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John and Loewner Ellipsoids

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Abstract John's ellipsoid criterion characterizes the unique ellipsoid of globally maximum volume contained in a given convex body C. In this article local and global maximum properties of the volume on the space of all ellipsoids in C are studied, where ultra maximality is a stronger version of maximality: the volume is nowhere stationary. The ellipsoids for which the volume is locally maximum, resp. locally ultra maximum are characterized. The global maximum is the only local maximum and for generic C it is an ultra maximum. The characterizations make use of notions originating from the geometric theory of positive quadratic forms. Part of these results generalize to the case where the ellipsoids are replaced by affine copies of a convex body D. In contrast to the ellipsoid case, there are convex bodies C and D, such that on the space of all affine images of D in C the volume has countably many local maxima. All results have dual counterparts. Extensions to the surface area and, more generally, to intrinsic volumes are mentioned.

Keywords John ellipsoid · Loewner ellipsoid · Eutaxy · Perfection · Maximum volume · Convex body

1 Introduction

Given a convex body C, that is, a compact convex set in Euclidean d-space $\mathbb{E}^d$ with non-empty interior, there is a unique ellipsoid of maximum volume in C, the *maximum* or *John ellipsoid* of C. Similarly, there is a unique ellipsoid of minimum volume containing C, the *minimum* or *Loewner ellipsoid* of C. These ellipsoids are important tools in convex geometry, optimization, affine differential geometry, the asymptotic

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theory of normed spaces, control theory, and statistics. For some historical remarks and references to the literature, including applications, see [2, 6, 8, 18].

The uniqueness of these ellipsoids seems to have been proved first by Danzer et al. [5] and the classical characterization of maximum ellipsoids is due to John [17] (sufficiency), and Pełczyński [19] and Ball [1] (necessity).

The aim of this article is to provide information on local and global maxima of the volume V on the space of all ellipsoids, resp. on the space of all affine images of a convex body D which are contained in C. Tools are the notions of eutaxy and perfect eutaxy, stemming from the geometric theory of positive definite quadratic forms. Ultra maximality is a stronger form of maximality. It will be shown that on the space of all ellipsoids contained in C the volume V is nowhere stationary. Local maximality, resp. local ultra maximality of V are characterized by eutaxy, resp. perfect eutaxy. The only local maximum is the global maximum (Theorem 1). On the space of all affine images of D which are contained in C the volume is nowhere stationary. Local maximality implies eutaxy and local ultra maximality is equivalent to perfect eutaxy. There are convex bodies C, D with infinitely many local maxima of the volume (Theorem 3). For generic convex bodies C the unique global maximum of V on the space of ellipsoids contained in C is an ultra maximum (Theorem 5). All these results have dual counterparts (Theorems 2, 4, 6). If the volume is replaced by the surface area or, more generally, by intrinsic volumes, similar results hold. One example is Theorem 7.

The idea of the proofs is to translate the problems from $\mathbb{E}^d$ into problems of separation of convex sets in $\mathbb{E}^{\frac{1}{2}d(d+3)}$ in the case of ellipsoids and in $\mathbb{E}^{d(d+1)}$ in the general case. This idea was used by Voronoĭ [27] to prove his famous criterion, stating that a positive definite quadratic form on $\mathbb{E}^d$ is extreme if and only if it is perfect and eutactic. More recent applications of this idea are to lattice packing and covering [12, 14], lattice zeta functions [4, 13, 20], geodesics on Riemannian manifolds of a Teichmüller space [16, 21–24], and to John type and minimum position problems [9, 11, 15].

For information on convex geometry see, e.g., [10, 25].

Let bd, cl, int, relint, pos, tr, I, O, B^d, S^{d-1}, $\mathcal{P}^d$, $N_C(u)$, ^{T}, V, $\cdot$, $\|\cdot\|$, h_C, $\angle$ stand for boundary, closure, interior, relative interior, positive (better: non negative) hull, trace, $d \times d$ unit and zero matrix, unit ball, unit sphere, the open convex cone of positive definite $d \times d$ matrices, normal cone of C at $u \in \mathrm{bd}\, C$, transposition, volume, inner product and Euclidean norm, support function of the convex body C, and angle.

2 Extrema of the Volume of Inscribed and Circumscribed Ellipsoids

Let C be a convex body. We study the local extrema of the volume on the space of ellipsoids which are contained in C, resp. contain C.

