2^n|$ is the volume of the unit Euclidean ball in $\mathbb{R}^n$. This inequality is sharp, and it generalizes the hyperplane inequality in dimensions up to four to the setting of arbitrary measures in place of volume. In order to prove this inequality, we first establish stability in the affirmative case of the Busemann–Petty problem for arbitrary measures in the following sense: if $\varepsilon > 0$, K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, $n \le 4$, and

$$\mu\big(K \cap \xi^\perp\big) \le \mu\big(L \cap \xi^\perp\big) + \varepsilon, \quad \forall \xi \in S^{n-1},$$

then

$$\mu(K) \le \mu(L) + \frac{n}{n-1} \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|} \operatorname{Vol}_n(K)^{1/n} \varepsilon.$$

Keywords Convex body · Hyperplane section · Measure

1 Introduction

The hyperplane problem of Bourgain [2, 3] asks whether there exists an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$

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$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le C \max_{\xi\in S^{n-1}} \mathrm{Vol}_{n-1}\bigl(K\cap\xi^\perp\bigr), \tag{1}$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to ξ. The problem is still open, with the best-to-date estimate $C \sim n^{1/4}$ established by Klartag [16], who slightly improved the previous estimate of Bourgain [4]. We refer the reader to recent papers [7, 8] for the history and current state of the hyperplane problem.

In the case where the dimension $n \le 4$, the inequality (1) can be proved with the best possible constant (see [11, Theorem 9.4.11]):

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|} \max_{\xi\in S^{n-1}} \mathrm{Vol}_{n-1}\bigl(K\cap\xi^\perp\bigr), \tag{2}$$

with equality when $K = B_2^n$ is the Euclidean ball. Here $|B_2^n| = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of B_2^n. Throughout the paper, we denote the constant in (2) by

$$c_n = \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|}.$$

Note that $c_n < 1$ for every $n \in \mathbb{N}$; this is an easy consequence of the log-convexity of the Γ-function.

Inequality (2) follows from the affirmative answer to the Busemann–Petty problem in dimensions up to four. The Busemann–Petty problem, posed in 1956 (see [6]), asks the following question. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that for every $\xi \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}\bigl(K\cap\xi^\perp\bigr) \le \mathrm{Vol}_{n-1}\bigl(L\cap\xi^\perp\bigr).$$

Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

The answer is affirmative if $n \le 4$ and negative if $n \ge 5$. The solution was completed at the end of the 90's as the result of a sequence of papers [1, 5, 9, 10, 12, 15, 17, 18, 23–28]; see [20, p. 3] or [11, p. 343] for the history of the solution. Applying the affirmative part of the solution to the case where $L = B_2^n$, one immediately gets (2).

In this article we prove that inequality (1) holds in dimensions up to four with arbitrary measure in place of volume. Let f be an even continuous non-negative function on $\mathbb{R}^n$, and denote by μ the measure on $\mathbb{R}^n$ with density f. For every closed bounded set $B \subset \mathbb{R}^n$ define

$$\mu(B) = \int_B f(x)\,dx.$$

Our extension of (2) is as follows.

Theorem 1 *If $2 \le n \le 4$ and K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$\mu(K) \le \frac{n}{n-1} c_n \max_{\xi\in S^{n-1}} \mu\bigl(K\cap\xi^\perp\bigr)\,\mathrm{Vol}_n(K)^{1/n}. \tag{3}$$

Moreover, the constant is the best possible, since there exists a sequence of measures μ_j with even continuous densities such that

$$\lim_{j\to\infty} \frac{\mu_j(B_2^n)}{\max_{\xi\in S^{n-1}} \mu_j(B_2^n\cap\xi^\perp)\operatorname{Vol}_n(B_2^n)^{1/n}} = \frac{n}{n-1}c_n.$$

Zvavitch [29] found a remarkable generalization of the Busemann–Petty problem to arbitrary measures, namely, one can replace volume by any measure with even continuous density in $\mathbb{R}^n$. In particular, if $n\le 4$, then for any origin-symmetric convex bodies K and L in $\mathbb{R}^n$ the inequalities

$$\mu\big(K\cap\xi^\perp\big)\le\mu\big(L\cap\xi^\perp\big),\quad \forall\xi\in S^{n-1}$$

imply

$$\mu(K)\le\mu(L).$$

Zvavitch also proved that this is generally not true if $n\ge 5$, namely, for any μ with strictly positive even continuous density there exist K and L providing a counterexample.

