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Hitting Simplices with Points in $\mathbb{R}^3$

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Abstract The so-called first selection lemma states the following: given any set P of n points in $\mathbb{R}^d$, there exists a point in $\mathbb{R}^d$ contained in at least $c_d n^{d+1} - O(n^d)$ simplices spanned by P, where the constant c_d depends on d. We present improved bounds on the first selection lemma in $\mathbb{R}^3$. In particular, we prove that $c_3 \geq 0.00227$, improving the previous best result of $c_3 \geq 0.00162$ by Wagner (On k-sets and applications. Ph.D. thesis, ETH Zurich, 2003). This makes progress, for the three-dimensional case, on the open problems of Bukh et al. (Stabbing simplices by points and flats. Discrete Comput. Geom., 2010) (where it is proven that $c_3 \leq 1/4^4 \approx 0.00390$) and Boros and Füredi (The number of triangles covering the center of an n-set. Geom. Dedic. 17(1):69–77, 1984) (where the two-dimensional case was settled).

Keywords Centerpoint · Selection lemma · Simplex

1 Introduction

The First Selection Lemma (see Chap. 9 in [7]) states the following: given any set P of n points in $\mathbb{R}^d$, there exists a point in $\mathbb{R}^d$ contained in at least $c_d n^{d+1} - O(n^d)$ sim-

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plices spanned by P. It is a fundamental result that has had several applications in discrete geometry [7]. In particular, it was used to construct weak ϵ-nets [2], which were crucially used in the solution of the famous Hadwiger–Debrunner conjecture [1].

The first selection lemma is closely linked to the following useful property of point sets: given any set P of n points in $\mathbb{R}^d$, there always exists a point q such that any closed halfspace containing q contains at least $n/(d+1)$ points of P. Such a point is called a *centerpoint*. In general, a point p has depth t w.r.t. P if any closed halfspace containing p contains at least t points of P. The centerpoint theorem guarantees a point of depth $n/(d+1)$. The depth of P is defined as the maximum depth, w.r.t. P, of any point $p \in \mathbb{R}^d$. Currently the best known bounds for c_d for $d \geq 3$ are achieved by using an *arbitrary* centerpoint.

In the past 27 years, the value of the constant c_d has been investigated in a series of papers. Unfortunately, there still remains a large gap between the current upper and lower bounds. Bárány [3] proved that $c_d \geq \frac{1}{d!(d+1)^{d+1}}$. For $d = 2$, this proves the existence of a point in $n^3/54$ of the triangles spanned by any point set of size n in the plane. This was improved to $n^3/27$ in [4] (see Bukh [5] for another proof, which is simple and elegant) and was later shown to be optimal in [6]. Bárány's bound was improved to $\frac{d^2+1}{(d+1)!(d+1)^{d+1}}$ by Wagner [8], who in fact showed that any point of depth τn is contained in at least the following number of simplices:

$$\frac{(d+1)\tau^d - 2d\tau^{d+1}}{(d+1)!} n^{d+1} - O\bigl(n^d\bigr). \tag{1}$$

The recent work of Bukh, Matoušek, and Nivasch [6] is devoted to the investigation of upper bounds on the value of c_d. In particular, they show that in $\mathbb{R}^d$, taking an arbitrary centerpoint cannot give a better bound than the one above (set $\tau = 1/(d+1)$). Their main result is an elegant construction of a point set P such that no point in $\mathbb{R}^d$ is contained in more than $(n/(d+1))^{d+1}$ simplices spanned by P. Furthermore, they conjecture that this is the right bound and leave improving the lower bound as the main open problem.

Our Result We make progress on the above problem, which was left as the main open problem in [6], by improving the bounds for the first selection lemma in $\mathbb{R}^3$ by a factor of 1.4. The previous best result, which follows from (1), proves that given any P, there exists a point contained in at least $0.00162n^4$ simplices (this can be any centerpoint). Our main result, presented in Sect. 2, is the following:

Theorem 1.1 *Let P be a set of n points in $\mathbb{R}^3$. Then there exists a point contained in at least $0.00227n^4$ simplices spanned by P.*

While our result does not settle the question, it shows the current lower bounds that are not tight and gives more strength to their conjecture. The technique we use can, in principle, be generalized to higher dimensions. However, there are some technical difficulties in proving a nontrivial lower bound for the function defined in Lemma 2.3. We solely concentrate on the three-dimensional problem since it is the smallest dimension in which the value of c_d is not known exactly.