Ellipsoids Contained in C

We begin with

Basic notions Following Voronoĭ, we identify a (real) $d \times d$ matrix $A = (a_{ik})$ with the point $(a_{11}, \dots, a_{1d}, a_{22}, \dots, a_{2d}, \dots, a_{dd})^T \in \mathbb{E}^{\frac{1}{2}d(d+1)}$ if A is symmetric and with the point $(a_{11}, \dots, a_{1d}, a_{21}, \dots, a_{2d}, \dots, a_{d1}, \dots, a_{dd})^T \in \mathbb{E}^{d^2}$ otherwise. For $d \times d$ matrices $A = (a_{ik})$, $B = (b_{ik})$, let the inner product and the norm be defined by $A \cdot B = \sum a_{ik} b_{ik}$, $\|A\| = (\sum a_{ik}^2)^{\frac{1}{2}}$. For $x, y \in \mathbb{E}^d$, let $x \otimes y = xy^T$ be the tensor product of x and y. Then the following identities hold:

$$x^T Ay = Ay \cdot x = A \cdot x \otimes y, \qquad x \otimes y \cdot z \otimes z = (x \cdot z)(y \cdot z),$$
$$\operatorname{tr} A = A \cdot I, \qquad \operatorname{tr} AB = A \cdot B^T.$$

Let $E = a + AB^d$ be an ellipsoid contained in C, where $a \in \mathbb{E}^d$. We may assume that $A \in \mathcal{P}^d$. The space of all ellipsoids contained in C can be identified with the set

$$\begin{aligned}
\mathcal{E} &= \bigl\{(a+h, A+H) : a+h+(A+H)B^d \subseteq C\bigr\} \cap \mathbb{E}^d \oplus \mathcal{P}^d \\
&= \bigl\{(a, A) + (h, H) : a+h+(A+H)u \in C \text{ for } u \in B^d\bigr\} \cap \mathbb{E}^d \oplus \mathcal{P}^d \\
&= \bigl\{(a, A) + (h, H) : a \cdot v + h \cdot v + Au \cdot v + Hu \cdot v \le h_C(v) \\
&\quad \text{for } u \in B^d, v \in S^{d-1}\bigr\} \cap \mathbb{E}^d \oplus \mathcal{P}^d \\
&= \Bigl((a, A) + \bigcap_{\substack{u \in B^d \\ v \in S^{d-1}}} \bigl\{(h, H) : a \cdot v + h \cdot v + A \cdot v \otimes u + H \cdot v \otimes u \le h_C(v)\bigr\}\Bigr) \\
&\quad \cap \mathbb{E}^d \oplus \mathcal{P}^d \\
&= \Bigl((a, A) + \bigcap_{\substack{u \in B^d \\ v \in S^{d-1}}} \bigl\{(h, H) : a \cdot v + h \cdot v + A \cdot (v \otimes u)' + H \cdot (v \otimes u)' \le h_C(v)\bigr\}\Bigr) \\
&\quad \cap \mathbb{E}^d \oplus \mathcal{P}^d,
\end{aligned}$$

where "$\,'\,$" is the orthogonal projection of $\mathbb{E}^{d^2}$ onto $\mathbb{E}^{\frac{1}{2}d(d+1)}$. Clearly,

(1) $\mathcal{E}$ is (in $\mathbb{E}^d \oplus \mathcal{P}^d \subseteq \mathbb{E}^{\frac{1}{2}d(d+3)}$) a closed convex set with $\operatorname{int} \mathcal{E} \neq \emptyset$.

The volume V is *stationary*, (*locally*) *maximum*, (*locally*) *ultra maximum*, or *globally maximum* at $E = (a, A) \in \mathcal{E}$ if

$$V(a+h, A+H) \left\{ \begin{aligned} &= V(a, A)\bigl(1 + o(\|(h, H)\|)\bigr) \\ &\le V(a, A) \\ &\le V(a, A)\bigl(1 - \text{const}\,\|(h, H)\|\bigr) \end{aligned} \right\}$$

$$\text{as } (h, H) \to (o, O), \qquad (a+h, A+H) \in \mathcal{E},$$

or,

$$V(a+h, A+H) \le V(a, A) \quad \text{for } (a+h, A+H) \in \mathcal{E},$$

respectively. Here $V(a, A)$ stands for $V(a + AB^d)$ and an inequality or equality holds as $(h, H) \to (o, O)$, if it holds for all (h, H) with sufficiently small norm. The expression 'const' denotes a positive constant independent of (h, H). If, in the following, 'const' appears several times in the same context, it may be different each time.

