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Tiling Lattices with Sublattices, I

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Abstract Call a coset C of a subgroup of $\mathbf{Z}^d$ a *Cartesian coset* if C equals the Cartesian product of d arithmetic progressions. Generalizing Mirsky–Newman, we show that a non-trivial disjoint family of Cartesian cosets with union $\mathbf{Z}^d$ always contains two cosets that differ only by translation. Where Mirsky–Newman's proof (for $d = 1$) uses complex analysis, we employ Fourier techniques. Relaxing the Cartesian requirement, for $d > 2$ we provide examples where $\mathbf{Z}^d$ occurs as the disjoint union of four cosets of distinct subgroups (with one not Cartesian). Whether one can do the same for $d = 2$ remains open.

Keywords Tiling · Lattices · Fourier analysis

In 1950, Paul Erdős conjectured that if a system of k arithmetic progressions, with distinct differences n_i, covers the natural numbers $\mathbf{N}$ (or the integers $\mathbf{Z}$), then

$$\sum_{i=1}^{k} \frac{1}{n_i} > 1.$$

Erdős credits Leon Mirsky and Donald Newman with the first proof (and also cites Harold Davenport and Richard Rado for independently finding the same proof just

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a bit later). Mirsky and Newman did not publish at the time, but Erdős credits them in a 1952 paper [2]. For Newman's own exposition, see his analytic number theory monograph [4]. For more history, see Soifer [5]. The Mirsky–Newman proof blends formal power series and complex analysis, and now serves as a standard example showing how such interactions solve combinatorial problems.

Nowadays one usually hears the Mirsky–Newman stated in the contrapositive: a non-trivial finite disjoint family of arithmetic progressions with union $\mathbf{Z}$ (so that $\sum_{i=1}^{k}\frac{1}{n_i}=1$) contains two progressions with the same common difference. Henceforth we refer to any nontrivial finite disjoint family $\mathcal{F}$ of sets as a *tiling* of its union $\bigcup\mathcal{F}$ and speak of the sets in $\mathcal{F}$ as *tiles*. Generally speaking, one may ask for groups G that possess no tiling by cosets of distinct subgroups. Mirsky–Newman says $G=\mathbf{Z}$ constitutes one such group, but examples we give below show that the groups $\mathbf{Z}^d$ for $d>2$ do not have this property (the case $d=2$ remains open).

We now frame our positive result extending Mirsky–Newman to $\mathbf{Z}^d$, and obtained by restricting the permissible tiles to cosets of certain special subgroups.[1] We call any coset $T=\mathbf{v}+L$ of a subgroup L of $\mathbf{Z}^d$ $(d\geq 1)$ a *Cartesian coset* if L has the form $a_1\mathbf{Z}\times\cdots\times a_d\mathbf{Z}$ for positive integers $a_1,\ldots,a_d$. Thus a Cartesian coset in $\mathbf{Z}^d$ equals the Cartesian product of d arithmetic progressions. (Note that L must have full rank.)

Theorem $\mathbf{Z}^d$ *admits no tiling using only Cartesian cosets no two of which are translates of one another.*

Ahead of the proof, we gather a few facts of Fourier analysis. For $\mathbf{C}$, the complex numbers, every $\mathbf{C}$-valued function on a finite abelian group equals a unique finite $\mathbf{C}$-linear combination of characters (i.e. homomorphisms to $\mathbf{C}^*$). For L a full-rank subgroup of $\mathbf{Z}^d$, we may regard L-periodic functions on $\mathbf{Z}^d$ as functions on $\mathbf{Z}^d/L$ which thus expand as $\mathbf{C}$-linear combinations of characters. These characters on $\mathbf{Z}^d/L$ pull back to $\mathbf{Z}^d$ as exponential functions $\mathbf{x}\mapsto\exp(2\pi i\mathbf{k}\cdot\mathbf{x})$ with $\mathbf{k}$ such that $\mathbf{k}\cdot L\subseteq\mathbf{Z}^d$. Adding an element of $\mathbf{Z}^d$ to $\mathbf{k}$ will not affect the exponential, so we standardize the components of $\mathbf{k}$ to lie in $[0,1)$. Given L, we write $\widehat{L}$ for the associated finite set of standardized vectors $\mathbf{k}$. E.g., if $L=2\mathbf{Z}\times 3\mathbf{Z}$ we have $\widehat{L}=\{(0,0),(0,1/3),(0,2/3),(1/2,0),(1/2,1/3),(1/2,2/3)\}$. Note that the cardinality of $\widehat{L}$ is precisely $[\mathbf{Z}^d:L]$.