2 Improving First Selection Lemma in $\mathbb{R}^3$

The bound of Wagner [8] improves with the depth of the point set P. Our simple idea is to show that when the depth of P is low, one can also get a better bound. For example, when $\mathrm{depth}(P) = n/(d+1)$, then the conjecture of [6] is in fact proven below. By combining the two approaches, one gets an overall improvement.

We will use the following lemma, which follows easily from a lemma proved in [4].

Lemma 2.1 *Given a set P of n points in $\mathbb{R}^d$, where* $\mathrm{depth}(P) = \tau n - 1$, *there exist a point p with depth $\tau n - 1$ and a set $\mathcal{H}$ of $d+1$ open halfspaces $\{h_1, \dots, h_{d+1}\}$ such that* (i) $|h_i \cap P| = \tau n$, (ii) *p lies on the boundary plane of each h_i, and* (iii) $h_1 \cup \cdots \cup h_{d+1}$ *cover the entire $\mathbb{R}^d$ except the point p.*

Proof Boros and Füredi [4] (Lemma 3) prove that given a point set P of size n and depth σ in $\mathbb{R}^d$, there exist a point p of depth σ and $d+1$ closed halfspaces $\eta_1, \dots, \eta_{d+1}$ which cover $\mathbb{R}^d$, have the point p on their boundary, and where $|\eta_i \cap P| = n - \sigma - 1$ (they actually prove this statement for $\mathbb{R}^2$, but as they also note in their paper, the generalization to $\mathbb{R}^d$ is straightforward). Applying this lemma with $\sigma = \tau n - 1$ and setting h_i to be the complement of η_i for $i = 1, \dots, d+1$ proves the required statement. □

We now prove a technical lemma which can be seen as a generalization of Carathéodory's theorem. Given a set P of points in $\mathbb{R}^d$, $\mathrm{conv}(P)$ denotes the convex hull of P.

Lemma 2.2 *Let $P = \{p_1, \dots, p_{d+2}\}$ be a set of $d+2$ points in $\mathbb{R}^d$. Then any point $x \in$* $\mathrm{conv}(P)$ *lies in at least two d-simplices spanned by P.*

Proof If x lies on any facet F of $\mathrm{conv}(P)$, then the (at least) two simplices spanned by P that contain F also contain x. Otherwise take any point of P, say p_1, and consider the ray emanating from p_1 and passing through x. This ray, after passing through x, intersects the boundary of $\mathrm{conv}(P)$ in a $(d-1)$-simplex spanned by d points, say P'. Then $P' \cup p_1$ contains x and has size $d+1$. Let p_i be the remaining point in $P \setminus (P' \cup \{p_1\})$. Repeating the same procedure of shooting a ray from p_i through x results in another d-simplex, with p_i as one of its points, that contains x. □

Given a set P of n points in $\mathbb{R}^3$, with $\mathrm{depth}(P) = \tau n - 1$, use Lemma 2.1 to get the point p and a set of four halfspaces $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$ satisfying the stated conditions. The rest of the paper will be devoted to proving that p is contained in a lot of simplices spanned by P (w.l.o.g. one can assume that $p \notin P$, otherwise the bound can only improve, as all the $\Theta(n^d)$ simplices defined by p contain p). For any halfspace h, let $\overline{h}$ be the complement halfspace of h, and ∂h be its boundary plane.

Define the following subsets of P for all $i, j = 1, \dots, 4$:

$$\begin{aligned} A_i &= P \cap \Big(\bigcap_{l \neq i} \overline{h}_l\Big) \cap h_i, \qquad B_{i,j} = P \cap \Big(\bigcap_{l \neq i,j} \overline{h}_l\Big) \cap h_i \cap h_j, \\ C_i &= P \cap \Big(\bigcap_{l \neq i} h_l\Big) \cap \overline{h}_i. \end{aligned} \tag{2}$$

Set $\alpha_i = |A_i|/n$, $\beta_{i,j} = |B_{i,j}|/n$, and $\gamma_i = |C_i|/n$. Our main lemma is the following.