The set $\mathcal{E}$ is the intersection of closed half spaces which depend continuously on the parameters $u \in B^d$, $v \in S^{d-1}$, and of $\mathbb{E}^d \oplus \mathcal{P}^d$. The support cone $\mathcal{S} = \operatorname{cl}\operatorname{pos}(\mathcal{E} - (a, A))$ of $\mathcal{E}$ at $E = (a, A) \in \mathcal{E}$ then is the intersection of those half spaces (translated by $-(a, A)$) which contain (a, A) on their boundary hyperplanes, i.e. for which the following holds:

$$a \cdot v + o \cdot v + A \cdot (v \otimes u)' + O \cdot (v \otimes u)' = (a + Au) \cdot v = h_C(v).$$

Since $w = a + Au \in E \subseteq C$, this equality is equivalent to the condition that

$$w = a + Au \in E \cap \operatorname{bd} C, \qquad v \in N_E(w).$$

Thus,

$$\mathcal{S} = \bigcap_{\substack{w \in E \cap \operatorname{bd} C \\ v \in N_E(w)}} \bigl\{(h, H) : h \cdot v + H \cdot (v \otimes u)' = (h, H) \cdot \bigl(v, (v \otimes u)'\bigr) \le 0\bigr\}.$$

Since the normal cone of $\mathcal{E}$ at (a, A) is the polar cone of $\mathcal{S}$, we get

$$\mathcal{N} = \operatorname{pos}\bigl\{\bigl(v, (v \otimes u)'\bigr) : w = a + Au \in E \cap \operatorname{bd} C,\ v \in N_E(w)\bigr\}.$$

Using these notions, we call the ellipsoid E *eutactic*, resp. *perfect eutactic* in C if

$$\bigl(o, A^{-1}\bigr) \in \mathcal{N}, \qquad \text{resp. } \bigl(o, A^{-1}\bigr) \in \operatorname{int} \mathcal{N}.$$

Clearly, this can be formulated in terms of finite positive linear combinations of expressions of the form $(v, v \otimes u)$. Actually, these notions are separation conditions for convex sets, as will become clear later on. We have chosen this terminology since these notions are closely related to the concepts of eutactic, resp. perfect and eutactic forms in the geometric theory of positive definite quadratic forms initiated by Voronoĭ.

Results John's ellipsoid criterion may be stated as follows.

(2) Let $E \subseteq C$. Then E is the unique ellipsoid of maximum volume in C if and only if E is eutactic in C.

The following result contains additional information.

Theorem 1 *Let C be a convex body, $\mathcal{E}$ the space of all ellipsoids contained in C, and $E \in \mathcal{E}$. Then the following statements hold*:

(i) *V is not stationary at any ellipsoid of $\mathcal{E}$.*
(ii) *V is locally maximum at E if and only if E is eutactic in C.*
(iii) *V is locally ultra maximum at E if and only if E is perfect eutactic in C.*
(iv) *The only local maximum of V on $\mathcal{E}$ is the global maximum.*

In view of statement (iv), other formulations of statements (ii) and (iii) would be more appropriate. The above formulation was chosen to see more clearly the differences between the situation for ellipsoids and the situation for affine images of a given convex body as described in Theorem 3.

Preliminaries Before beginning with the proof, some tools are collected together. Well-known properties of the *discriminant surface* $\{A \in \mathcal{P}^d : \det A = 1\}$, of algebraic number theory, yield the following proposition, see [10], p. 437:

(3) Let $E = (a, A)$ be an ellipsoid in $\mathcal{C}$. Then

$$\mathcal{D} = \big((a, A) + \big\{(h, H) : \det(A + H) \geq \det A\big\}\big) \cap \big(\mathbb{E}^d \oplus \mathcal{P}^d\big)$$

is an unbounded, closed smooth convex body with $\operatorname{int} \mathcal{D} \neq \emptyset$ and interior normal vector

$$\big(o, A^{-1}\big) \quad \text{at } (a, A) \in \operatorname{bd} \mathcal{D}.$$

From [9] and proposition (3) we take the following result:

(4) $$\det(I + H) = 1 + \operatorname{tr} H + O\big(\|H\|^2\big) = 1 + H \cdot I + O\big(\|H\|^2\big)$$
$$\text{as } H \to O, \quad H \in \mathbb{E}^{d^2}$$
$$= 1 + \operatorname{tr} H + O\big(\|H\|^2\big) \leq 1 + \operatorname{tr} H \quad \text{as } H \to O, \quad H \in \mathbb{E}^{\frac{1}{2}d(d+1)}.$$

Proof of Theorem 1 (i) Since $\operatorname{int} \mathcal{E} \neq \emptyset$ by (1), there is a point $(a + h, A + H) \in \operatorname{int} \mathcal{E}$ which is not contained in the hyperplane $\{(a + h, A + H) : A^{-1} \cdot H = 0\}$. Noting that $\mathcal{E}$ is convex by (1) and that $(a, A) \in \mathcal{E}$ it follows that

$$(a + \tau h, A + \tau H) \in \mathcal{E} \quad \text{for } 0 \leq \tau \leq 1.$$

Then

$$V(a + \tau h, A + \tau H) = V\big((A + \tau H)B^d\big) = V\big(AB^d\big) \det\big(I + \tau A^{-1} H\big)$$
$$= V(a, A)\big(1 + \tau A^{-1} \cdot H + O\big(\tau^2\big)\big) \quad \text{as } \tau \to 0+$$

by (4). By our choice of (h, H) holds $A^{-1} \cdot H \neq 0$. Thus V is not stationary at $E = (a, A)$.