By analogy with the volume case, one would expect that this result immediately implies (3). The argument, however, does not work in this setting, because the measure μ of sections of the Euclidean ball does not have to be a constant. Instead, to prove (3) we establish stability in the affirmative part of Zvavitch's result in the following sense,

Theorem 2 *Let f be an even non-negative continuous function on $\mathbb{R}^n$, $2\le n\le 4$, let μ be the measure with density f, let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and let $\varepsilon>0$. Suppose that for every $\xi\in S^{n-1}$,*

$$\mu\big(K\cap\xi^\perp\big)\le\mu\big(L\cap\xi^\perp\big)+\varepsilon. \tag{4}$$

Then

$$\mu(K)\le\mu(L)+\frac{n}{n-1}c_n\operatorname{Vol}_n(K)^{1/n}\varepsilon. \tag{5}$$

Interchanging K and L, we get

Corollary 1 *Under the conditions of Theorem 2, we have*

$$\begin{aligned}&\big|\mu(K)-\mu(L)\big|\\ &\quad\le\frac{nc_n}{n-1}\max_{\xi\in S^{n-1}}\big|\mu\big(K\cap\xi^\perp\big)-\mu\big(L\cap\xi^\perp\big)\big|\max\big\{\operatorname{Vol}_n(K)^{\frac1n},\operatorname{Vol}_n(L)^{\frac1n}\big\}.\end{aligned} \tag{6}$$

Proof of Theorem 1 To prove the inequality (3) simply put $L=\emptyset$ in Corollary 1. To show that the constant in (3) is sharp, let $K=B_2^n$ and, for every $j\in N$, let f_j be a non-negative continuous function on $[0,1]$ supported in $(1-\frac1j,1)$ and such that

$\int_0^1 f_j(t)\,dt = 1$. Let μ_j be the measure on $\mathbb{R}^n$ with density $f_j(|x|_2)$, where $|x|_2$ is the Euclidean norm. We have

$$\mu_j\big(B_2^n\big) = \big|S^{n-1}\big| \int_0^1 r^{n-1} f_j(r)\,dr,$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in $\mathbb{R}^n$. For every $\xi \in S^{n-1}$,

$$\mu_j\big(B_2^n \cap \xi^{\perp}\big) = \big|S^{n-2}\big| \int_0^1 r^{n-2} f_j(r)\,dr.$$

Clearly,

$$\lim_{j\to\infty} \frac{\int_0^1 r^{n-1} f_j(r)\,dr}{\int_0^1 r^{n-2} f_j(r)\,dr} = 1.$$

The result follows from the equality (use the formula $\Gamma(x+1) = x\Gamma(x)$)

$$\frac{|S^{n-1}|}{|S^{n-2}||B_2^n|^{1/n}} = \frac{n}{n-1} c_n. \qquad \square$$

It remains to prove Theorem 2. Note that stability in the original Busemann–Petty problem (for volume) was established in [21]. We discuss the relation between different stability estimates in the end of the paper.

2 Preliminaries

We use the techniques of the Fourier approach to sections of convex bodies; see [20] and [22] for details. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$.

Suppose that f is a locally integrable complex-valued function on $\mathbb{R}^n$ with *power growth at infinity*, i.e. there exists a number $\beta > 0$ so that

$$\lim_{|x|_2\to\infty} \frac{f(x)}{|x|_2^{\beta}} = 0.$$

Then f represents a distribution acting by integration: for every $\phi \in \mathcal{S}$,

$$\langle f, \phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx.$$

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ.