Lemma 2.3 *Let P be a set of n points in $\mathbb{R}^3$, with* $\operatorname{depth}(P) = \tau n - 1$. *Then, there exists a point contained in at least $g(P) \cdot n^4$ simplices spanned by P, where*

$$g(P) = \Big(\prod_i \alpha_i\Big) + \Big(\sum_{i<j} \beta_{i,j} \frac{\prod_l \alpha_l}{\max(\alpha_i, \alpha_j)}\Big) + \Big(\sum_i \gamma_i \frac{\prod_l \alpha_l}{\max_{l \neq i} \alpha_l}\Big). \tag{3}$$

Proof Let p be the point from Lemma 2.1, together with the four halfspaces $h_1, \dots, h_4$. We first show that the simplex spanned by any four points, one from each of A_i, will always contain p.

Claim 2.4 *Let $p_1, p_2, p_3, p_4 \in P$ be four points of P such that $p_i \in A_i$. Then the simplex spanned by these four points contains p.*

Proof Assume for contradiction that $\operatorname{conv}(\{p_1, p_2, p_3, p_4\})$ does not contain p. Then there exists a hyperplane h that separates p from $\operatorname{conv}(\{p_1, p_2, p_3, p_4\})$ and does not contain p. For $i \in \{1, 2, 3, 4\}$, define q_i to be the point $pp_i \cap h$. By definition, each h_i passes through p, contains p_i and does not contain any other point p_j with $j \neq i$. Note that then each h_i also contains q_i. Moreover, h_i does not contain q_j with $i \neq j$ because h_i does not contain p_j and has p only on its boundary. By Radon's theorem [7, p. 9] applied to $\{q_1, q_2, q_3, q_4\}$ lying on the plane h, there exist disjoint sets Q_1, Q_2 and a point s such that $Q_1 \cup Q_2 = \{q_1, q_2, q_3, q_4\}$ and $s \in \operatorname{conv}(Q_1) \cap \operatorname{conv}(Q_2)$. Since s lies on h, $s \neq p$. By convexity, any halfspace that contains s must also contain at least one point from both Q_1 and Q_2. As $\mathcal{H}$ covers $\mathbb{R}^3 \setminus \{p\}$, there exists an i such that $s \in h_i$. But this gives a contradiction, as then this h_i must contain at least one point from both Q_1 and Q_2, and so contain some point q_j with $j \neq i$. □

The total number of such simplices is $n^4 \cdot \prod_i \alpha_i$, which is the first term in (3). Call any such simplex a *basic simplex*, i.e., a simplex on $p_1, p_2, p_3, p_4 \in P$ is basic iff $p_i \in A_i$ for all i. All other simplices are called nonbasic.

Now we use basic simplices, which always contain p, to prove the existence of several other simplices which must also contain p.

Claim 2.5 *Let $P' = \{p_1, p_2, p_3, p_4, p_5\} \subset P$ be five points of P such that $p_k \in A_k$, $k = 1, \dots, 4$, and $p_5 \in B_{i,j}$ for any $1 \leq i < j \leq 4$. Then the simplex spanned either by $P' \setminus p_i$ or by $P' \setminus p_j$ contains p.*

Proof By Claim 2.4, the basic simplex spanned by p_1, p_2, p_3, p_4 contains p. Therefore, by Lemma 2.2, at least one other simplex spanned by P' must contain p. Note that this simplex must have p_5 as one of its points. Also, it must contain p_k, where $k \neq i, j$, since the plane ∂h_k separates $P' \setminus p_k$ from p, as from the definitions (2) it follows that $p_l \in \overline{h}_k$ for all $l \neq k$ and $p_5 \in \overline{h}_k$ since $p_5 \in B_{i,j}$. So for this second simplex, the only possible choice is for the fourth vertex, which can be either p_i or p_j. □

For any fixed i, j, there are $n^5(\beta_{i,j} \prod_i \alpha_i)$ 5-tuples as in Claim 2.5, and each produces one d-simplex containing p. Each such d-simplex may be double-counted at most $n \max(\alpha_i, \alpha_j)$ times, so the total number of distinct d-simplices of the type in Claim 2.5 containing p is at least $n^4 \frac{\beta_{i,j} \prod_i \alpha_i}{\max(\alpha_i, \alpha_j)}$, which when summed over all $i < j$, forms the second term in (3).