(ii) The following statements are equivalent:

V has a local maximum at $(a, A) \in \mathcal{E}$.

$\Leftrightarrow$ $V(a + h, A + H) \leq V(A, H)$ for $(a + h, A + H) \in \mathcal{E} \cap \mathcal{U}$ where $\mathcal{U}$ is a suitable (convex) neighborhood of (a, A) in $\mathbb{E}^d \oplus \mathcal{P}^d$.

$\Leftrightarrow$ $\det(A + H) \leq \det A$ for $(a + h, A + H) \in \mathcal{E} \cap \mathcal{U}$.

$\Leftrightarrow$ The convex sets $\mathcal{D}$, $\mathcal{E} \cap \mathcal{U}$ have the point (a, A) in common and their interiors are disjoint.

$\Leftrightarrow$ The convex sets $\mathcal{D}, \mathcal{E}$ touch at (a, A).

$\Leftrightarrow$ The interior normal vector (o, A^{-1}) of $\mathcal{D}$ at (a, A) is contained in the normal cone $\mathcal{N}$ of $\mathcal{E}$ at (a, A).

$\Leftrightarrow$ $(o, A^{-1}) \in \mathcal{N}$ i.e., E is eutactic in C.

(iii) The following statements are equivalent:

V has a local ultra maximum at (a, A).
$\Leftrightarrow$ $V(a+h, A+H) \le V(a, A)(1-\text{const}\,\|(h, H)\|)$ as $(h, H) \to (o, O)$, $(a+h, A+H) \in \mathcal{E}$.
$\Leftrightarrow$ $\det(A+H) \le \det A\,(1-\text{const}\,\|(h, H)\|)$ as $(h, H) \to (\sigma, O)$, $(a+h, A+H) \in \mathcal{E}$.
$\Leftrightarrow$ $\det(I + A^{-1}H) = 1 + \operatorname{tr} A^{-1}H + O(\|A^{-1}H\|^2) \le 1 - \text{const}\,\|(h, H)\|$ for $(a + h, A + H) \in \mathcal{E} \cap \mathcal{U}$ where $\mathcal{U} \subseteq \mathbb{E}^d \oplus \mathcal{P}^d$ is a suitable convex neighborhood of (a, A).
$\Leftrightarrow$ $A^{-1} \cdot H \le -\text{const}\,\|(h, H)\|$ for $(a+h, A+H) \in \mathcal{E} \cap \mathcal{U} \subseteq (a, A) + \mathcal{S}$.
$\Leftrightarrow$ $A^{-1} \cdot H \le -\text{const}\,\|(h, H)\|$ for $(h, H) \in \mathcal{S}$.
$\Leftrightarrow$ $\cos\angle((o, -A^{-1}), (h, H)) = \frac{-A^{-1}H}{\|A^{-1}\|\,\|(h,H)\|} \ge \frac{\text{const}}{\|A^{-1}\|} = \text{const} > 0$ for $(h, H) \in \mathcal{S}$.
$\Leftrightarrow$ $\angle((o, -A^{-1}), (h, H)) \le \alpha(= \arccos(\text{const})) < \frac{\pi}{2}$ for $(h, H) \in \mathcal{S}$.
$\Leftrightarrow$ $\{(h, H) : \angle((o, -A^{-1}), (h, H)) \le \alpha\} \supseteq \mathcal{S}$.
$\Leftrightarrow$ $\{(h, H) : \angle((o, A^{-1}, (h, H)) \le \frac{\pi}{2} - \alpha\} \subseteq \mathcal{S}^* = \mathcal{N}$.
$\Leftrightarrow$ $(o, A^{-1}) \in \operatorname{int}\mathcal{N}$, i.e., E is perfect eutactic in C.