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if

$$\big\langle f(x), \phi(x/\alpha)\big\rangle = |\alpha|^{n+p}\langle f, \phi\rangle$$

for every test function ϕ and every $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.

We say that a distribution is *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ. Schwartz's generalization of Bochner's theorem (see, for example, [14, p. 152]) states that a distribution is positive definite if and only if it is the Fourier transform of a tempered measure on $\mathbb{R}^n$. Recall that a (non-negative, not necessarily finite) measure μ is called tempered if

$$\int_{\mathbb{R}^n} \big(1 + |x|_2\big)^{-\beta} \, d\mu(x) < \infty$$

for some $\beta > 0$.

For an origin-symmetric convex body K in $\mathbb{R}^n$ we denote by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}, \qquad x \in \mathbb{R}^n$$

the norm in $\mathbb{R}^n$ generated by K. Our definition of a convex body assumes that the origin is an interior point of K. If $0 < p < n$, then $\| \cdot \|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents an even homogeneous of degree $-p$ distribution. If $\| \cdot \|_K^{-p}$ represents a positive definite distribution for some $p \in (0, n)$, then its Fourier transform is a tempered measure which is at the same time a homogeneous distribution of degree $-n+p$. One can express such a measure in polar coordinates, as follows.

Proposition 1 ([20, Corollary 2.26]) *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and $p \in (0, n)$. The function $\| \cdot \|_K^{-p}$ represents a positive definite distribution on $\mathbb{R}^n$ if and only if there exists a finite Borel measure μ_0 on S^{n-1} so that for every even test function ϕ,*

$$\int_{\mathbb{R}^n} \|x\|_K^{-p} \phi(x) \, dx = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1} \hat{\phi}(t\xi) \, dt \right) d\mu_0(\xi).$$

The following proposition was proved in [12] (see [20, Corollary 4.9]).

Proposition 2 *If $2 \leq n \leq 4$ and K is any origin-symmetric convex body in $\mathbb{R}^n$, then the function $\| \cdot \|_K^{-1}$ represents a positive definite distribution.*

For any even continuous function f on the sphere S^{n-1} and any non-zero number $p \in \mathbb{R}$, we denote by $f \cdot r^p$ the extension of f to an even homogeneous function of degree p on $\mathbb{R}^n$ defined as follows. If $x \in \mathbb{R}^n$, then $x = r\theta$, where $r = |x|_2$ and $\theta = x/|x|_2$. We put

$$f \cdot r^p(x) = f(\theta) r^p.$$

It was proved in [20, Lemma 3.7] that the Fourier transform of $f \cdot r^{-n+1}$ is equal to another continuous function g on S^{n-1} extended to an even homogeneous of degree -1 function $g \cdot r^{-1}$ on the whole of $\mathbb{R}^n$ (in fact, g is the spherical Radon transform of f, up to a constant). This is why we can remove smoothness conditions in the Parseval formula on the sphere [20, Corollary 3.23] and formulate it as follows.

Proposition 3 *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Suppose that $\|\cdot\|_K^{-1}$ is a positive definite distribution, and let μ_0 be the finite Borel measure on S^{n-1} that corresponds to $\|\cdot\|_K^{-1}$ by Proposition 1. Then for any even continuous function f on S^{n-1},*

$$\int_{S^{n-1}} \big(f \cdot r^{-n+1}\big)^\wedge(\theta)\, d\mu_0(\theta) = \int_{S^{n-1}} \|\theta\|_K^{-1} f(\theta)\, d\theta. \tag{7}$$

Finally, we need a formula from [29], expressing the measure of a section in terms of the Fourier transform. This formula generalizes the corresponding result for volume; see [19].

Proposition 4 ([29]) *Let K be an origin-symmetric star body in $\mathbb{R}^n$, then, for every $\xi \in S^{n-1}$,*

$$\mu\big(K \cap \xi^\perp\big) = \frac{1}{\pi}\left(|x|_2^{-n+1} \int_0^{|x|_2/\|x\|_K} t^{n-2} f\left(\frac{tx}{|x|_2}\right) dt\right)^\wedge (\xi),$$

where the Fourier transform of the function of $x \in \mathbb{R}^n$ in the right-hand side is a continuous homogeneous of degree -1 function on $\mathbb{R}^n \setminus \{0\}$.