Claim 2.6 *Let $P' = \{p_1, p_2, p_3, p_4, p_5\} \subset P$ be five points of P such that $p_k \in A_k$, $k = 1, \dots, 4$, and $p_5 \in C_i$ for any $1 \le i \le 4$. Then there is a two-element subset $P'' \subset P' \setminus \{p_5, p_i\}$ such that the simplex* $\operatorname{conv}(\{p_5, p_i\} \cup P'')$ *contains p.*

Proof As in Claim 2.5, at least one nonbasic simplex spanned by P' must contain p, with p_5 as one of its points. Also, it must contain p_i: the plane ∂h_i separates $P' \setminus p_i$ from p, as $P' \setminus p_i \subseteq \overline{h}_i$. The other two vertices of this second simplex must therefore be a subset of the remaining three vertices in P'. □

By similarly eliminating the double-counting, the d-simplices from Claim 2.6 form the third term of $g(P)$. Finally, note that no two simplices are counted twice in $g(P)$, since each contains exactly one point from a distinct region (one of $B_{i,j}$ or C_i). □

Note that we only have these two constraints on the nonnegative variables α_i, $\beta_{i,j}$, and γ_i:

$$\tau = \frac{|h_i \cap P|}{n} = \alpha_i + \sum_{j \neq i} \beta_{i,j} + \sum_{j \neq i} \gamma_j \quad \text{for each } i = 1, \dots, 4; \tag{4}$$

$$\sum_i \alpha_i + \sum_{i<j} \beta_{i,j} + \sum_i \gamma_i = 1, \quad \text{as } \mathcal{H} \text{ covers } \mathbb{R}^3 \setminus \{p\} \text{ and } p \notin P. \tag{5}$$

It remains to show that regardless of the distribution of the points in the disjoint sets A_i, $B_{i,j}$, and C_j, and therefore the values of the variables satisfying (4) and (5), the quantity $g(P)$ is always bounded suitably from below.

Lemma 2.7 *Let P be a set of n points in $\mathbb{R}^3$, with* $\operatorname{depth}(P) = \tau n - 1$, *and $g(P)$ as in Lemma 2.3. If $\tau \le 0.3$, then $g(P) \ge \tau(1 - 3\tau)^2(5\tau - 1)$.*

Proof Using the fact that $\alpha_i \le \tau$, we get

$$
\begin{aligned}
g(P) &= \left(\prod_i \alpha_i\right) + \left(\sum_{i<j} \beta_{i,j} \frac{\prod_l \alpha_l}{\max(\alpha_i, \alpha_j)}\right) + \left(\sum_i \gamma_i \frac{\prod_l \alpha_l}{\max_{l \ne i} \alpha_l}\right) \\
&\ge \left(\prod_i \alpha_i\right) + \left(\sum_{i<j} \beta_{i,j} \frac{\prod_l \alpha_l}{\tau}\right) + \left(\sum_i \gamma_i \frac{\prod_l \alpha_l}{\tau}\right) \\
&= \left(\prod_i \alpha_i\right)\left(1 + \frac{\sum_{i<j} \beta_{i,j} + \sum_i \gamma_i}{\tau}\right). \qquad (6)
\end{aligned}
$$

Summing up (4) for all four halfspaces and then subtracting (5), we get

$$\sum_{i<j} \beta_{i,j} + 2\sum_i \gamma_i = 4\tau - 1. \qquad (7)$$

Therefore, $\sum_{i<j} \beta_{i,j} + \sum_i \gamma_i \le 4\tau - 1$. This fact, together with (4), implies that $1 - 3\tau \le \alpha_i \le \tau$ for $i = 1, \dots, 4$. Assuming that $\tau \le 0.3$, we can show the following:

Claim 2.8 *The bound in* (6) *subject to constraints* (4) *and* (5) *is minimized when* $\sum_i \gamma_i = 0$ *or equivalently, when* $\sum_{i<j} \beta_{i,j} + \sum_i \gamma_i = 4\tau - 1$.