(iv) Statement (iv) is a consequence of the proof of statement (ii): Going backwards from the last statement, omitting every mention of the neighborhood $\mathcal{U}$ one arrives at the first statement with 'local' replaced by 'global'. □

Ellipsoids Containing C

The inclusion $C \subseteq E = a + AB^d$ is equivalent to the inclusion $-A^{-1}a + A^{-1}C \subseteq B^d$. The mapping

$$(a, A) \to (b, B) = \left(-A^{-1}a, A^{-1}\right) \quad \text{for } (a, A) \in \mathbb{E}^d \oplus \mathcal{P}^d$$

is a diffeomorphism of $\mathbb{E}^d \oplus \mathcal{P}^d$ onto itself and maps $\det A$ onto $1/\det A$. Thus the properties that on the space $\mathcal{F}$ of all ellipsoids which contain C the volume V is stationary, locally minimum, locally ultra minimum or globally minimum at $E = (a, A)$ are equivalent to the properties that on the space $\mathcal{A}$ of all affine images of C which are contained in B^d the volume is stationary, locally maximum, locally ultra maximum, or globally maximum at $b + BC$, $b = -A^{-1}a$, $B = A^{-1}$. This makes it plausible that there is a dual version of Theorem 1 and that the proof should be similar to that of Theorem 1.

Basic notions Let $E = a + AB^d \supseteq C$ be an ellipsoid, $(a, A) \in \mathbb{E}^d \oplus \mathcal{P}^d$. We identify the space of all ellipsoids containing C with the set

$$\mathcal{F} = \left\{(a+h, A+H) : a+h+(A+H)B^d \supseteq C\right\} \cap \mathbb{E}^d \oplus \mathcal{P}^d.$$

Using this set, define *stationarity*, *local minimality*, *local ultra minimality*, *global minimality* of V at E, in analogy to the earlier definitions. This is equivalent to the following: Let $b + BC \subseteq B^d$ be the affine image of C where $(b, B) = (-A^{-1}a, A^{-1}) \in \mathbb{E}^d \oplus \mathcal{P}^d$. Identify the space of all affine images of C contained in B^d with the convex set

$$\begin{aligned}\mathcal{A} &= \left\{(b+h, B+H) : b+h+(B+H)C \subseteq B^d\right\} \cap \mathbb{E}^d \oplus \mathcal{P}^d \\ &= \left((b, B) + \bigcap_{\substack{u\in C\\ v\in S^{d-1}}} \left\{(h, H) : b\cdot v + h\cdot v + B\cdot (v\otimes u)' + H\cdot (v\otimes u)' \le 1\right\}\right) \\ &\qquad \cap \mathbb{E}^d \oplus \mathcal{P}^d.\end{aligned}$$

Using $\mathcal{A}$, define *stationarity*, *local maximality*, etc., of V at $b+BC$ as earlier for $\mathcal{E}$. To define eutaxy and perfect eutaxy, we consider the support cone and the normal cone of $\mathcal{A}$ at (b, B):

$$\mathcal{S} = \bigcap_{\substack{v\in (b+BC)\cap \operatorname{bd} B^d\\ v=b+Bu}} \left\{(h, H) : h\cdot v + H\cdot (v\otimes u)' = (h, H)\cdot \left(v, (v\otimes u)'\right) \le 0\right\}$$

$$\mathcal{N} = \operatorname{pos}\left\{\left(v, (v\otimes u)'\right) : v = (b+BC)\cap \operatorname{bd} B^d,\ v = b+Bu\right\}.$$

Then C is said to be *eutactic*, resp. *perfect eutactic* in E if

$$(o, A) = \left(o, B^{-1}\right) \in \mathcal{N}, \quad \text{resp. } (o, A) = \left(o, B^{-1}\right) \in \operatorname{int}\mathcal{N}.$$

Results The dual counterpart of Theorem 1 is as follows.

Theorem 2 *Let C be a convex body, $\mathcal{F}$ the space of all ellipsoids containing C and $E \in \mathcal{F}$. Then the following statements hold*:

(i) *V is not stationary at any ellipsoid of $\mathcal{F}$.*
(ii) *V is locally minimum at E if and only if C is eutactic in E.*
(iii) *V is locally ultra minimum at E if and only if C is perfect eutactic in E.*
(iv) *The only local minimum of V on $\mathcal{F}$ is the global minimum.*

Using the above remarks, the proof, in essence, is the same as that of Theorem 1.

3 Extrema of the Volume of Inscribed and Circumscribed Affine Images

A natural extension of John's criterion is to consider instead of ellipsoids affine images of a convex body. This extension has been studied by Giannopoulos, Perissinaki and Tsolomitis [7], Bastero and Romance [3], Gordon, Litvak, Meyer and Pajor [8] and the author and Schuster [15].

Let C and D be convex bodies where $D \subseteq C$. In the following we investigate local extremum properties of the volume on the space of all affine images of D in C.