3 Stability

The following elementary fact was used by Zvavitch [29] in his generalization of the Busemann–Petty problem.

Lemma 1 *Let $a, b > 0$ and let α be a non-negative function on $(0, \max\{a, b\}]$ so that the integrals below converge. Then*

$$\int_0^a t^{n-1}\alpha(t)\, dt - a \int_0^a t^{n-2}\alpha(t)\, dt \le \int_0^b t^{n-1}\alpha(t)\, dt - a \int_0^b t^{n-2}\alpha(t)\, dt. \tag{8}$$

Proof The inequality (8) is equivalent to

$$a \int_a^b t^{n-2}\alpha(t)\, dt \le \int_a^b t^{n-1}\alpha(t)\, dt.$$

Note that the latter inequality also holds in the case $a \ge b$. □

The measure of a body can be expressed in polar coordinates as follows:

$$\mu(K) = \int_K f(u)\, du = \int_{S^{n-1}} \left(\int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)\, dt\right) dx. \tag{9}$$

In particular, if $f = 1$ we get the polar formula for volume:

$$n\,\mathrm{Vol}_n(K) = \int_{S^{n-1}} \|x\|_K^{-n}\, dx. \tag{10}$$

We are ready to prove Theorem 2.

Proof of Theorem 2 First, we rewrite the condition (4) using Proposition 4:

$$\begin{aligned}&\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} t^{n-2} f\left(\frac{tx}{|x|_2}\right)dt\right)^{\wedge}(\xi)\\ &\quad\le \left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_L}} t^{n-2} f\left(\frac{tx}{|x|_2}\right)dt\right)^{\wedge}(\xi) + \pi\varepsilon\end{aligned} \tag{11}$$

for each $\xi \in S^{n-1}$.

By Proposition 2, the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution. Let μ_0 be the measure on S^{n-1} corresponding to this positive definite distribution by Proposition 1. Integrating (11) over S^{n-1} with respect to the measure μ_0 and applying the spherical Parseval formula, Proposition 3, we get

$$\begin{aligned}&\int_{S^{n-1}}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} t^{n-2} f\left(\frac{tx}{|x|_2}\right)dt\right)^{\wedge}(\xi)\, d\mu_0(\xi)\\ &\quad\le \int_{S^{n-1}}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_L}} t^{n-2} f\left(\frac{tx}{|x|_2}\right)dt\right)^{\wedge}(\xi)\, d\mu_0(\xi) + \pi\varepsilon\int_{S^{n-1}} d\mu_0(\xi),\end{aligned}$$

and

$$\begin{aligned}&\int_{S^{n-1}} \|x\|_K^{-1}\left(\int_0^{\|x\|_K^{-1}} t^{n-2} f(tx)\, dt\right)dx\\ &\quad\le \int_{S^{n-1}} \|x\|_K^{-1}\left(\int_0^{\|x\|_L^{-1}} t^{n-2} f(tx)\, dt\right)dx + \pi\varepsilon\int_{S^{n-1}} d\mu_0(\xi).\end{aligned} \tag{12}$$

Applying Lemma 1 with $a = \|x\|_K^{-1}$, $b = \|x\|_L^{-1}$, $\alpha(t) = f(tx)$, we get

$$\begin{aligned}&\int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)\, dt - \|x\|_K^{-1}\int_0^{\|x\|_K^{-1}} t^{n-2} f(tx)\, dt\\ &\quad\le \int_0^{\|x\|_L^{-1}} t^{n-1} f(tx)\, dt - \|x\|_K^{-1}\int_0^{\|x\|_L^{-1}} t^{n-2} f(tx)\, dt, \quad \forall x \in S^{n-1},\end{aligned}$$

so

$$\int_{S^{n-1}}\left(\int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)\, dt\right)dx - \int_{S^{n-1}} \|x\|_K^{-1}\left(\int_0^{\|x\|_K^{-1}} t^{n-2} f(tx)\, dt\right)dx$$

$$\leq \int_{S^{n-1}} \left(\int_0^{\|x\|_L^{-1}} t^{n-1} f(tx)\, dt \right) dx - \int_{S^{n-1}} \|x\|_K^{-1} \left(\int_0^{\|x\|_L^{-1}} t^{n-2} f(tx)\, dt \right) dx. \tag{13}$$