Proof Suppose that $\sum_i \gamma_i = \epsilon + \epsilon_1$, where $\gamma_1 = \epsilon_1 > 0$. We show that the variables $\alpha_i, \beta_{i,j}, \gamma_1$ can be readjusted to new values $\alpha'_i, \beta'_{i,j}, \gamma'_1$ to give a smaller value in (6), while still satisfying all the constraints in (4) and (5), and where $\gamma'_1 = 0$. As long as $\sum_i \gamma_i > 0$, we can iteratively apply this procedure for all $\gamma_j > 0$ to make $\sum_i \gamma_i = 0$ without increasing the lower bound. At each such step, a value of $\gamma_j > 0$, for some j, is set to 0, so this procedure finishes after at most 4 steps.

Set $\gamma'_1 = 0$, $\beta'_{i,j} = \beta_{i,j} + \frac{2\epsilon_1}{3}$ for all $i, j \ne 1$, and $\alpha'_i = \alpha_i - \epsilon_1/3$ for all $i \ne 1$. One can verify that (4) and (5) still hold. Therefore, (7) also holds; in particular, it follows that $1 - 3\tau \le \alpha'_i \le \tau$ for each i, and so all new α'_i variables are still nonnegative as $\tau \le 0.3$. Now simple calculation shows that the function of (6) can only decrease. For completeness sake, we present the explicit computations:

$$\left(\prod_i \alpha'_i\right)\left(1 + \frac{\sum \gamma'_i + \sum \beta'_{i,j}}{\tau}\right) \le \left(\prod_i \alpha_i\right)\left(1 + \frac{\sum \gamma_i + \sum \beta_{i,j}}{\tau}\right).$$

Note that $\sum_{i<j} \beta_{i,j} + \sum_i \gamma_i = 4\tau - 1 - \epsilon - \epsilon_1$ and $\sum_{i<j} \beta'_{i,j} + \sum_i \gamma'_i = 4\tau - 1 - \epsilon$. So

$$\left(\prod_{i \ne 1} \alpha_i - \frac{\epsilon_1}{3}\right)\left(\frac{5\tau - 1 - \epsilon}{\tau}\right) \le \left(\prod_{i \ne 1} \alpha_i\right)\left(\frac{5\tau - 1 - \epsilon - \epsilon_1}{\tau}\right),$$

$$\prod_{i \ne 1}\left(1 - \frac{\epsilon_1}{3\alpha_i}\right) \le 1 - \frac{\epsilon_1}{5\tau - 1 - \epsilon}.$$

Since each α_i, $i \neq 1$, can be at most $\tau - \epsilon_1$ (from (4)), we have to prove

$$\left(1 - \frac{\epsilon_1}{3(\tau - \epsilon_1)}\right)^3 = 1 - \frac{3\epsilon_1}{3(\tau - \epsilon_1)} + 3\left(\frac{\epsilon_1}{3(\tau - \epsilon_1)}\right)^2 - \left(\frac{\epsilon_1}{3(\tau - \epsilon_1)}\right)^3$$
$$\leq 1 - \frac{\epsilon_1}{5\tau - 1 - \epsilon}.$$

By dropping the negative cubic term and simplifying, we have to show that

$$1 - \frac{3\epsilon_1}{3(\tau - \epsilon_1)} + 3\left(\frac{\epsilon_1}{3(\tau - \epsilon_1)}\right)^2 \leq 1 - \frac{\epsilon_1}{5\tau - 1 - \epsilon},$$
$$-\frac{1}{(\tau - \epsilon_1)} + \frac{\epsilon_1}{3(\tau - \epsilon_1)^2} \leq -\frac{1}{5\tau - 1 - \epsilon},$$
$$-(5\tau - 1 - \epsilon) + \frac{\epsilon_1(5\tau - 1 - \epsilon)}{3(\tau - \epsilon_1)} \leq -(\tau - \epsilon_1),$$
$$4\tau \geq 1 + \epsilon - \epsilon_1 + \frac{\epsilon_1(5\tau - 1 - \epsilon)}{3(\tau - \epsilon_1)}.$$

Since $\sum_{i<j} \beta_{i,j} + 2\sum_i \gamma_i = 4\tau - 1$, we have $\sum \gamma_i = \epsilon + \epsilon_1 \leq (4\tau - 1)/2$. So $4\tau \geq 2\epsilon + 1$, and it remains to show that

$$\frac{\epsilon_1(5\tau - 1 - \epsilon)}{3(\tau - \epsilon_1)} \leq \epsilon_1 \quad \text{or equivalently,} \quad 2\tau \leq 1 + \epsilon - 3\epsilon_1.$$