Affine images of D in C

Again, we begin with

Basic Notions Identify the space of all affine images of D which are contained in C with the convex set

$$\begin{aligned}\mathcal{A} &= (o, I) + \big\{(h, H) : h + (I + H)D \subseteq C\big\} \subseteq \mathbb{E}^{d+d^2} = \mathbb{E}^{d(d+1)}\\ &= (o, I) + \bigcap_{\substack{u\in D\\ v\in S^{d-1}}} \big\{(h, H) : h \cdot v + (I + H) \cdot v \otimes u \le h_C(v)\big\} \subseteq \mathbb{E}^{d(d+1)}.\end{aligned}$$

The support and the normal cone of $\mathcal{A}$ at (o, I) are then

$$\mathcal{S} = \bigcap_{\substack{u\in D\cap \operatorname{bd} C\\ v\in N_C(u)}} \big\{(h, H) : h \cdot v + H \cdot v \otimes u \le 0\big\},$$

$$\mathcal{N} = \operatorname{pos}\big\{(v, v \otimes u) : u \in D \cap \operatorname{bd} C,\ v \in N_C(u)\big\}.$$

The maximum properties of V are defined in the obvious way. *Eutaxy* and *perfect eutaxy* here mean that

$$(o, I) \in \mathcal{N}, \quad \text{resp. } (o, I) \in \operatorname{int}\mathcal{N}.$$

Results Part of the results for ellipsoids hold in this more general case. However, there are substantial differences.

Theorem 3 *Let C, D be convex bodies, $D \subseteq C$, and $\mathcal{A}$ the space of all affine images of D contained in C. Then the following statements hold*:

(i) *V is not stationary at any affine image of $\mathcal{A}$.*
(ii) *If V is locally maximum at D, then D is eutactic in C.*
(iii) *V is locally ultra maximum at D if and only if D is perfect eutactic in C.*
(iv) *For certain convex bodies C, D the volume has countably many local maxima on $\mathcal{A}$.*

Outline of the Proof and Remarks The proof of statements (i)–(iii) follows closely the proof of the corresponding statements of Theorem 1 and, therefore, is omitted.

The reason why we could not prove that in statement (ii) eutaxy implies local maximality is the following: Eutaxy means that the internal normal vector of the smooth body $\{(o + h, I + H) : \det(I + H) \ge 1\}$ at the point (o, I) is contained in the normal cone of the convex body $\mathcal{A}$ at (o, I). Since $\{(o, H) : \det(I + H) \ge 1\}$ is not convex, this does not warrant that the two bodies have disjoint interiors in a suitable neighborhood of (o, I) and thus does not imply local maximality of V.

(iv) We describe a construction of a pair of convex bodies C, D, such that V has countably many local maxima on the space of the affine images of D which are contained in C:

Let $C_1 = B^d$ and D_1 a convex polytope such that $D_1 \subseteq C_1$ provides a local maximum of V. Let $F_1 = D_1 \cap \operatorname{bd} C_1$. Choose for each $f \in F_1$ a closed cap $N_f \subseteq \operatorname{bd} C_1$ such that the system $\mathcal{N}_1$ of these caps is pairwise disjoint and

$$D_1 \subseteq \operatorname{conv}\bigl(F_1 \cup (\operatorname{bd} C_1 \backslash \mathcal{N}_1)\bigr).$$

Next choose a rotation R_1 such that $F_1 \cap R_1 F_1 = \emptyset$ and for each $f \in F_1$ holds $R_1 f \in \operatorname{relint} N_f$. For each $f \in F_1$ choose a closed spherical neighborhood $M_f \subseteq \operatorname{relint} N_f$ of $R_1 f$ such that $f \notin M_f$ and let $\mathcal{M}_1$ be the system of these neighborhoods. Now choose $1 - 1/2^n < \varrho_n < 1$, $n = 1$, so close to 1 that the following hold: let

$$C_2 = \operatorname{conv}\bigl(F_1 \cup (\operatorname{bd} C_1 \backslash \mathcal{N}_1) \cup \varrho_1 \mathcal{M}_1\bigr),$$

then

$$\begin{aligned} &F_1 \cup (\operatorname{bd} C_1 \backslash \mathcal{N}_1) \cup \mathcal{M}_1 \subseteq \operatorname{bd} C_2, \\ &D_1, D_2 = \varrho_1 R_1 D_1 \subseteq C_2 \text{ provide local maxima of } V \\ &\quad \text{among all affine images in } C_2. \end{aligned}$$

Repeat this step with $C_2, D_2, F_2 = \varrho_1 R_1 F_1, \mathcal{N}_2 = \varrho_1 \mathcal{M}_1$ instead of $C_1, D_1, F_1, \mathcal{N}_1$, etc.

In this way we get a sequence of convex bodies $C_1 \supseteq C_2 \supseteq \dots$ and a sequence of affine images of D, say $D_1, D_2, \dots$, such that the images $D_1, D_2, \dots$, are contained in the convex body $\bigcap C_n$ and provide local maxima of V on the space of all affine images of D in $\bigcap C_n$. □

Affine images of D containing C

If $a + AD$ is an affine image of D which contains C, then $b + BC$, $b = -A^{-1}$, $B = A^{-1}$, is an affine image of C contained in D. Thus the following result, in essence, is a reformulation of Theorem 3. We state it for completeness and without giving definitions.