Adding inequalities (12) and (13) and using the polar formula (9), we get

$$\int_{S^{n-1}} \left(\int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)\, dt \right) dx$$
$$\leq \int_{S^{n-1}} \left(\int_0^{\|x\|_L^{-1}} t^{n-1} f(tx)\, dt \right) dx + \pi \varepsilon \int_{S^{n-1}} d\mu_0(\xi),$$

and

$$\mu(K) \leq \mu(L) + \pi \varepsilon \int_{S^{n-1}} d\mu_0(\xi).$$

It remains to estimate the integral in the right-hand side of the latter inequality. For this we use the formula for the Fourier transform (in the sense of distributions; see [13, p. 194]):

$$\left(|x|_2^{-n+1}\right)^\wedge(\xi) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})} |\xi|_2^{-1}.$$

Using Parseval's formula again, Proposition 3, and Hölder's inequality, we obtain

$$\begin{aligned} \pi \varepsilon \int_{S^{n-1}} d\mu_0(\xi) &= \frac{\pi \varepsilon \Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{S^{n-1}} \left(|\cdot|_2^{-n+1}\right)^\wedge(\xi)\, d\mu_0(\xi) \\ &= \frac{\pi \varepsilon \Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{S^{n-1}} \|x\|_K^{-1}\, dx \\ &\leq \frac{\pi \varepsilon \Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \left(\int_{S^{n-1}} \|x\|_K^{-n}\, dx \right)^{1/n} \left|S^{n-1}\right|^{\frac{n-1}{n}}. \end{aligned}$$

Now use the polar formula for volume (10) and note that

$$\frac{\pi \Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \left|S^{n-1}\right|^{\frac{n-1}{n}} n^{1/n} = \frac{n}{n-1} c_n.$$

□

Stability in the original Busemann–Petty problem was studied in [21], where it was shown that if the dimension $n \leq 4$, then for any origin-symmetric convex bodies K and L in $\mathbb{R}^n$ and every $\varepsilon > 0$, the inequalities

$$\mathrm{Vol}_{n-1}\left(K \cap \xi^\perp\right) \leq \mathrm{Vol}_{n-1}\left(L \cap \xi^\perp\right) + \varepsilon, \quad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + c_n \varepsilon. \tag{14}$$

This is stronger than what Theorem 2 provides in the case of volume. In fact, if μ in Theorem 2 is volume ($f \equiv 1$), then (5) reads

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L) + \frac{n\varepsilon}{n-1} c_n \,\mathrm{Vol}_n(K)^{1/n},$$

which follows from (14) by the Mean Value Theorem applied to the function $h(t) = t^{n/(n-1)}$. However, Theorem 2 works for arbitrary measures, while the approach of [21] does not allow this degree of generality.

Theorem 2 does not hold true in dimensions greater than four, simply because the answer to the Busemann–Petty problem in these dimensions is negative. However, the statement of this theorem becomes correct in all dimensions if we assume in addition that K is an intersection body (see [20, Chap. 4] for the definition and properties of intersection bodies). It was proved in [17] that an origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution. The proof of Theorem 2 remains exactly the same in this case. Corollary 1 holds in all dimensions under an additional assumption that K and L are both intersection bodies, and the result of Theorem 1 is valid in all dimensions when K is an intersection body (the unit Euclidean ball is an intersection body, so the inequality (3) is sharp in the class of intersection bodies in every dimension). Note that Proposition 2 means that every origin-symmetric convex body in $\mathbb{R}^n$, $n \le 4$ is an intersection body. The latter is no longer true if $n \ge 5$.

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