Now one can verify that $2\tau \leq 1 - 3\epsilon_1$, since $\epsilon_1 \leq (4\tau - 1)/2$ and $\tau \leq 0.3$. □

It follows from Claim 2.8 that

$$g(P) \geq \left(\prod_i \alpha_i\right)\left(1 + \frac{4\tau - 1}{\tau}\right) = \left(\prod_i \alpha_i\right)\left(\frac{5\tau - 1}{\tau}\right).$$

Claim 2.8, together with (5), also implies that $\sum \alpha_i = 2 - 4\tau$. As $\alpha_i \in [1 - 3\tau, \tau]$, the term $\prod \alpha_i$ is minimized when, say, $\alpha_1 = \alpha_2 = \tau$ and $\alpha_3 = \alpha_4 = 1 - 3\tau$ (and then $\beta_{3,4} = 4\tau - 1$).

Claim 2.9 $\prod \alpha_i$ *is minimized when* $\alpha_1 = \alpha_2 = \tau$ *and* $\alpha_3 = \alpha_4 = (1 - 3\tau)$.

Proof Recall that each α_i lies in the closed interval $[(1 - 3\tau), \tau]$. If each $\alpha_i > (1 - 3\tau)$, pick the smallest of them, say α_4, set $\alpha_4' = \alpha_4 - \epsilon$ for a small enough $\epsilon > 0$, and add this excess to any other variable that is less than τ, say α_3 (there always exists another variable less than τ, else $\sum \alpha_i > (1 - 3\tau) + 3\tau = 1$, a contradiction). Then $(\alpha_3 + \epsilon)(\alpha_4 - \epsilon) < \alpha_3\alpha_4$ since $\alpha_4 \leq \alpha_3$, minimizing the product further. Similarly, α_3 is also $(1 - 3\tau)$ in the configuration minimizing $\prod \alpha_i$. So we get that $\alpha_1 + \alpha_2 = (2 - 4\tau) - 2(1 - 3\tau) = 2\tau$. As each α is at most τ, this forces $\alpha_1 = \alpha_2 = \tau$. □

It can be verified that all the constraints are satisfied, and so we get the required lower bound for $g(P)$:

$$g(P) \geq \left(\prod_i \alpha_i\right)\left(\frac{5\tau-1}{\tau}\right) \geq \tau^2(1-3\tau)^2\frac{5\tau-1}{\tau}. \qquad \square$$

We can now complete the proof of our main result.

Proof of Theorem 1.1 Wagner [8] proved that any point of depth τn in $\mathbb{R}^3$ is contained in at least $f(\tau)n^4$ simplices spanned by P, where $f(\tau) = (4\tau^3 - 6\tau^4)/4!$ and $0.25 \leq \tau \leq 0.5$. If P has depth at least τn, where $\tau \geq 0.2889$, then as $f'(\tau) \geq 0$ for $\tau \in [0.25, 0.5]$, we can deduce that $f(\tau) \geq f(0.2889) = 0.00227$, and so there exists a point lying in at least $0.00227n^4$ simplices spanned by P.

Otherwise, as depth is always an integer, P has depth at most $\tau n - 1$, where $\tau \leq 0.2889$. By Lemma 2.7, we can conclude that there exists a point lying in $g(\tau)n^4$ simplices, where $g(\tau) = \tau(1-3\tau)^2(5\tau-1)$. As $g'(\tau) \leq 0$ for $\tau \in [0.25, 0.3]$, we can deduce that $g(\tau) \geq g(0.2889) = 0.00227$, and so there exists a point lying in at least $0.00227n^4$ simplices spanned by P. $\square$

3 Conclusion

The conjecture of [6] that there always exists a point contained in at least $(n/(d+1))^{d+1}$ simplices spanned by any n points in $\mathbb{R}^d$ is an elegant one. So far, it has only been proven in $\mathbb{R}^2$ [4], and in this paper, we have made a step toward the optimal bound for $\mathbb{R}^3$. This indicates that the current lower bounds are not tight and gives more strength to their conjecture. The full conjecture, however, is still open.

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