Theorem 4 *Let C, D be convex bodies, $C \subseteq D$ and $\mathcal{B}$ the space of all affine images of D which contain C. Then the following statements hold*:

(i) *V is not stationary at any affine image of $\mathcal{B}$.*
(ii) *If V is locally minimum at D, then C is eutactic in D.*
(iii) *V is locally ultra minimum at D if and only if C is perfect eutactic in D.*
(iv) *For certain convex bodies C, D the volume has countably many local minima on $\mathcal{B}$.*

4 John and Loewner Ellipsoids of Generic Convex Bodies

There are convex bodies where the volume is locally ultra maximum for the John ellipsoid and convex bodies for which this does not hold and similarly for the Loewner ellipsoid. This leads to the question, what is the situation for generic convex bodies?

Generic Convex Bodies A topological space is *Baire* if any of its meager subsets has dense complement, where a set is *meager* or *of first category* if it is a countable union of nowhere dense sets. A version of Baire's category theorem says that each locally compact space is Baire. When speaking of *most*, *typical*, or *generic* elements of a Baire space, all elements are meant, with a meager set of exceptions. By Blaschke's selection theorem the space $\mathcal{C}$ of all convex bodies, endowed with its natural topology, is locally compact and, thus, Baire. See [10], Sects. 5.1, 13.1.

John ellipsoids of generic C

The result The answer to the above question is, surprisingly, in favor of ultra maximality.

Theorem 5 *For a generic convex body C the volume on the space of all ellipsoids contained in C is ultra maximum at the John ellipsoid.*

Proof A result of the author [9] is as follows:

(5) The maximum ellipsoid E of a generic convex body C meets $\operatorname{bd} C$ in precisely $\frac{1}{2}d(d+3)$ points. The family of support half spaces of E (and thus of C) at these points is *irreducible*.

The latter means that the intersection of a proper subfamily of these half spaces contains ellipsoids of larger volume than E. In view of this result it its sufficient for the proof of the Theorem to show the following:

(6) Let $C \in \mathcal{C}$ and $E = a + AB^d$ be the maximum ellipsoid of C such that $E \cap \operatorname{bd} C = \{w_1, \dots, w_n\} = W$, say, where $n = \frac{1}{2}d(d+3)$ and the family of support half spaces of E at the points of W is irreducible. Then V is ultra maximum at E.

For the proof of (7), by Theorem 1(iii) it is sufficient to show that

(7) E is perfect eutactic in C, i.e. $(o, A^{-1}) \in \operatorname{int} \mathcal{N}$, where $\mathcal{N} = \operatorname{pos}\{(v, (v \otimes u)') : w = a + Au \in W,\ v \in N_E(w)\}$.

Since E is the maximum ellipsoid of C, Theorem 1(ii) yields that

(8) E is eutactic in C, i.e. $(o, A^{-1}) \in \mathcal{N}$.

For the proof of (7) we assume the contrary. Taking into account (8), then either $\operatorname{int} \mathcal{N} = \emptyset$ or $\operatorname{int} \mathcal{N} \neq \emptyset$ and $(o, A^{-1}) \in \operatorname{bd} \mathcal{N}$ holds. In both cases there is a proper subset $Z \subsetneq W$ such that

(9) $(o, A^{-1}) \in \operatorname{relint} \operatorname{pos}\{(v, (v \otimes u)') : w = a + Au \in Z, v \in N_E(w)\}$.

(In the first case this is a consequence of Carathéodory's theorem, in the second case this follows from the fact that all proper faces of the simplicial cone $\mathcal{N} = \operatorname{pos}\{(v, (v \otimes u)') : w = a + Au \in W,\ v \in N_E(w)\}$ are of the form $\operatorname{pos}\{(v, (v \otimes u)') : w = a + Au \in Z,\ v = N_E(w)\}$ where $Z \subsetneq W$.) The set of ellipsoids $(a + h, A + H)$ which are contained in the intersection of the half spaces

$$\big\{x : x \cdot v \le h_C(v)\big\}, \qquad w = a + Au \in Z, \qquad v \in N_E(w)$$

is represented by the following set:

$$(a, A) + \bigcap_{\substack{w=a+Au\in Z\\ v\in N_E(w)}} \bigl\{(h, H) : h \cdot v + H \cdot (v \otimes u)' \le 0\bigr\}.$$