Given an L-periodic function $f:\mathbf{x}\mapsto\sum_{\mathbf{k}\in\widehat{L}}c_{\mathbf{k}}\exp(2\pi i\mathbf{k}\cdot\mathbf{x})$, write $\widehat{f}(\mathbf{k})=c_{\mathbf{k}}$. One can write $\widehat{f}(\mathbf{k})$ as $\frac{1}{[\mathbf{Z}^d:L]}\sum_{\mathbf{x}}f(\mathbf{x})\exp(-2\pi i\mathbf{k}\cdot\mathbf{x})$ where $\mathbf{x}$ varies over a set of coset representatives. If f is L-periodic, then $\widehat{f}(\mathbf{k})$ vanishes for $\mathbf{k}\notin\widehat{L}$.

As usual, for each $S\subseteq\mathbf{Z}^d$ we have an indicator function χ_S, with $\chi_S(v)=1$ if $v\in S$, and $=0$ otherwise.

By symmetry, the sum of all characters on $\mathbf{Z}^d/L$ vanishes everywhere except the identity, where the value equals the cardinality of $\mathbf{Z}^d/L$; thus summing the corresponding exponentials and dividing by the index $[\mathbf{Z}^d:L]$ of L in $\mathbf{Z}^d$ yields χ_L. So $\widehat{\chi_L}=1/[\mathbf{Z}^d:L]$ on $\widehat{L}$, 0 elsewhere.

[1] Where we strengthen the hypothesis (requiring tiles to be Cartesian), Sun [6] generalizes Mirsky–Newman to $\mathbf{Z}^d$ by weakening the conclusion: every tiling of $\mathbf{Z}^d$ contains two tiles, cosets respectively of subgroups L and L' giving rise to isomorphic quotients of $\mathbf{Z}^d$. Whether or not we can require that L and L' differ merely by a rotation (in $O(d)$) we do not know.

$\chi_{\mathbf{v}+L}(\mathbf{x}) = \chi_L(\mathbf{x}-\mathbf{v})$, so $\widehat{\chi_{\mathbf{v}+L}}(\mathbf{k}) = (1/[\mathbf{Z}^d : L])\exp(-2\pi i\mathbf{k}\cdot\mathbf{v})$ on $\widehat{L}$, 0 elsewhere.

Proof Suppose that $\mathbf{Z}^d = \cup T_j$ gives a tiling by Cartesian cosets $T_j = \mathbf{v}_j + L_j$. Tiling implies $\sum_j \chi_{T_j} = \chi_{\mathbf{Z}^d}$ and thus $\sum_j \widehat{\chi_{T_j}} = \widehat{\chi_{\mathbf{Z}^d}}$. For $L_j = a_1\mathbf{Z}\times\cdots\times a_d\mathbf{Z}$, $\mathbf{k}_j := (1/a_1,\dots,1/a_d) \in \widehat{L_j}$. Take L_m with $a_1\cdots a_d$ maximal. Then $\widehat{\chi_{T_m}}(\mathbf{k}_m) \neq 0$ even though $\widehat{\chi_{\mathbf{Z}^d}}(\mathbf{k}_m) = 0$. So $\widehat{\chi_{T_j}}(\mathbf{k}_m) \neq 0$ for some $j \neq m$. Our choice of L_m implies $L_j = L_m$. □

Note We know no previous Fourier-theoretic proof of Mirsky–Newman. However, Henry Cohn has pointed out that in the case $d = 1$, the Fourier coefficients in our proof are precisely the residues at the poles of the terms in the partial fraction decomposition that occurs in the Mirsky–Newman power series proof. Indeed, the link between Fourier decomposition and partial fraction decomposition is discussed in Chap. 7 of the book of Beck and Robins [1]. This link extends to higher dimensions, and we intend to write a follow-up article that re-proves the main result of the present paper using d-variable generating functions. For a preview, see the Feldman–Propp–Robins preprint [3] listed in the References.

The four cosets

$$(0,1,0) + 2\mathbf{Z}\times 2\mathbf{Z}\times\mathbf{Z},$$

$$(0,0,1) + \mathbf{Z}\times 2\mathbf{Z}\times 2\mathbf{Z},$$

$$(1,0,0) + 2\mathbf{Z}\times\mathbf{Z}\times 2\mathbf{Z},$$

$$(2\mathbf{Z}\times 2\mathbf{Z}\times 2\mathbf{Z}) \cup ((1,1,1) + 2\mathbf{Z}\times 2\mathbf{Z}\times 2\mathbf{Z})$$

tile $\mathbf{Z}^3$, and no two of them are translates of one another; this shows that for $d = 3$, the Theorem becomes false if the Cartesian requirement is dropped. Furthermore, taking this example $\times\mathbf{Z}^{d-3}$ gives a d-dimensional counterexample for every $d > 3$.

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