Since $Z \subsetneq W$ and W is irreducible, the intersection of these half spaces contains an ellipsoid $(a + h, A + H)$ of volume greater than the volume of the ellipsoid $E = (a, A)$. Hence

$$\det(A + H) > \det A.$$

Since A is symmetric and positive definite, it can be represented in the form $A = A^{-\frac{1}{2}} A^{-\frac{1}{2}}$, where $A^{-\frac{1}{2}}$ is symmetric and positive definite. Since $A^{-\frac{1}{2}} H A^{-\frac{1}{2}}$ is symmetric and

$$\det\bigl(I + A^{-\frac{1}{2}} H A^{-\frac{1}{2}}\bigr) > 1,$$

it follows from (4) that

(10)

$$\begin{aligned} 0 < \operatorname{tr} A^{-\frac{1}{2}} H A^{-\frac{1}{2}} &= A^{-\frac{1}{2}} \cdot A^{-\frac{1}{2}} H = \bigl(A^{-\frac{1}{2}}\bigr)^T \cdot \bigl(A^{-\frac{1}{2}} H\bigr)^T \\ &= A^{-\frac{1}{2}} \cdot H A^{-\frac{1}{2}} = I \cdot A^{-\frac{1}{2}} A^{-\frac{1}{2}} H = A^{-1} \cdot H = \operatorname{tr} A^{-1} H. \end{aligned}$$

On the other hand we have

$$h \cdot v + H \cdot (v \otimes u)' \le 0 \quad \text{for } w = a + Au \in Z,\ v \in N_E(w).$$

Thus by (9), there are $\lambda_w > 0$ for $w \in Z$, such that

$$\bigl(o, A^{-1}\bigr) = \sum_{w\in Z} \lambda_w \bigl(v, (v \otimes u)'\bigr).$$

Then,

$$\begin{aligned} \operatorname{tr} A^{-1} H = A^{-1} \cdot H &= \bigl(o, A^{-1}\bigr) \cdot (h, H) \\ &= \sum_{w\in Z} \lambda_u \bigl(h \cdot v + H \cdot (v \otimes u)'\bigr) \le 0, \end{aligned}$$

a contradiction to (10). The proof of (7) and thus of (6) and thus of Theorem 5 is complete. □

Loewner Ellipsoids of Generic C

We give the following result without proof.

Theorem 6 *For a generic convex body C the volume on the space of all ellipsoids which contain C is ultra minimum at the Loewner ellipsoid.*

5 Extensions

Let C be an o-symmetric convex body and $E = AB^d$, $A \in \mathcal{P}^d$, an o-symmetric ellipsoid contained in C. The space of all o-symmetric ellipsoids which are contained in C can be identified with the set

$$\mathcal{E} = \left(A + \bigcap_{\substack{u \in B^d \\ v \in S^{d-1}}} \left\{(h, H) : A \cdot (v \otimes u)' + H \cdot (v \otimes u)' \le h_C(v)\right\}\right) \cap \mathcal{P}^d.$$

In analogy to Sect. 2 define *stationarity, (local) maximality, (local) ultra maximality* and *global maximality* of the surface area S at the ellipsoid $E \in \mathcal{E}$.

Let $\mathcal{N}$ be the polar cone of the support cone of $\mathcal{E}$ at E as in Sect. 2. The set

$$\mathcal{S} = \left\{A + H : S(A + H) \ge S(A)\right\} \cap \mathcal{P}^d$$

of all o-symmetric ellipsoids with surface area at least $S(A)$ is an unbounded, smooth convex body in $\mathcal{P}^d$, see the remark on p. 335 in [11] or the article [26]. The ellipsoid E then is said to be S-*eutactic*, resp. S-*perfect eutactic* in C, if a normal vector of $\mathcal{S}$ at E is contained in $\mathcal{N}$, resp. in $\operatorname{int}\mathcal{N}$. Unfortunately, we are not aware of an explicit formula for a normal vector of the unbounded convex body $\mathcal{S}$ at the point E. Thus S-eutaxy and S-perfect eutaxy are difficult to check.

As an example of an extension of Theorem 1, we present the following result; similar results hold with S replaced by general intrinsic volumes, $\mathcal{E}$ by the space $\mathcal{F}$ of circumscribed ellipsoids, and where the assumption of o-symmetry is deleted.

Theorem 7 *Let C be an o-symmetric convex body, $\mathcal{E}$ the space of all o-symmetric ellipsoids which are contained in C and $E \in \mathcal{E}$. Then the following statements hold*:

(i) *S is not stationary at any ellipsoid of $\mathcal{E}$.*
(ii) *S is locally maximum at E if and only if E is S-eutactic in C.*
(iii) *S is locally ultra maximum at E if and only if E is S-perfect eutactic in C.*
(iv) *The only local maximum of S on $\mathcal{E}$ is the* (*unique*) *global minimum.*

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